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# ITERATIVE REFINEMENT OF FINITE ELEMENT APPROXIMATIONS FOR ELLIPTIC PROBLEMS (*)

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Résumé — *On présente une extrapolation itérative d'approximations de problèmes elliptiques par des éléments finis de bas degré*

Abstract — *An iterative refinement of low-degree finite element approximations for elliptic problems is presented*

**1.** We will consider the boundary value problem

$$\Delta u + \sum a_i \frac{\partial u}{\partial x_i} + bu = -f \quad \text{in} \quad \Omega ,$$

$$u = 0 \qquad \text{on } \partial\Omega . \tag{1}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$ sufficiently smooth. We will adopt the standard notations (*cf.* Gilbarg-Trudinger, 1977). Especially $(.,.)$ respective $(.,.)_1$ denote the $L_2(\Omega)$-inner-product respective the Dirichlet integral and $\| \cdot \|_k$ the norm in $H_k = W_2^k(\Omega)$.

The weak formulation of problem (1) is $u \in \overset{\circ}{H}_1$ and

$$(u, v)_1 = (\sum a_i u |_i + bu + f, v) \quad \text{for} \quad v \in \overset{\circ}{H}_1 . \tag{2}$$

Our basic assumption is : problem (1) resp. (2) has a unique solution $u$ to $f \in H_0$ with $u \in \overset{\circ}{H}_1 \cap H_2$ and $\| u \|_2 \leqslant c \| f \|$. Now let $S_h$ be the space of linear finite

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elements with isoparametric modifications in the boundary elements such that $S_h \subset \overset{\circ}{H}_1$ holds true. Due to an argument of Schatz (1974) for $h$ sufficiently small the Galerkin-approximation $u^0 = u_h \in S_h$ defined by

$$(u^0, \chi)_1 = (\sum a_i u^0 |_i + bu^0 + f, \chi) \quad \text{for} \quad \chi \in S_h \tag{3}$$

is uniquely defined. The error estimate

$$\| u - u^0 \| + h \| u - u^0 \|_1 \leqslant ch^2 \| u \|_2 \tag{4}$$

is well known.

In Lin Qun (1978), (1980) we introduced a refinement of $u^0$ on th basis of the additional assumption : to $F \in H_0$ given the solution of

$$\begin{aligned} - \Delta U &= F \quad \text{in} \quad \Omega , \\ U &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{5}$$

resp. $U \in \overset{\circ}{H}_1$ and

$$(U, v)_1 = (F, v) \quad \text{for} \quad v \in \overset{\circ}{H}_1 \tag{6}$$

is computable. Then given $u^0$ we can compute $\bar{u}^0$ defined by $\bar{u}^0 \in \overset{\circ}{H}_1$ and

$$(\bar{u}^0, v)_1 = (\sum a_i u^0 |_i + bu^0 + f, v) \quad \text{for} \quad v \in \overset{\circ}{H}_1 . \tag{7}$$

This leads to a higher accuracy in the $H_1$-norm :

$$\| u - \bar{u}^0 \|_1 \leqslant ch^2 \| u \|_2 . \tag{8}$$

Of course $\bar{u}^0$ is not an element of $S_h$.

Following a suggestion of Nitsche (private communication) we construct starting with the pair $(u^0, \bar{u}^0)$ iterates $(u^{\nu+1}, \bar{u}^{\nu+1})$ for $\nu \geqslant 0$ defined

$$u^{\nu+1} = \bar{u}^\nu + \varphi^\nu \tag{9}$$

with $\varphi^\nu \in S_h$ and

$$\begin{aligned} &(\varphi^\nu, \chi)_1 - (\sum a_i \varphi^\nu |_i + b\varphi^\nu, \chi) = \\ &\qquad = (\sum a_i(\bar{u}^\nu - u^\nu) |_i + b(\bar{u}^\nu - u^\nu), \chi) \quad \text{for} \quad \chi \in S_h \end{aligned} \tag{10}$$

and on the other hand by ($\nu \geqslant 0$)

$$(\bar{u}^\nu, v)_1 = (\sum a_i u^\nu |_i + bu^\nu + f, v) \quad \text{for} \quad v \in \overset{\circ}{H}_1 . \tag{11}$$

In Section 3 we give the proof of :

THEOREM 1 : *Let $(u^\nu, \bar{u}^\nu)$ be defined as above. Then*

$$\| u - u^\nu \| + \| u - \bar{u}^\nu \|_1 \leqslant (ch)^{\nu+2} \| u \|_2$$

*is valid.*

**2.** Our proof is based on the following operator frame work (*cf.* Chatelin, 1981, Hackbusch, 1981). Let us consider the equation

$$u = Ku + y \tag{12}$$

in a Banach-space $X$ with $K$ being a linear compact operator. Further let $S$ be an approximating subspace and $P : X \to S$ a bounded projection onto $S$. The standard Galerkin solution is defined by

$$u^0 = PKu^0 + Py\,. \tag{13}$$

Now we construct iterates $\bar{u}^\nu$ and $u^{\nu+1}$ in the way

$$\bar{u}^\nu = Ku^\nu + y\,, \tag{14}$$

$$u^{\nu+1} = \bar{u}^\nu + r^\nu \tag{15}$$

with $r^\nu$ defined by

$$r^\nu = PKr^\nu + PK(\bar{u}^\nu - u^\nu)\,. \tag{16}$$

*Remark* 1 : $d^\nu = \bar{u}^\nu - u^\nu = Ku^\nu - u^\nu + y$ is the defect of the $\nu$-th iterate. Therefore $r^\nu$ may be interpreted as the Galerkin-solution to the right hand side $Kd^\nu$.

*Remark* 2 : The approximations $\bar{u}^0$ are also considered in Sloan (1976), but the higher iterates introduced there differ from ours.

LEMMA 1 : *Suppose that $K$ is compact, 1 is not an eigenvalue of $K$ and $\kappa : = \| (I - P)\, K \|$ is sufficiently small.*

*Then $(I - PK)^{-1}$ exists as a bounded operator in $X$ and the Galerkin solutions are well defined. Moreover*

$$u - u^\nu = (I - PK)^{-1}\,(I - P)\,K(u - u^{\nu-1})\,. \tag{17}$$

*Proof* : Since $(I - K)^{-1}$ is bounded for $\kappa$ small enough also $(I - PK)^{-1}$ is bounded. As a consequence the Galerkin solution is uniquely defined. The identity

$$(I - K)^{-1} = (I - PK)^{-1} + (I - PK)^{-1}\,(I - P)\,K(I - K)^{-1} \tag{18}$$

will be useful. The solution $u$ of (12) may be written in the form

$$u = (I - PK)^{-1} y + (I - PK)^{-1} (I - P) Ku . \tag{19}$$

Because of our construction we have

$$\begin{aligned} u^{\nu+1} &= Ku^{\nu} + y + (I - PK)^{-1} PK(Ku^{\nu} + y - u^{\nu}) \\ &= (I - PK)^{-1} y + (I - PK)^{-1} (I - P) Ku^{\nu} . \end{aligned} \tag{20}$$

Subtraction of (20) from (19) gives (17).

*Remark* 3 : We mention that under our assumptions also $(I - KP)^{-1}$ exists and the recurrence relation

$$u - \bar{u}^{\nu} = (I - KP)^{-1} K(I - P)(u - \bar{u}^{\nu-1}) \tag{21}$$

is valid. The proof is omitted.

By our assumptions $\| u^0 \|$ is bounded by a multiple of $\| y \|$. Because of

$$\| (I - PK)^{-1} \| \leqslant \frac{\gamma}{1 - \kappa\gamma} \tag{22}$$

with $\gamma$ being the norm of $\| (I - K)^{-1} \|$ we conclude from lemma 1 :

COROLLARY 1 : *Let* $\kappa = \| (I - P) K \|$ *be less than the half of*

$$\gamma^{-1} = \| (I - K)^{-1} \|^{-1} .$$

*Then error-estimates of the type*

$$\| u - u^{\nu} \| \leqslant c \left\{ \frac{\kappa\gamma}{1 - \kappa\gamma} \right\}^{\nu} \| y \| \tag{23}$$

*hold true.*

**3.** Now we come back to the situation discussed in section 1. We identify $X$ with the Hilbertspace $H_0 = L_2(\Omega)$. Since we want to work with the Ritz-method we have to impose the condition $S \subseteq \overset{\circ}{H}_1$. For simplicity we focuss our attention to the case : $S = S_h$ is the space of linear finite elements with isoparametric modifications along the boundary. Further let $P = R_h$ be the standard Ritz-projection defined by $Pu \in S_h$ and

$$(Pu, \chi)_1 = (u, \chi)_1 \quad \text{for} \quad \chi \in S_h . \tag{24}$$

The operator $K$ is defined by

$$w = Kv \Leftrightarrow w \in \overset{\circ}{H}_1 \quad \text{and} \quad (w, g)_1 = (v, -\sum (a_i g)|_i + bg) \quad \text{for} \quad g \in \overset{\circ}{H}_1 . \tag{25}$$

Under suitable conditions concerning the regularity of $a_i$, $b$ and since the original problem (1) resp. (2) is assumed to be uniquely solvable $K$ is a bounded operator from $H_0$ into $\overset{\circ}{H}_1$ and hence compact as mapping of $H_0$ into itself.

By duality the error-estimate

$$\| u - Pu \| \leqslant ch \| u \|_1 \tag{26}$$

is a consequence of (4). Because of

$$\| (I - P) Kv \| \leqslant ch \| Kv \|_1 \leqslant c' h \| v \| \tag{27}$$

we find

$$\kappa = \kappa_h = \| (I - P) K \| \leqslant ch \tag{28}$$

with some constant $c$.

The estimates derived in section 2 lead to

$$\| u - u^\nu \| \leqslant (ch)^\nu \| u - u^0 \| \tag{29}$$

and because of (4) to

$$\| u - u^\nu \| \leqslant (ch)^{\nu + 2} \| u \|_2 . \tag{30}$$

Finally the terms $\| u - \bar{u}^\nu \|_1$ are bounded in the same way since by definition

$$u - \bar{u}^\nu = K(u - u^\nu) . \tag{31}$$

This completes the proof of theorem 1.

**4.** In this section we consider the model problem

$$\begin{aligned} - \Delta u &= f(., u) \quad \text{in} \quad \Omega \\ u &= 0 \qquad\quad \text{on} \quad \partial\Omega \end{aligned} \tag{32}$$

in two or three space dimensions. The weak formulation of (32) is : Find $u \in \overset{\circ}{H}_1$ such that

$$(u, v)_1 = (f(u), v) \quad \text{for} \quad v \in \overset{\circ}{H}_1 . \tag{33}$$

Our assumptions are :

(i) $f(x, z)$ is twice continuously differentiable with respect to $z \in \mathbb{R}$ and

$$|f_{zz}(x, z)| \tag{34}$$

is uniformly bounded.

(ii) For $z = u(x) \in C^0(\overline{\Omega})$ the functions $f(x, u(x))$, $f_z(x, u(x))$ anf $f_{zz}(x, u(x))$ are in $C^0(\overline{\Omega})$.

(iii) $u$ is an isolated solution of (32), i.e. the linear problem

$$(w, g)_1 = (f'(u)\, w, g) \quad \text{for} \quad g \in \overset{\circ}{H}_1 \tag{35}$$

admits only $w = 0$ in $\overset{\circ}{H}_1$.

Now let $u^0 = u_h \in S_h$ be the solution of the corresponding Galerkin-problem

$$(u^0, \chi)_1 = (f(u^0), \chi) \quad \text{for} \quad \chi \in S_h\,. \tag{36}$$

Corresponding to section 1 we define the iterates $\overline{u}^\nu$ for $\nu \geqslant 0$ by

$$(\overline{u}^\nu, g)_1 = (f(u^\nu), g) \quad \text{for} \quad g \in \overset{\circ}{H}_1\,, \tag{37}$$

and

$$u^{\nu+1} = \overline{u}^\nu + \varphi^\nu \tag{38}$$

with $\varphi^\nu \in S_h$ and

$$(\varphi^\nu, \chi)_1 = (f'(u^0)(\varphi^\nu + \overline{u}^\nu - u^\nu), \chi) \quad \text{for} \quad \chi \in S_h\,. \tag{39}$$

The counterpart of theorem 1 is :

THEOREM 2 : *Let $(u^\nu, \overline{u}^\nu)$ be defined as above. Then*

$$\| u - u^\nu \| + \| u - \overline{u}^\nu \|_2 \leqslant c_1 (c_2\, h^2)^{\nu+1} \tag{40}$$

*is valid. The constants $c_1$, $c_2$ depend on u and bounds of $f_z$, $f_{zz}$ but are independent of h and* $\nu$.

*Proof* : Let $K : H_0 \to \overset{\circ}{H}_1 \cap H_2$ be the inverse of the Laplacian defined by

$$w = Kv \Leftrightarrow (w, g)_1 = (v, g) \quad \text{for} \quad g \in \overset{\circ}{H}_1\,, \tag{41}$$

and let $P = R_h$ be the Ritz operator defined by

$$\Phi = Pv \Leftrightarrow \Phi \in S_h \quad \text{and} \quad (\Phi, \chi)_1 = (v, \chi)_1 \quad \text{for} \quad \chi \in S_h\,. \tag{42}$$

Problem (32) is equivalent to $u = Kf(u)$. We may rewrite this in the form

$$(I - PKf'(u^0))\, u = Kf(u) - PKf'(u^0)\, u\,. \tag{43}$$

In terms of $K$ and $P$ the iterates $\bar{u}^\nu$ and $\varphi^\nu$ have the representation

$$\bar{u}^\nu = Kf(u^\nu)\,, \tag{44}$$

$$(I - PKf'(u^0))\, \varphi^\nu = PKf'(u^0)\,(\bar{u}^\nu - u^\nu)\,. \tag{45}$$

This leads to

$$(I - PKf'(u^0))\, u^{\nu+1} = Kf(u^\nu) - PKf'(u^0)\, u^\nu\,. \tag{46}$$

By comparison of (43) and (46) and by adding and subtracting appropriate terms we come to

$$\begin{aligned}(I - PKf'(u^0))\,(u^{\nu+1} - u) &= (I - P)\, Kf'(u^0)\,(u^\nu - u) + \\ &+ K\,\{ f(u^\nu) - f(u) - f'(u)\,(u^\nu - u) + f'(u) - f'(u^0) \}\,(u^\nu - u)\,. \end{aligned} \tag{47}$$

The Ritz operator $P$ is the orthogonal projection in $H_1$ onto $S = S_h$. For $v, w \in H_0$ arbitrary we get

$$\begin{aligned}((I - P)\, Kv, w) &= ((I - P)\, Kv, Kw)_1 \\ &= ((I - P)\, Kv, (I - P)\, Kw)_1 \\ &\leqslant ch^2 \,\| Kv \|_2 \,\| Kw \|_2 \leqslant ch^2 \,\| v \| \,\| w \|\,. \end{aligned} \tag{48}$$

This implies that the norm of $(I - P)\, K$ as mapping of $H_0$ into $H_0$ is bounded by $ch^2$. Next let a be a continuous function and $v, w \in H_0$. Then also $K(avw)$ is in $H_0$ and

$$\| K(avw) \| \leqslant c \,\| v \| \,\| w \|\,. \tag{49}$$

This follows from

$$\| K(avw) \|_0 = \sup \{ (Kavw, g) \mid \| g \| = 1 \} \tag{50}$$

and

$$(K(avw), g) = (v, \{ aKg \}\, w) \tag{51}$$

in combination with Sobolev's embedding lemma.

For $h$ small enough the initial Galerkin solution $u^0$ is " near " to $u$. Because of our assumption (iii) then the operator $I - PKf'(u^0)$ will have a bounded inverse.

By the aid of these arguments we derive from the recurrence relation (47) the corresponding error bound

$$\| u^{\nu+1} - u \| \leqslant c_3 h^2 \| u^{\nu} - u \| + c_4 \| u^{\nu} - u \|^2 + \\ + c_5 \| u^0 - u \| \| u^{\nu} - u \| . \quad (52)$$

For the sake of clarity we have numbered the constants. Since an estimate of the type

$$\| u^0 - u \| \leqslant ch^2 \quad (53)$$

holds true anyway we derive from (52)

$$\| u^{\nu+1} - u \| \leqslant c_6 h^2 \| u^{\nu} - u \| + c_4 \| u^{\nu} - u \|^2 . \quad (54)$$

Because of (53) by complete inductions there is a constant $c_7$ such that for $h \leqslant h_0$ with $h_0$ chosen appropriate the relation

$$\| u^{\nu+1} - u \| \leqslant c_7 h^2 \| u^{\nu} - u \| \quad (55)$$

holds true (55) together with (53) lead to the error bound stated in theorem 2 for $u^{\nu} - u$.

Because of

$$\bar{u}^{\nu} - u = K(f(u^{\nu}) - f(u)) \quad (56)$$

we come to

$$\begin{aligned} \| \bar{u}^{\nu} - u \|_2 &\leqslant c \| f(u^{\nu}) - f(u) \| \\ &\leqslant c \| u^{\nu} - u \| . \end{aligned} \quad (57)$$

*Remark* 3 : Whereas assumption (iii) is essential the two preceding ones can be reduced.

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