

REVUE FRANÇAISE D'INFORMATIQUE ET DE RECHERCHE OPÉRATIONNELLE. SÉRIE ROUGE

D. DE WERRA

Equitable colorations of graphs

Revue française d'informatique et de recherche opérationnelle. Série rouge, tome 5, n° R3 (1971), p. 3-8

http://www.numdam.org/item?id=M2AN_1971__5_3_3_0

© AFCET, 1971, tous droits réservés.

L'accès aux archives de la revue « Revue française d'informatique et de recherche opérationnelle. Série rouge » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

EQUITABLE COLORATIONS OF GRAPHS (*)

par D. de WERRA (1)

Abstract. — *An edge coloration of a graph is a coloration of its edges in such a way that no two edges of the same colour are adjacent. We generalize this concept by introducing the notion of equitable coloration, i.e., coloration of the edges of a graph such that if $f_i(x)$ denotes the number of edges with colour i which are adjacent to vertex x , we have $|f_i(x) - f_j(x)| \leq 1$ for every vertex x and every pair of colours i, j . Equitable colorations are also defined for hypergraphs,*

Finally some results on edge colorations are generalized to the case of equitable colorations.

1. Coloration of Hypergraphs

A hypergraph $H = (X, U)$ consists of a finite set X of vertices $x_1, \dots, x_n$ and a family U of nonempty edges $U_j (j = 1, \dots, m)$ satisfying $\bigcup_{j=1}^m U_j = X$.

A hypergraph H is *unimodular* if its edge incidence matrix A ($a_{ij} = 1$ if $x_i \in U_j$ or 0 otherwise) is totally unimodular. The *subhypergraph* of $H = (X, U)$ spanned by a subset $Y \subset X$ is the hypergraph $H(Y) = (Y, U(Y))$ where $U(Y) = \{U_j \cap Y | U_j \cap Y \neq \emptyset\}$. An *equitable k -coloration* E of $H = (X, U)$ is a partition of X into k subsets $F_1, \dots, F_k$ such that for every edge U_j

$$| |U_j \cap F_p| - |U_j \cap F_q| | \leq 1 \quad \forall p, q \in \{1, \dots, k\}$$

The result of Camion [1] and Ghouila-Houri [2] about totally unimodular matrices may be formulated in terms of hypergraphs as follows [3] [4] :

Lemma : A hypergraph H is unimodular if and only if all its subhypergraphs have an equitable bicoloration.

We have the following :

Theorem 1 : A unimodular hypergraph H has an equitable k -coloration for any k .

(*) This research was supported by a grant from the National Research Council of Canada.

(1) Department of Management Sciences, University of Waterloo, (Canada).

(1) Actuellement. Département de Mathématiques École Polytechnique Fédérale de Lausanne (Suisse).

Proof : Given a coloration E of the vertices of H with k colours (E is not necessarily an equitable k -coloration), for each edge U_j we define a vector $E(j) = (f_1^j, f_2^j, \dots, f_k^j)$ where f_p^j is the number of vertices of U_j which have colour p . Let $F_p \subset X$ be the subset of vertices which have colour p . For every edge let $e(j) = \max_{p,q} (f_p^j - f_q^j) \geq 0$; let $e^* = \max_j e(j)$. If $e^* < 2$, E is an equitable k -coloration of H . Otherwise, let U_j be an edge such that $e(j) = e^* = f_p^j - f_q^j$. We consider the subgraph H' spanned by $F_p \cup F_q$. It follows from the lemma that H' has an equitable 2-coloration E' ; we colour its vertices with 2 colours p and q in such a way that $|f_p^j - f_q^j| \leq 1$ for every U_j . The values f_r^j are unchanged for $r \neq p, q$ and for every U_j . Thus at least one value $e(j)$ is such that the number of pairs p, q with $|f_p^j - f_q^j| \leq e(j) - 1$ has increased by at least one unit and the other $e(j)$ have not increased. This procedure can be repeated until $e^* < 2$. We get thus an equitable k -coloration of H . End of proof.

A *transversal* of a hypergraph $H = (X, U)$ is a subset of vertices T such that $T \cap U_j \neq \emptyset$ for $j = 1, \dots, m$. The following corollary is a slight generalization of a theorem in Berge [3].

Corollary 1 : Let $H = (X, U)$ be a unimodular hypergraph and $k = \min_j |U_j|$ the minimal cardinality of its edges. The set X of vertices of H may be partitioned into k transversals.

Proof : Consider an equitable k -coloration of H where $k = \min_j |U_j|$; such a k -coloration exists from theorem 1. Clearly in each edge there will be at least one vertex of each colour. Hence the subsets $F_1, \dots, F_k$ defined by the k -coloration are transversals.

Following Berge [3], we call *strong chromatic number* of $H = (X, U)$ the smallest integer k such that there exists a partition of X into subsets $F_1, \dots, F_k$ with $|F_i \cap U_j| \leq 1$ $i = 1, \dots, k$. The next corollary is due to Berge [5].

Corollary 2 : The strong chromatic number of a unimodular hypergraph is equal to the maximal cardinality of its edges.

Proof : Let $k = \max_j |U_j|$ and consider an equitable k -coloration of H ; let F_i be the set of vertices with colour i for $i = 1, \dots, k$. Obviously

$$|F_i \cap U_j| \leq 1 \quad \begin{matrix} i = 1, \dots, k \\ j = 1, \dots, m \end{matrix}$$

We can also apply theorem 1 to graphs; an equitable k -coloration of a graph is then a coloration of its edges with k colours such that for each vertex x , we have :

$$|f_p(x) - f_q(x)| \leq 1 \quad \forall p, q \in \{1, \dots, k\}$$

where $f_p(x)$ denotes the number of edges with colour p which are adjacent to x .

Corollary 3 : A bipartite graph $G = (X, U)$ has an equitable k -coloration for any k .

Proof : This result is obtained by applying theorem 1 to the hypergraph H obtained as follows : its vertices are the edges of G and its edges are the sets of edges which are adjacent to the same vertex of G . H is unimodular since its edge incidence matrix is the transposed matrix of the edge incidence matrix of G .

When applied to the case of graphs, corollary 1 becomes the theorem of Gupta [3] : If $G = (X, U)$ is a bipartite graph with minimum degree k , then there exists a partition of U into k spanning subsets of edges $H_1, \dots, H_k$. (H_i is a spanning subset if the edges in H_i meet all vertices of G .)

Moreover corollary 2 gives the well-known result : the chromatic index of a bipartite graph is equal to the maximum degree of the vertices in G (the chromatic index of G is by definition the smallest k such that the edges of G may be partitioned into k subsets of nonadjacent edges).

2. P-bounded colorations

We will now generalize some results about edge colorations. A p -bounded k -coloration E of a graph G is a partition of its edges into k nonempty subsets $F_1, \dots, F_k$ such that for any vertex x : $|f_j(x) - f_i(x)| \leq p$ for $i, j = 1, \dots, k$ where $f_j(x)$ is the number of edges of F_j which are adjacent to x . An equitable k -coloration is thus a 1-bounded k -coloration. Let $E = \{F_1, \dots, F_k\}$ be a p -bounded k -coloration and $f_1 \geq \dots \geq f_k$ the cardinalities of $F_1, \dots, F_k$ respectively.

Theorem 2 : If the sequence $(f_1, \dots, f_k)$ corresponds to a p -bounded k -coloration of G , then any sequence $f'_1, \dots, f'_k$ with :

- a) $f'_1 \geq \dots \geq f'_k$
- b) $\sum_{i=1}^l f'_i \leq \sum_{i=1}^l f_i \quad l = 1, \dots, k-1$
- c) $\sum_{i=1}^k f'_i = \sum_{i=1}^k f_i$

corresponds to a p -bounded k -coloration of G .

Proof : A) We first prove that any couple of subsets F_i, F_j in E with $f_i - f_j = K \geq 2$ may be replaced by two subsets F'_i, F'_j with $f'_i - f'_j = K - 2$. $E_{ij} = (F_i, F_j)$ is a p -bounded bicoloration of $G_{ij} = (X, F_i \cup F_j)$; we consider any edge u in G_{ij} and construct an alternating path P containing u (i.e., the

edges of which belong alternately to F_i and F_j); we extend the path P as far as possible; we obtain thus either an alternating circuit (with even length) or an alternating open path. We remove P from G_{ij} and repeat the same construction with another edge u , until all edges in G_{ij} are removed.

Since $f_i - f_j = K \geq 2$, there is at least one alternating path P in which the first edge and last edge belong to F_i ; we interchange the edges of $P \cap F_i$ and $P \cap F_j$.

Let x and y be the endpoints of P . Since P terminates at x with an edge in F_i we have $f_i(x) \geq f_j(x) + 1$; by interchanging the edges of P we get

$$f_j(x) \leq f_i'(x) = f_i(x) - 1 \leq f_i(x)$$

$$f_j(x) \leq f_j'(x) = f_j(x) + 1 \leq f_i(x)$$

The same inequalities hold for y . Furthermore, for all vertices $z \neq x, y$ we have $f_i'(z) = f_i(z)$ and $f_j'(z) = f_j(z)$. So we obtain a new p -bounded bicolouration (F_i', F_j') with $f_i' - f_j' = K - 2$.

B) By successive applications of the above described procedure we can obtain p -bounded k -colorations corresponding to any sequence $(f_1', \dots, f_k')$ satisfying a), b) and c). This ends the proof.

Theorem 2 is a generalization of a result which appears in Folkman and Fulkerson [6]. (Their theorem corresponds to the case where $p = 1$ and k is at least equal to the chromatic index of G .) We raise now and answer the following question : given a graph G , what is the smallest value p such that G has a p -bounded k -coloration for any k ? From corollary 3, we know that if G is bipartite, then the minimum value of p is $p = 1$. If G is not bipartite, it is not the case : a triangle has for instance no equitable 2-coloration. (Clearly for any k not less than the chromatic index of G , there is a 1-bounded k -coloration of G .)

Théorème 3 : Let G be any graph; for any k , G has a 2-bounded k -coloration.

Proof : The theorem is true for a graph G with one edge. Suppose that it is true for graphs with at most $m - 1$ edges; we will show that it is also true for graphs with m edges. Let G be a graph with m edges; let us remove from G an edge u joining vertices x and y . By our induction hypothesis, $G' = G - u$ has a 2-bounded k -coloration for any k . Given some integer k , let $F_1, \dots, F_k$ be the subsets of edges defined by such a k -coloration of G' . There exist 2 integers $a, b \geq 0$ such that

$$a \leq f_i(x) \leq a + 2 \quad \text{for } i = 1, \dots, k$$

$$b \leq f_i(y) \leq b + 2 \quad \text{for } i = 1, \dots, k$$

We can assume that there is at least one colour, say q , such that $f_q(x) = a$ (otherwise a is replaced by $a + 1$); similarly there is one colour r such that $f_r(y) = b$. We have to examine the following cases :

A) There is a colour s with $f_s(x) < a + 2$ and $f_s(y) < b + 2$. Then u may be introduced into F_s and $F_1, \dots, F_k$ is a 2-bounded k -coloration of G .

B) For every colour s with $f_s(x) < a + 2$ we have $f_s(y) = b + 2$ and for every colour t with $f_t(y) < b + 2$ we have $f_t(x) = a + 2$. Let us consider colours q and r ; we have $q \neq r$ (otherwise we are in case A).

We determine an alternating chain C starting at x with an r -edge (i.e., an edge in F_r) and whose edges are alternately r -edges and q -edges. We extend chain C as far as possible. Then 2 cases may occur :

B1) The last vertex in C is y ; so the last edge in C is a q -edge (because if we arrive at y with an r -edge we can introduce one more q -edge into C since $f_q(y) = b + 2 > f_r(y) = b$). By interchanging the q -edges and the r -edges in C we obtain a 2-bounded k -coloration of G' with $f_q(x) = f_r(x) = a + 1$ and $f_q(y) = f_r(y) = b + 1$. So u may be introduced into F_q (or F_r) and $F_1, \dots, F_k$ is a 2-bounded k -coloration of G .

B2) The last vertex in C is $z \neq y$. Again by interchanging the q -edges and the r -edges in C we obtain a 2-bounded k -coloration of G' with $f_r(x) = a + 1$, $f_r(y) = b$ (if C ends for instance with a q -edge we have $f_r(z) + 2 \geq f_q(z) > f_r(z)$) and after having interchanged the r -edges and the q -edges, we still have $|f_r(z) - f_q(z)| \leq 2$.

We can now introduce edge u into F_r and we still obtain a 2-bounded k -coloration of G .

We have examined all possible cases and the proof is completed.

We now define an *odd cycle* as a connected graph containing an odd number of edges and such that all degrees are even.

Theorem 4 : A connected graph G has an equitable bicolouration if and only if it is not an odd cycle.

Proof : A) Suppose G is an odd cycle; for any equitable bicolouration $\{F_1, F_2\}$ we must have $f_1(x) = f_2(x)$ at each vertex x . Hence, F_1 and F_2 have the same cardinality; but this is not possible since G contains an odd number of edges.

B) Conversely if G is not an odd cycle, then from Euler's theorem, the edges of G may be partitioned into a unique even cycle (if all degrees are even) or into one or more chains joining 2 vertices with odd degrees. By coloring the edges in each chain (or in the unique cycle if all degrees are even) alternately with colours 1 and 2 we obtain an equitable bicolouration of G .

Necessary and sufficient conditions for a graph G to have an equitable k -coloration ($k > 2$) are much more difficult to obtain (this would in fact solve the four color problem). However we can formulate :

Proposition : If in a connected graph G all degrees are multiples of k and if the number of edges is not a multiple of k , then G has no equitable k -coloration.

Proof as in theorem 4, A.

However even if all degrees and the number of edges in a connected graph G are multiples of k , G may not have an equitable k -coloration for $k > 2$.

REFERENCES

- [1] P. CAMION, *Characterization of totally unimodular matrices*, Proc. Am. Math. Soc., vol. 16, n° 5, october 1965.
- [2] A. GHOUILA - HOURI, *Caractérisation des matrices totalement uni-modulaires*, C. R. Acad. Sci. Paris, vol. 254, 1192.
- [3] C. BERGE, *Théorie des graphes et des hypergraphes*, Dunod, Paris, 1970.
- [4] C. BERGE, *Sur certains hypergraphes généralisant les graphes bipartites*, pp. 119-133 in *Combinatorial Theory and its Applications*, Balatonfüred (P. Erdős, A. Renyi, V. T. Sos, ed.) North-Holland Publishing Co., Amsterdam, London, 1970.
- [5] C. BERGE, *Six papers on graph theory*. Indian Statistical Institute, Calcutta, 1963.
- [6] J. FOLKMAN and D. R. FULKERSON, *Edge Colorings in Bipartite Graphs*, Chapter 31 in *Combinatorial Mathematics and their Applications* (R. C. Bose and T. A. Dowling, ed.), University of North Carolina Press, Chapel Hill, 1969).
- [7] D. de WERRA, *Some Combinatorial Problems Arising in Scheduling*, Canad. Op. Res. Soc. Journal, vol. 8, n° 3, november 1970.