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Anna Kazeykina

Solitons et comportement en grand temps des solutions d'une équation multidimensionnelle intégrable

Résumé

L'équation de Novikov-Veselov est un analogue (2+1)-dimensionnel de l'équation classique de Korteweg-de Vries, intégrable via la transformation de diffusion inverse pour l'équation de Schrödinger bidimensionnelle stationnaire. Dans cet exposé on présente quelques résultats récents sur l'existence et l'absence de solitons algébriquement localisés pour l'équation de Novikov-Veselov ainsi que quelques résultats sur le comportement en grand temps des "inverse scattering" solutions de cette équation.

Abstract

Novikov-Veselov equation is a (2+1)-dimensional analog of the classic Korteweg-de Vries equation integrable via the inverse scattering translform for the 2-dimensional stationary Schrödinger equation. In this talk we present some recent results on existence and absence of algebraically localized solitons for the Novikov-Veselov equation as well as some results on the large time behavior of the "inverse scattering solutions" for this equation.

1. Introduction

In this talk we are concerned with the Novikov-Veselov equation (NV):

$$\begin{aligned} &\partial_t v = 4\mathrm{Re}(4\partial_z^3 v + \partial_z(vw) - E\partial_z w), \\ &\partial_{\bar z} w = -3\partial_z v, \quad v = \bar v, \quad E \in \mathbb{R}, \\ &v = v(x,t), \quad w = w(x,t), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \end{aligned} \tag{1.1}$$

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where the following notations are used

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \partial_{\bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Equation (1.1) is the most natural (from the mathematical point of view) $(2+1)$-dimensional analogue of the classic Korteweg-de Vries equation. If $v = v(x_1, t)$, $w = w(x_1, t)$, then NV reduces to KdV. In addition, NV is integrable via the inverse scattering transform (IST) for the 2-dimensional stationary Schrödinger equation at fixed energy:

$$L\psi = E\psi, \quad L = -4\partial_z\partial_{\bar z} + v(z), \quad z = x_1 + ix_2, \quad x \in \mathbb{R}^2, \quad E = E_{fix}. \tag{1.2}$$

Equation (1.1) was contained implicitly in [21] as an equation possessing the following representation

$$\frac{\partial(L-E)}{\partial t} = [L - E, A] + B(L - E), \tag{1.3}$$

where L is the operator of the corresponding scattering problem and A, B are some appropriate differential operators. For the particular case of the 2-dimensional Schrödinger operator L as in (1.2), the explicit form of A and B and the corresponding evolution equation (1.1), with its higher order analogues, were given in [26], [27], where equation (1.1) was also studied in the periodic setting.

As $E \to \pm\infty$ in (1.1), NV reduces to another renowned $(2 + 1)$-dimensional analogue of KdV equation, Kadomtsev-Petviashvili equation (KPI and KPII, respectively). Note also that there exists a bidimensional generalization of the Miura transform which maps solutions of the modified Novikov-Veselov equation (a $(2+1)$-dimensional analogue of mKdV and a member of the Davey-Stewartson II integrable hierarchy) to solutions of the Novikov-Veselov equation (see [2]). It is also worth mentioning that the stationary Novikov-Veselov equaton at $E = 0$ describes isothermically asymptotic surfaces in projective differential geometry (see [7]) and the dispersionless Novikov-Veselov at $E = 0$ was derived in the framework of nonlinear optics (see [19]).

In this talk we will study the behavior of regular, sufficiently localized solutions of (1.1) satisfying the following conditions:

$$\begin{aligned} &v, w \in C(\mathbb{R}^2 \times \mathbb{R}), \quad v(\cdot, t) \in C^3(\mathbb{R}^2) \quad \forall t \in \mathbb{R}; \\ &|v(x)| < q(1 + |x|)^{-2-\varepsilon}, \quad \varepsilon > 0,\ q > 0; \\ &w \to 0,\ \text{as } |x| \to \infty. \end{aligned}$$

Our major interest will be devoted to soliton solutions of (1.1). We will say that solution (v, w) of (1.1) is a soliton (or, in other words, a travelling wave), if $v(x, t) = V(x - ct)$ for a certain $c \in \mathbb{R}^2$.

The presentation of results is organized as follows. In section 2 we recall, in particular, some known notions and results from the direct and inverse scattering theory for the stationary two-dimensional Schrödinger equation. In section 3 we present the results on the large-time asymptotic behavior of the so-called "inverse scattering solutions" for the Novikov-Veselov equation at nonzero energy. In section 4 we discuss some recent results on existence and absence of algebraically localized solitons for the Novikov-Veselov equation. Finally, in section 5 we sketch the proof of absence of sufficiently localized solitons for the Novikov-Veselov equation.

2. Direct and inverse scattering for the two-dimensional stationary Schrödinger equation

Consider the stationary two-dimensional Schrödinger equation:

$$L\psi = E\psi, \quad L = -4\partial_z\partial_{\bar z} + v(z), \quad z = x_1 + ix_2, \quad x \in \mathbb{R}^2 \tag{2.1}$$

with a potential v satisfying the following conditions

$$\begin{aligned} &v(x) = \overline{v(x)}, \quad v(x) \in L^\infty(\mathbb{R}^2), \\ &|\partial_{x_1}^{j_1}\partial_{x_2}^{j_2} v(x)| < \frac{q}{(1+|x|)^{2+\varepsilon}} \text{ for some } q > 0,\ \varepsilon > 0, \quad j_1, j_2 \in \mathbb{N}\cup 0,\ j_1 + j_2 \leqslant 3. \end{aligned} \tag{2.2}$$

Scattering theory for (2.1) under condition (2.2) was developed in the works of P.G. Grinevich, S.V. Manakov, R.G. Novikov, S.P. Novikov during the several last decades (the summary of related results can be found in [13, 24, 12, 11]; see also references therein). Important contributions for the case $E = 0$ were made in [1, 29, 23].

In this section we will mostly present the notions and results of the scattering theory for the two-dimensional stationary Schrödinger equation at nonzero energy $E \neq 0$. At the end of the section we will briefly discuss some particular features of the problem at $E = 0$.

Direct scattering

It is known that for each $k \in \mathbb{R}^2$, such that $k^2 = E > 0$, there exists a unique continuous solution $\psi^+(x,k)$ of the two-dimensional Schrödinger equation (2.1) with the following asymptotic behavior:

$$\psi^+(x,k) = e^{ikx} - i\pi\sqrt{2\pi}\, e^{-\frac{i\pi}{4}} f\left(k, |k|\frac{x}{|x|}\right)\frac{e^{i|k||x|}}{\sqrt{|k||x|}} + o\left(\frac{1}{\sqrt{|x|}}\right), \tag{2.3}$$

as $|x| \to \infty$, for a certain f which is not known a priori. Function $\psi^+(x,k)$ is the classical scattering eigenfunction of equation (2.1). Function $f = f(k,l)$, $(k,l) \in \{k \in \mathbb{R}^2, l \in \mathbb{R}^2 \colon k^2 = l^2 = E\}$, arising in (2.3), is called the scattering amplitude of potential v. If $f(k,l) \equiv 0$ at the fixed energy $E > 0$, then the corresponding potential v is called transparent. In this talk we will mostly be concerned with transparent potentials, since the solitons of the Novikov-Veselov equation are transparent potentials at $E > 0$ (see [25]).

In [12] it was shown that there exist non trivial transparent potentials of equation (2.1) at $E > 0$ from the Schwartz class. This results implies, in particular, that the scattering amplitude f is not sufficient for reconstruction of potential v satisfying conditions (2.2). Thus, some additional scattering data need to be introduced.

For $\forall k \in \Sigma_E$, where

$$\begin{aligned} &\Sigma_E = \{k \in \mathbb{C}^2 \colon k^2 = E,\ \mathrm{Im} k \neq 0\},\ \text{if } E > 0, \\ &\Sigma_E = \{k \in \mathbb{C}^2 \colon k^2 = E\},\ \text{if } E < 0, \end{aligned} \tag{2.4}$$

we consider a solution $\psi(x,k)$ of equation (2.1) with the following asymptotic behavior:

$$\psi(x,k) = e^{ikx}(1 + o(1)), \quad |x| \to \infty. \tag{2.5}$$

This type of scattering solutions for (2.1) was first introduced by L.D. Faddeev (see [6]) and is usually referred to as Faddeev's exponentially growing solutions of (2.1), or complex geometrical optics solutions. Considering the following members in the asymptotic expansion for ψ, we obtain:

$$\psi(x,k) = e^{ikx} - \pi \mathrm{sgn}(\mathrm{Im}(k_2\bar{k}_1))e^{ikx}\left(\frac{a(k)}{-k_2x_1+k_1x_2} + \frac{e^{-2i\mathrm{Re}(kx)}b(k)}{-\bar{k}_2x_1+\bar{k}_1x_2} + o\left(\frac{1}{|x|}\right)\right), \tag{2.6}$$

as $|x|\to\infty$, for certain a, b. Functions $a(k)$, $b(k)$ are called nonphysical or Faddeev's scattering data for potential v.

Note that the scattering amplitude $f(k,l)$ satisfies the following relation

$$f(k,l) = \left(\frac{1}{2\pi}\right)^2 \iint_{\mathbb{R}^2} e^{-ilx}v(x)\psi^+(x,k)dx_1dx_2, \tag{2.7}$$

while the additional scattering data $a(k)$, $b(k)$ satisfy the following formulas:

$$a(k) = \left(\frac{1}{2\pi}\right)^2 \iint_{\mathbb{R}^2} e^{-ikx}v(x)\psi(x,k)dx_1dx_2, \tag{2.8}$$

$$b(k) = \left(\frac{1}{2\pi}\right)^2 \iint_{\mathbb{R}^2} e^{i\bar{k}x}v(x)\psi(x,k)dx_1dx_2. \tag{2.9}$$

In Born approximation ($\| v \| \ll E$) the scattering amplitude f represents the Fourier transform of v inside the ball $B_{2\sqrt{E}}(0)$, while b gives the Fourier transform of v outside the ball $B_{2\sqrt{E}}(0)$ (which gives the intuition behind the fact that both f and b are necessary to define a potential v at $E>0$ uniquely).

Note that the set Σ_E is two-dimensional, thus it is convenient to perform the following parameterization of Σ_E:

$$\lambda = \frac{k_1+ik_2}{\sqrt{E}}, \quad k_1 = \frac{\sqrt{E}}{2}\left(\lambda + \frac{1}{\lambda}\right), \quad k_2 = \frac{i\sqrt{E}}{2}\left(\frac{1}{\lambda} - \lambda\right). \tag{2.10}$$

Note that the above mapping transforms the set $\{k\in\mathbb{R}^2, k^2=E\}$ into the unit circle $|\lambda|=1$.

Define $z = x_1+ix_2$. In terms of z, λ asymptotics of ψ becomes:

$$\psi(z,\lambda) = \exp\left(\frac{i\sqrt{E}}{2}\left(\lambda\bar{z} + \frac{z}{\lambda}\right)\right)\mu(z,\lambda), \quad \mu(z,\lambda) = 1+o(1), \text{ as } |z|\to\infty \tag{2.11}$$

and relations (2.7), (2.8), (2.9) take the form

$$f(\lambda,\lambda') = \left(\frac{1}{2\pi}\right)^2 \iint_{\mathbb{C}} \exp\left(-\frac{i\sqrt{E}}{2}\left(\lambda'\bar{\zeta} + \frac{\zeta}{\lambda'}\right)\right)v(\zeta)\psi^+(\zeta,\lambda)d\mathrm{Re}\zeta d\mathrm{Im}\zeta,$$

$$a(\lambda) = \left(\frac{1}{2\pi}\right)^2 \iint_{\mathbb{C}} \exp\left(-\frac{i\sqrt{E}}{2}\left(\lambda\bar{\zeta} + \frac{\zeta}{\lambda}\right)\right)v(\zeta)\psi(\zeta,\lambda)d\mathrm{Re}\zeta d\mathrm{Im}\zeta,$$

$$b(\lambda) = \left(\frac{1}{2\pi}\right)^2 \iint_{\mathbb{C}} \exp\left(\frac{i\sqrt{E}\mathrm{sgn}(E)}{2}\left(\bar{\lambda}\zeta + \frac{\bar{\zeta}}{\bar{\lambda}}\right)\right)v(\zeta)\psi(\zeta,\lambda)d\mathrm{Re}\zeta d\mathrm{Im}\zeta,$$

where $\lambda' = (l_1+il_2)/\sqrt{E}$.

Function $\mu(z,\lambda)$, appearing in (2.11), can be defined as a solution of an integral equation which, in terms of function $m(z,\lambda) = (1+|z|)^{-(2+\varepsilon/2)}\mu(z,\lambda)$, is written as follows:

$$m(z,\lambda) = (1+|z|)^{-(2+\varepsilon/2)} + \\ + \iint_{\mathbb{C}} (1+|z|)^{-(2+\varepsilon/2)} g(z-\zeta,\lambda) \frac{v(\zeta)}{(1+|\zeta|)^{-(2+\varepsilon/2)}} m(\zeta,\lambda) d\mathrm{Re}\zeta d\mathrm{Im}\zeta \quad (2.12)$$

with

$$g(z,\lambda) = -\left(\frac{1}{2\pi}\right)^2 \iint_{\mathbb{C}} \frac{e^{\frac{i}{2}(p\bar{z}+\bar{p}z)}}{p\bar{p}+\sqrt{E}(\lambda\bar{p}+p/\lambda)} d\mathrm{Re}p d\mathrm{Im}p,$$

where $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$ and, if $E > 0$, then $|\lambda| \neq 1$.

The integral operator of this equation is a Hilbert-Schmidt operator. Thus we can consider its modified Fredholm determinant

$$\ln\Delta(\lambda) = \mathrm{Tr}(\ln(I - H(\lambda)) + H(\lambda)), \quad (2.13)$$

where $H(\lambda)$ is the integral operator of the direct scattering equation (2.12) (see [9] for the precise sense of this definition). The basic properties of Δ (as well as some properties of the Faddeev's scattering data) are given in section 5.

Define $\mathcal{E} = \{\lambda \in \mathbb{C} \colon \Delta(\lambda) = 0\}$ the set of values of the spectral parameter for which the integral equation (2.12) is not uniquely solvable. If $\mathcal{E} = \varnothing$ or, in other words, if the solution $\psi(x,k)$ of (2.1) with asymptotics (2.5) exists for $\forall k \in \Sigma_E$, we say that the equations of direct scattering are everywhere solvable (or the scattering data are everywhere well-defined). In particular, if potential v satisfies an appropriate "small norm" condition, then the equations of direct scattering are everywhere solvable. The scattering data $b(k)$, if they are everywhere well-defined, are sufficient to reconstruct potential v, transparent at $E > 0$.

Inverse scattering

Let $E < 0$ or let $E > 0$ and v be a transparent potential. Let T denote the unit circle on the complex plane. Function $\mu(z,\lambda)$ defined via (2.11), where $\psi(z,\lambda)$ is the Faddeev's solution of (2.1), satisfies the following properties:

- $\mu(z,\lambda)$ is continuous with respect to λ on $\mathbb{C}\backslash\mathcal{E}$; (2.14)

- $\mu(\lambda)$ satisfies the following $\bar{\partial}$-equation for $\lambda \in \mathbb{C}\backslash(0 \cup T \cup \mathcal{E})$:

$$\frac{\partial\mu(z,\lambda)}{\partial\bar{\lambda}} = r(z,\lambda)\overline{\mu(z,\lambda)}, \text{ where} \quad (2.15a)$$

$$r(z,\lambda) = \exp\left\{\frac{-i\sqrt{E}}{2}\left(1+(\mathrm{sgn}E)\frac{1}{\lambda\bar{\lambda}}\right)\left((\mathrm{sgn}E)z\bar{\lambda}+\bar{z}\lambda\right)\right\} r(\lambda) \text{ and} \quad (2.15b)$$

$$r(\lambda) = \frac{\pi}{\bar{\lambda}}\mathrm{sgn}(\lambda\bar{\lambda}-1)b(\lambda); \quad (2.15c)$$

- $\mu \to 1$ as $|\lambda| \to 0$, $|\lambda| \to \infty$. (2.16)

Equations (2.14)–(2.16) form the basis of the inverse scattering problem for the two-dimensional stationary Schrödinger equation at nonzero energy.

The inverse scattering problem consists in reconstructing potential v from an appropriately chosen set of scattering data. Reconstruction of a potential v at $E < 0$ or reconstruction of a transparent potential v at $E > 0$ from the scattering data b reduces to solving the linear problem (2.14)–(2.16) or, equivalently, the linear integral equation

$$\mu(z,\lambda) = 1 - \frac{1}{\pi}\iint_{\mathbb{C}} r(z,\zeta)\overline{\mu(z,\zeta)}\frac{d\mathrm{Re}\zeta\, d\mathrm{Im}\zeta}{\zeta-\lambda}. \tag{2.17}$$

The solvability of the above equation is based on the following statement (see [30]):

Statement. *Let*

$$r(\lambda) \in L_p(D), \quad |\lambda|^{-2} r(1/\lambda) \in L_p(D) \quad \textit{for } 2 < p < 4,$$

where D is the unit disk on the complex plane. Then (2.17) is uniquely solvable in $C(\mathbb{C})$ for $\forall z \in \mathbb{C}$.

Given $\mu(z,\lambda)$, the solution of (2.14)–(2.16), the potential v can be reconstructed as follows:

$$v(z) = 2i\sqrt{E}\frac{\partial \mu_{-1}(z)}{\partial z}, \quad z = x_1 + ix_2,$$

where $\mu_{-1}(z)$ is defined via the following expansion

$$\mu(z,\lambda) = 1 + \frac{\mu_{-1}(z)}{\lambda} + o\left(\frac{1}{|\lambda|}\right), \text{ as } \lambda \to \infty.$$

Remark. Reconstruction of a *nontransparent* potential from the scattering data b and the scattering amplitude f requires solving a nonlocal Riemann-Hilbert problem instead of $\bar\partial$-problem and will not be discussed here.

Evolution of scattering data driven by the Novikov-Veselov equation

Suppose that potential v in the Schrödinger equation (2.1) depends on an additional parameter t, $v = v(x,t)$. If $v(x,t)$ satisfies the Novikov-Veselov equation, then the evolution of the scattering data with respect to the additional parameter t is given by very simple relations.

Lemma. *Let (v,w) satisfy the Novikov-Veselov equation at $E \neq 0$. Then the evolution of Faddeev's scattering data for v is described in the following way:*

$$a(\lambda,t) = a(\lambda,0), \tag{2.18}$$

$$b(\lambda,t) = \exp\left\{i(\sqrt{E})^3\left(\lambda^3 + \frac{1}{\lambda^3} + (\mathrm{sgn}E)\left(\bar\lambda^3 + \frac{1}{\bar\lambda^3}\right)\right)t\right\} b(\lambda,0). \tag{2.19}$$

If in addition $E > 0$, then the evolution of the scattering amplitude for v is described as follows

$$f(\lambda,\lambda',t) = \exp\left\{i(\sqrt{E})^3\left(\lambda^3 + \frac{1}{\lambda^3} - (\lambda')^3 - \frac{1}{(\lambda')^3}\right)t\right\} f(\lambda,\lambda',0). \tag{2.20}$$

Note, in particular, that if $v(x,0)$ is transparent, then $v(x,t)$ is transparent for $\forall t$.

Formulas (2.18)-(2.20) represent an analogue of the Gardner-Greene-Kruskal-Miura relation for KdV.

Thus the inverse scattering method for solving the Cauchy problem for the Novikov-Veselov equation can be summarized as follows. Given the initial data $v(x,0)$ solve the direct scattering problem to find the scattering data $\{b(k,0)\}$ (or $\{f(k,l,0), b(k,0)\}$ if $v(x,0)$ is nontransparent at $E>0$). Apply the Gardner-Greene-Kruskal-Miura type relation to find how the scattering data evolve with time t. Given the scattering data $\{b(k,t)\}$ (or $\{f(k,l,t), b(k,t)\}$ for a nontransparent solution $v(x,t)$ at $E>0$) at any moment of time t solve the inverse problem to find the solution of the Novikov-Veselov equation $v(x,t)$ at that moment of time. If all the steps of this procedure are feasible (i.e. the equations of direct and inverse scattering are solvable), we will call the solution, obtained via this procedure, the "inverse scattering solution".

Remarks on the scattering transform for the Schrödinger equation at zero energy

For the two-dimensional Schrödinger equation at fixed zero energy

$$\begin{aligned} &L\psi = 0, \quad L = -\Delta + v, \\ &\Delta = 4\partial_z\partial_{\bar z}, \quad v = v(x), \quad x \in \mathbb{R}^2, \end{aligned} \tag{2.21}$$

with a potential v satisfying conditions (2.2), the notion of scattering amplitude is not defined. The Faddeev's scattering functions are defined as solutions $\psi(z,\lambda)$, $\lambda \in \mathbb{C}$, of (2.21) having the following asymptotics

$$\psi(z,\lambda) = e^{i\lambda z}\mu(z,\lambda), \quad \mu(z,\lambda) = 1 + \bar{o}(1), \text{ as } |z| \to \infty. \tag{2.22}$$

Function $\mu(z,\lambda)$, defined by (2.22), can be also represented as the solution of the following integral equation

$$\mu(z,\lambda) = 1 + \iint\limits_{\mathbb{C}} g(z-\xi,\lambda)v(\xi)\mu(\xi,\lambda)d\mathrm{Re}\xi d\mathrm{Im}\xi, \text{ where} \tag{2.23}$$

$$g(z,\lambda) = -\left(\frac{1}{4\pi}\right)^2 \iint\limits_{\mathbb{C}} \frac{e^{\frac{i}{2}(p\bar z + \bar p z)}}{p\bar p + 2p\lambda} d\mathrm{Re}p d\mathrm{Im}p, \tag{2.24}$$

for $z \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash 0$. Similarly to the case of nonzero energy we denote by $\mathcal{E}$ the set of values of λ for which the equation (2.23) is not uniquely solvable. Note that function $g(z,\lambda)$, defined in (2.24), has a logarithmic singularity at $\lambda = 0$ and thus function $\mu(z,\lambda)$ is not generally defined for $\lambda = 0$ even for arbitrarily small values of v.

For $\lambda \in \mathbb{C}\backslash(\mathcal{E}\cup 0)$ the Faddeev's scattering data for the potential v are defined as follows:

$$a(\lambda) = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{\mathbb{C}} v(z)\mu(z,\lambda)d\mathrm{Re}z d\mathrm{Im}z, \tag{2.25}$$

$$b(\lambda) = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{\mathbb{C}} e^{i\lambda z + i\bar\lambda\bar z}v(z)\mu(z,\lambda)d\mathrm{Re}z d\mathrm{Im}z. \tag{2.26}$$

Proposition 1. *Let v satisfy conditions (2.2). Then*

1. *$\mu(z,\lambda)$ is a well-defined and continuous function of λ on $\mathbb{C}\backslash(\mathcal{E}\cup 0)$;*

2. *$\mu(\lambda)$ satisfies the following $\bar\partial$-equation:*

$$\frac{\partial\mu(z,\lambda)}{\partial\bar\lambda} = \frac{\pi}{\bar\lambda}e^{-i(\lambda z+\bar\lambda\bar z)}b(\lambda)\overline{\mu(z,\lambda)}, \quad z\in\mathbb{C}, \quad \lambda\in\mathbb{C}\backslash(\mathcal{E}\cup 0); \tag{2.27}$$

3. *$\mu(z,\lambda)\to 1$, as $\lambda\to\infty$;*

4. *the scattering data $a(\lambda)$, $b(\lambda)$ are continuous on $\mathbb{C}\backslash(\mathcal{E}\cup 0)$;*

5. *if v satisfies the Novikov-Veselov equation at zero energy, then the evolution of the scattering data is described by the following formulas:*

$$a(\lambda,t) = a(\lambda,0),$$
$$b(\lambda,t) = e^{8i(\lambda^3+\bar\lambda^3)t}b(\lambda,0).$$

Items 1 - 3 of Proposition 1 (together with an additional condition on the behavior of $\mu(z,\lambda)$ at $\lambda=0$) form the inverse scattering problem for the two-dimensional stationary Schrödinger equation at zero energy. The evolution relations of item 5 of Proposition 1 provide the possibility to use the inverse scattering method to integrate the Novikov-Veselov equation at zero energy (i.e. to construct its “inverse scattering solutions”).

3. Large time behavior of “inverse scattering solutions” for the Novikov-Veselov equation

Theorem 1 ([17]). *Suppose that*

- *$(v(x,t),w(x,t))$ is a solution of the Novikov-Veselov equation at $E>0$;*
- *$v(\cdot,0)\in\mathcal{S}(\mathbb{R}^2)$;*
- *$w(\cdot,0)\in C^\infty(\mathbb{R}^2)$, $w(x,0)=o(1)$, $|x|\to\infty$;*
- *$v(\cdot,0)$ is transparent;*
- *the equations of direct scattering for $v(\cdot,0)$ are everywhere solvable (e.g. if $v(\cdot,0)$ satisfies a “small norm” condition).*

Then

$$|v(x,t)| \leqslant \frac{\mathrm{const}(v(\cdot,0))\ln(3+|t|)}{1+|t|}$$

uniformly on $x\in\mathbb{R}^2$ for $\forall t\in\mathbb{R}$.

This theorem implies, in particular, that the large-time asymptotics of transparent $v(x,t)$, satisfying the assumptions of the theorem, does not contain isolated solitons. Solitons can only be generated by singularities of the scattering data.

Note that, by contrast, in the $(1+1)$-dimensional case (KdV equation) there exist reflectionless solitons that are smooth and exponentially localized. These solitons are generated by the poles of the extension of the scattering function to the complex plane (similarly to the $(2+1)$-dimensional case).

An analogous, in a certain sense, result holds for the case of negative energy.

Theorem 2 ([14]). *Suppose that*

- $(v(x,t), w(x,t))$ *is a solution of the Novikov-Veselov equation at* $E < 0$*;*
- $v(\cdot, 0) \in \mathcal{S}(\mathbb{R}^2)$*;*
- $w(\cdot, 0) \in C^\infty(\mathbb{R}^2)$*,* $w(x,0) = o(1)$*,* $|x| \to \infty$*;*
- *equations of direct scattering for* $v(\cdot,0)$ *are everywhere solvable (e.g. if* $v(\cdot,0)$ *satisfies the "small norm" condition).*

Then

$$|v(x,t)| \leqslant \frac{\mathrm{const}(v(\cdot,0)) \ln(3+|t|)}{(1+|t|)^{3/4}}$$

uniformly on $x \in \mathbb{R}^2$ *for* $\forall t \in \mathbb{R}$*.*

This estimate is optimal (for certain initial data $v(x,0)$ *there exist lines* $x = ct$ *along which the asymptotics of* $v(x,t)$*, as* $t \to \infty$*, is exactly* $\frac{\mathrm{const}}{(1+|t|)^{3/4}}$*).*

4. Solitons of the Novikov-Veselov equation

Case of nonzero energy

Theorems 1, 2 demonstrate that solitons in the asymptotics of the solution of the Novikov-Veselov equation can be generated by the singularities of the scattering data. This is precisely what happens for the Grinevich-Zakharov solutions, the first explicit solutions of the Novikov-Veselov equation constructed by P.G. Grinevich, V.E. Zakharov (see [10]).

Grinevich-Zakharov solutions are defined as follows

$$v(x,t) = -4\partial_z \partial_{\bar{z}} \ln \det A,$$
$$w(x,t) = 12\partial_z^2 \ln \det A,$$

where $A = (A_{lm})$ is a $4N \times 4N$ matrix:

$$A_{ll} = \frac{iE^{1/2}}{2}\left(\bar{z} - \frac{z}{\lambda_l^2}\right) - 3iE^{3/2}t\left(\lambda_l^2 - \frac{1}{\lambda_l^4}\right) - \gamma_l,$$
$$A_{lm} = \frac{1}{\lambda_l - \lambda_m} \text{ for } l \neq m,$$

and $\lambda_1, \ldots, \lambda_{4N}$, $\gamma_1, \ldots, \gamma_{4N}$ are complex numbers such that

$$\lambda_j \neq 0, \quad |\lambda_j| \neq 1, \quad j = 1, \ldots, 4N, \quad \lambda_l \neq \lambda_m \text{ for } l \neq m,$$
$$\lambda_{2j} = -\lambda_{2j-1}, \quad \gamma_{2j-1} - \gamma_{2j} = \frac{1}{\lambda_{2j-1}}, \quad j = 1, \ldots, 2N,$$
$$\lambda_{4j-1} = \frac{1}{\bar{\lambda}_{4j-3}}, \quad \gamma_{4j-1} = \bar{\lambda}_{4j-3}^2 \bar{\gamma}_{4j-3}, \quad j = 1, \ldots, N.$$

Grinevich-Zakharov solutions are rational, nonsingular solutions of NV at $E > 0$, localized as $O\left(|x|^{-2}\right)$, $|x| \to \infty$. It turns out that they are also N-soliton solutions of NV.

Theorem 3 ([18]). *Let (v,w) be a Grinevich-Zakharov solution of NV. Then the asymptotic behavior of (v,w) is described as follows:*

$$v \sim \sum_{k=1}^{N} \nu_k(\xi_k), \quad w \sim \sum_{k=1}^{N} \omega_k(\xi_k) \quad \textit{as } t \to \infty, \tag{4.1}$$

where $\xi_k = z - c_{4k}t$ and $c_l = 6E\left(\bar{\lambda}_l^2 + \frac{1}{\lambda_l^2} + \frac{\lambda_l^2}{\bar{\lambda}_l^2}\right)$. Functions ν_k, ω_k are defined by formulas

$$\nu_k = -4\partial_z\partial_{\bar{z}} \ln\det A^{(k)}, \quad \omega_k = 12\partial_z^2 \ln\det A^{(k)}$$

and $A^{(k)}$ is a 4×4 submatrix of A defined by $A^{(k)} = \{A_{lm}\}_{l,m=4(k-1)+1}^{4k}$.

Relation (4.1) is understood in the following sense:

$$\lim_{t\to\infty} v = \lim_{t\to\infty} \sum_{k=1}^{N} \nu_k(\xi_k) \quad \text{for } \xi = z - ct \text{ fixed, where}$$

$$\lim_{t\to\infty} \nu_k(\xi_k) = \begin{cases} 0, & \text{for } \xi = z - ct \text{ fixed, } c \neq c_{4k}, \\ \nu_k(\xi), & \text{for } \xi = z - c_{4k}t \text{ fixed.} \end{cases}$$

Note that Grinevich-Zakharov solutions exhibit the same asymptotic behavior when $t \to -\infty$ and $t \to +\infty$, which means that Grinevich-Zakharov solitons do not interact at all. This is in contrast with the (1+1)-dimensional case, where interaction of solitons results in a phase shift. Note that a similar completely elastic interaction of solitons also takes place for the Kadomtsev-Petviashvili I solitons (see [22]).

Grinevich-Zakharov solutions are localized as $O(|x|^{-2})$, $|x| \to \infty$. This localization is almost the strongest possible for the solitons of NV at nonzero energy.

Theorem 4 ([16]). *Suppose that*

- *$(v(x,t), w(x,t))$ is a solution of the Novikov-Veselov equation at $E \neq 0$;*
- *$v, w \in C(\mathbb{R}^2, \mathbb{R})$, $v(\cdot, t) \in C^4(\mathbb{R}^3)$;*
- *$|\partial_x^j v(x,t)| \leqslant \frac{q(t)}{(1+|x|)^{3+j+\varepsilon}}$, $j = (j_1, j_2) \in (\mathbb{N}\cup 0)^2$, $j_1 + j_2 \leqslant 4$ for certain $\varepsilon > 0$, $q(t) \geqslant 0$; $w(x,t) \to 0$, $|x| \to \infty$;*
- *$v(x,t) = V(x - ct)$ (let v be a soliton).*

Then $v \equiv 0$, $w \equiv 0$.

The sketch of the proof of this Theorem, based on the inverse scattering method, is given in section 5.

Note that a similar result on absence of algebraically localized solitons for KPI, KPII and their generalizations was obtained in [3] using techniques which do not exploit the integrability of KPI, KPII.

Case of zero energy

As was noted in section 2, the case of zero energy represents the most difficult case for analysis since the scattering data for the two-dimensional Schrödinger equation at $E = 0$ have, in general, a logarithmic singularity at the origin. However, there exists a class of potentials for which the scattering data are everywhere well-defined: potentials of conductivity type.

Potential $v \in L^p(\mathbb{R}^2)$, $1 < p < 2$, is called a potential of conductivity type, if $v = \gamma^{-1/2}\Delta\gamma^{1/2}$ for a certain real-valued positive function $\gamma \in L^\infty(\mathbb{R}^2)$, such that $\gamma \geqslant \delta_0 > 0$, $\nabla\gamma^{1/2} \in L^p(\mathbb{R}^2)$. This type of potentials arises naturally when Gelfand-Calderón conductivity problem (see [8, 4]) is studied via the inverse scattering problem for the two-dimensional Schrödinger equation at $E = 0$.

The Calderón problem consists in finding a conductivity γ in a domain Ω given operator Λ_γ that associates the flux current $\Lambda_\gamma f$ on the boundary $\partial\Omega$ to the voltage f on $\partial\Omega$:

$$\begin{cases} \nabla(\gamma\nabla u) = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = f, \\ \Lambda_\gamma f = \gamma\nabla u \cdot \nu|_{\partial\Omega}. \end{cases} \tag{4.2}$$

Here ν denotes the normal vector to the boundary $\partial\Omega$.
One of the first and most studied strategies to solve Calderón's problem is to substitute $\tilde{u} = u \cdot \gamma^{-1/2}$ into (4.2) to obtain

$$\begin{cases} (-\Delta + v)\tilde{u} = 0 \text{ in } \Omega, \\ \tilde{u}|_{\partial\Omega} = \tilde{f}, \\ \Lambda_v \tilde{f} = \nabla\tilde{u} \cdot \nu|_{\partial\Omega} \end{cases} \tag{4.3}$$

with $\Lambda_v = \gamma^{-1/2}(\Lambda_\gamma + \frac{1}{2}\frac{\partial\gamma}{\partial\nu})\gamma^{-1/2}$ and $v = \Delta(\gamma^{1/2})\gamma^{-1/2}$. Note that the potential in the thus obtained Gelfand problem (4.3) is exactly a conductivity type potential.

In addition to being a physically natural type of potentials, the conductivity type potentials exhibit some interesting mathematical properties. It was proved in [20] that the property of conductivity is preserved by the Novikov-Veselov equation at $E = 0$. Further, solutions of conductivity type of the Novikov-Veselov equation form exactly the image of solutions of the modified Novikov-Veselov equation under the two-dimensional Miura transform (see [2, 28]). Last but not least, it was discovered in [23] that the scattering data for the Schrödinger operator with a potential of conductivity type are nonsingular. The last property of the conductivity type potentials allows us to prove the following result.

Theorem 5 ([15])**.** *Suppose that*

- $(v(x,t), w(x,t))$ *is a solution of the Novikov-Veselov equation at* $E = 0$*;*
- $v, w \in C(\mathbb{R}^2, \mathbb{R})$*,* $v(\cdot, t) \in C^3(\mathbb{R}^3)$*;*
- $|\partial_x^j v(x,t)| \leqslant \frac{q(t)}{(1+|x|)^{2+\varepsilon}}$*,* $|j| \leqslant 3$ *for certain* $\varepsilon > 0$*,* $q(t) \geqslant 0$*;* $w(x,t) \to 0$*,* $|x| \to \infty$*;*
- $v(x,t) = V(x - ct)$ *(let* v *be a soliton);*
- v *is of conductivity type.*

Then $v \equiv 0$, $w \equiv 0$.

Note that there exist, however, localized nonsingular solitons solutions of the Novikov-Veselov equation at zero energy which are not of conductivity type. In [5] the following nonsingular algebraically localized stationary soliton of the Novikov-Veselov equation at $E = 0$ is presented:

$$v(z,t) = -\frac{3840 z\bar{z}}{(15z^2\bar{z}^2+8)^2}, \quad w(z,t) = -\frac{10800\bar{z}^4 z^2 - 5760\bar{z}^2}{(15z^2\bar{z}^2+8)^2}$$

Note that this solution is localized as $|v(z,t)| = O(|z|^{-6})$, $|w(z,t)| = O(|z|^{-2})$ as $|z| \to \infty$, $\forall t$.

5. Proof of absence of algebraically localized solitons for the Novikov-Veselov equation at $E \neq 0$

We devote this last section to the proof of Theorem 4. We start by giving some extra notions and results from the scattering theory for the two-dimensional stationary Schrödinger equation at $E \neq 0$ which will be used in the proof.

Additional notions and results from the scattering theory for the 2d Schrödinger equation

Proposition 2. *Let v be a potential in equation (2.1) satisfying conditions (2.2). Let Δ be the modified Fredholm determinant of equation (2.12) defined via (2.13). Let a, b be the Faddeev's scattering data for v. Let, finally, T denote the unit circle on the complex plane. The following properties hold.*

1. $\Delta \in C(\bar{D}_+)$, $\Delta \in C(\bar{D}_-)$, *where* $D_+ = \{\lambda \in \mathbb{C}\colon |\lambda| < 1\}$, $D_- = \{\lambda \in \mathbb{C}\colon |\lambda| > 1\}$;

2. $\Delta(\lambda) \to 1$ *as* $|\lambda| \to \infty$, $|\lambda| \to 0$;

3. Δ *is a real-valued function;*

4. $\Delta(\lambda)$ *satisfies the following* $\bar{\partial}$*-equation*

$$\frac{\partial \Delta}{\partial \bar{\lambda}} = -\frac{\pi \operatorname{sgn}(\lambda\bar{\lambda}-1)}{\bar{\lambda}} \left(a\left(-(\operatorname{sgn} E)\frac{1}{\bar{\lambda}}\right) - \hat{v}(0) \right) \Delta, \tag{5.1}$$

   where $\hat{v}(0) = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{\mathbb{C}} v(\zeta) d\mathrm{Re}\zeta d\mathrm{Im}\zeta$, $\lambda \in \mathbb{C}\backslash(T \cup \mathcal{E} \cup 0)$;

5. $\Delta(\lambda) = \Delta\left(-(\operatorname{sgn} E)\frac{1}{\bar{\lambda}}\right)$, $\lambda \in \mathbb{C}\backslash 0$;

6. *if* $E < 0$ *or if* $E > 0$ *and* v *is transparent, then* $\Delta \equiv \mathrm{const}$ *on* T;

7. *scattering data* $a(\lambda)$, $b(\lambda)$ *are continuous functions on* $\mathbb{C}\backslash\mathcal{E}$;

8. $\hat{v}(0) = \lim\limits_{\lambda\to\infty} a(\lambda)$, *where* $\hat{v}(0) = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{\mathbb{C}} v(z) d\mathrm{Re}z d\mathrm{Im}z$;

9. *if $E<0$ and $T\cap\mathcal{E}=\varnothing$, then $a\equiv b$ on T; if $E>0$, v is transparent and $T\cap\mathcal{E}=\varnothing$, then $a\equiv 0$ on T.*

In the proof of Theorem 4 we will also need some additional scattering data. Suppose that

$$|\partial_{x_1}^{j_1}\partial_{x_2}^{j_2}v(x)|<q(1+|x|)^{-3-\varepsilon}\text{ for some } q>0,\ \varepsilon>0,\ \text{where } j_1,j_2\in\mathbb{N}\cup 0,\ j_1+j_2\leqslant 3.$$

Define $\varphi(x,k)$, $k\in\Sigma_E$, where Σ_E is given in (2.4), as a solution of (2.1) at nonzero energy with the following asymptotics

$$\varphi(x,k)=e^{ikx}\nu(x,k),\quad \nu(x,k)=k_1x_2-k_2x_1+o(1)\text{ as }|x|\to\infty.$$

The additional "scattering data" are defined as follows:

$$\alpha(k)=\left(\frac{1}{2\pi}\right)^2\iint\limits_{\mathbb{R}^2}e^{-ikx}v(x)\varphi(x,k)dx_1dx_2,$$
$$\beta(k)=\left(\frac{1}{2\pi}\right)^2\iint\limits_{\mathbb{R}^2}e^{i\bar{k}x}v(x)\varphi(x,k)dx_1dx_2.$$

In terms of $z=x_1+ix_2$ and λ given by (2.10) the asymptotics of φ and the definitions of the additional scattering data become:

$$\varphi(z,\lambda)=\exp\left(\frac{i\sqrt{E}}{2}\left(\lambda\bar{z}+\frac{z}{\lambda}\right)\right)\left(\frac{i\sqrt{E}}{2}\left(\lambda\bar{z}-\frac{1}{\lambda}z\right)+o(1)\right),\ \text{as }|z|\to\infty,$$
$$\alpha(\lambda)=\left(\frac{1}{2\pi}\right)^2\iint\limits_{\mathbb{C}}\exp\left(-\frac{i\sqrt{E}}{2}\left(\lambda\bar{z}+\frac{z}{\lambda}\right)\right)v(\zeta)\varphi(\zeta,\lambda)d\mathrm{Re}\zeta d\mathrm{Im}\zeta,$$
$$\beta(\lambda)=\left(\frac{1}{2\pi}\right)^2\iint\limits_{\mathbb{C}}\exp\left(\frac{i\sqrt{E}\mathrm{sgn}(E)}{2}\left(\bar{\lambda}z+\frac{\bar{z}}{\bar{\lambda}}\right)\right)v(\zeta)\varphi(\zeta,\lambda)d\mathrm{Re}\zeta d\mathrm{Im}\zeta.$$

Lemma. *Let (v,w) satisfy NV. Then the evolution of the additional scattering data for v is described in the following way:*

$$\alpha(\lambda,t)=\alpha(\lambda,0)+3i(\sqrt{E})^3\left(\lambda^3-\frac{1}{\lambda^3}\right)a(\lambda,0)t,\tag{5.2}$$
$$\beta(\lambda,t)=\exp\left\{i(\sqrt{E})^3\left(\lambda^3+\frac{1}{\lambda^3}+(\mathrm{sgn}E)\left(\bar{\lambda}^3+\frac{1}{\bar{\lambda}^3}\right)\right)t\right\}\times\tag{5.3}$$
$$\times\left(\beta(\lambda,0)+3i(\sqrt{E})^3\left(\lambda^3-\frac{1}{\lambda^3}\right)b(\lambda,0)t\right).\tag{5.4}$$

Finally, we present a lemma which describes the scattering data corresponding to a shifted potential.

Lemma. *Let $v(z)$ be a potential with the corresponding scattering data $S(\lambda)$, $\lambda\in\mathbb{C}\backslash(\mathcal{E}\cup 0)$, $S(\lambda)=\{a(\lambda),b(\lambda),\alpha(\lambda),\beta(\lambda)\}$. Then the scattering data $S_\eta(\lambda)$ for the potential $v_\eta(z)=v(z-\eta)$ are defined for $\lambda\in\mathbb{C}\backslash(\mathcal{E}\cup 0)$ and are related to $S(\lambda)$ by* *the following formulas:*

$$a_\eta(\lambda) = a(\lambda), \tag{5.5}$$

$$b_\eta(\lambda) = \exp\left\{\frac{i\sqrt{E}}{2}\left(1+(\mathrm{sgn}E)\frac{1}{\lambda\bar\lambda}\right)\left((\mathrm{sgn}E)\bar\lambda\eta+\lambda\bar\eta\right)\right\}b(\lambda), \tag{5.6}$$

$$\alpha_\eta(\lambda) = \alpha(\lambda)+\frac{i\sqrt{E}}{2}\left(\lambda\bar\eta-\frac{1}{\lambda}\eta\right)a(\lambda), \tag{5.7}$$

$$\beta_\eta(\lambda) = \exp\left\{\frac{i\sqrt{E}}{2}\left(1+(\mathrm{sgn}E)\frac{1}{\lambda\bar\lambda}\right)\left((\mathrm{sgn}E)\bar\lambda\eta+\lambda\bar\eta\right)\right\}\times \tag{5.8}$$

$$\times\left(\beta(\lambda)+\frac{i\sqrt{E}}{2}\left(\lambda\bar\eta-\frac{1}{\lambda}\eta\right)b(\lambda)\right). \tag{5.9}$$

Sketch of the proof of Theorem 4

The key idea of the proof is to compare the dynamics of the scattering data arising from the fact that $v(x,t)$ is a soliton and from the fact that $v(x,t)$ is a solution of the Novikov-Veselov equation. In particular, comparing expressions (2.19) and (5.6) and using the continuity of b on $\mathbb{C}\backslash\mathcal{E}$ and the fact that functions λ^3, $\bar\lambda^3$, λ, $\bar\lambda$, 1, $\frac{1}{\lambda}$, $\frac{1}{\bar\lambda}$, $\frac{1}{\lambda^3}$, $\frac{1}{\bar\lambda^3}$ are linearly independent in the neighborhood of any point $\lambda\in\mathbb{C}\backslash(T\cup 0)$, we obtain that $b\equiv 0$ on $\mathbb{C}\backslash\mathcal{E}$. In a similar way in [25] the following result was obtained:

Theorem 6. *Let v be a soliton solution of NV at $E>0$. Then v is a transparent potential.*

(Similarly, in dimension 1 the solitons of KdV are reflectionless potentials.)

The next step in the proof is to study the set $\mathcal{E}$, where the scattering data are not well-defined. For that purpose, we use the properties of the modified Fredholm determinant Δ (see Proposition 2) and, in particular, equation (5.1). Our next goal will be to prove that Faddeev's scattering data a, appearing as a coefficient in equation (5.1), vanishes on $\mathbb{C}\backslash\mathcal{E}$.

Comparing (2.18) and (5.5) does not give any new information about the behavior of a. However, comparing the dynamics of the additional scattering data (5.2) and (5.7) gives the following formula for a:

$$a(\lambda,0)=\frac{3i(\sqrt{E})^3\left(\lambda^3-\frac{1}{\lambda^3}\right)\hat v(0)}{3i(\sqrt{E})^3\left(\lambda^3-\frac{1}{\lambda^3}\right)-\frac{i\sqrt{E}}{2}\left(\lambda\bar c-\frac{1}{\lambda}c\right)}. \tag{5.10}$$

The expression in the denominator of (5.10) has been analysed in [17], where, in particular, it was shown that

the denominator of (5.10) has at least two roots on the unit circle for $\forall c\in\mathbb{C}$. (5.11)

Consider two cases.

I $\Delta\neq 0$ on T

In this case a, b are well-defined on T, $b\equiv 0$ on T, and thus from item 9 of Proposition 2 it follows that $a\equiv 0$ on T. From (5.10) and property (5.11) we obtain that it can only be true if $\hat v(0)=0$, and thus $a\equiv 0$ on $\mathbb{C}\backslash\mathcal{E}$.

II $\Delta = 0$ on T

In this case consider an angle φ, such that the ray $\gamma_\varphi = \{re^{i\varphi}, r \geqslant 0\}$ does not pass through any of the roots of the denominator of (5.10). Recall that $\Delta(0) = 1$ and consider the following segment l_φ on the ray γ_φ:

$$l_\varphi = \{re^{i\varphi}, r \leqslant r' \leqslant 1\colon \Delta(re^{i\varphi}) > 0,\, \Delta(r'e^{i\varphi}) = 0\}.$$

Note that l_φ is a finite segment and it lies inside the unit disk, since $\Delta = 0$ on T. In addition, on segment l_φ expression (5.10) is finite.

From equation (5.1), real-valuedness of Δ and continuity of Δ it follows that Δ can be represented in the following form

$$\Delta(\lambda) = \left|\exp \int\limits_0^{\bar\lambda} \frac{\pi \operatorname{sgn}(\zeta\bar\zeta - 1)}{\bar\zeta} \left(a\left(-(\operatorname{sgn} E)\frac{1}{\bar\zeta}\right) - \hat v(0)\right) d\zeta \right|^2$$

for any $\lambda \in l_\varphi$, where the integration is performed along the ray γ_φ. In particular, this representation implies that $\Delta \neq 0$ on l_φ. However, the latter is in contradiction with the fact that $\Delta = 0$ at one of the endpoints of l_φ.

We have shown that case II cannot hold and from case I it follows that $a \equiv 0$ on $\mathbb{C}\backslash\mathcal{E}$. Consequently, Δ is holomorphic on $\mathbb{C}\backslash(T \cup \mathcal{E} \cup 0)$, i.e. Δ is a function holomorphic everywhere except for the set where it vanishes. Using the properties of Δ given in Proposition 2 and some simple complex analysis, it is not difficult to show afterwards that $\Delta \equiv 1$ and hence $\mathcal{E} = \varnothing$.

The inverse problem for reconstructing v reduces to the following: find μ holomorphic on $\mathbb{C}\backslash 0$, continuous on $\mathbb{C}$, such that $\mu \to 1$ as $\lambda \to 0, \infty$. Evidently, $\mu \equiv 1$ and, hence, $v \equiv 0$. Theorem is proved.

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