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Accurate Spectral Asymptotics for Periodic Operators

Victor Ivrii

Abstract

Asymptotics with sharp remainder estimates are recovered for number $\mathbf{N}(\tau)$ of eigenvalues of operator $A(x,D) - tW(x,x)$ crossing level E as t runs from 0 to τ, $\tau \to \infty$. Here A is periodic matrix operator, matrix W is positive, periodic with respect to first copy of x and decaying as second copy of x goes to infinity, E either belongs to a spectral gap of A or is one its ends. These problems are first treated in papers of M.Sh.Birman, M.Sh.Birman-A.Laptev and M.Sh.Birman-T.Suslina.

0. Results.

The main goal of this paper is to obtain sharp remainder estimates for spectral asymptotics derived in papers of M.Sh.Birman [B1-3], M.Sh.Birman-A.Laptev [BL1,2], M.Sh.Birman-A.Laptev-T.Suslina [BLS], and M.Sh.Birman-T.Suslina [BS]. The second goal is to generalize their results.

Let us consider in $\mathbb{R}^d$ $D\times D$-matrix operator $A(x,D) = A^w(x,D)$ with the Weyl symbol $A(x,\xi) = \sum_{|\sigma|\le m} a_\sigma(x,x)\xi^\sigma$ where

(H1) $A(x,y,D)$ are periodic with period lattice $\Gamma = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \dots \mathbb{Z}e_d$ and transformation matrices $\{T_1,\dots,T_d\}$ with respect to x, which means that

$$A(x+e_j,y,D) = T_j^* A(x,y,D) T_j \qquad \forall j=1,\dots,d \tag{0.1}$$

with unitary commuting matrices T_j.

Moreover, let

$$|\nabla_x^\alpha \nabla_y^\beta \big(a_\sigma(x,y) - \bar{a}_\sigma(x)\big)| \le C\langle y\rangle^{-\delta-|\beta|} \quad \forall\alpha : |\alpha|\le K \ \forall|\beta| : |\beta|\le K \tag{0.2}$$

$$|\nabla^\alpha \bar{a}_\sigma| \le C \qquad \forall\alpha : |\alpha| \le K \tag{0.3}$$

with $\delta > 0$ and large enough $K = K(m,d,\mu)$.

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Further we assume that A is elliptic:

$$|A(x,\xi)v| \geq \epsilon_0\big(|\xi|^m - C\big)|v| \qquad \forall x,\xi \in \mathbb{R}^d \ \forall v \in \mathbb{C}^D \tag{0.4}$$

Furthermore, we assume that

$$A(x,\xi)^\dagger = A(x,\xi) \qquad \forall (x,\xi) \tag{0.5}$$

where † means Hermitian conjugation.

Then both operators A and $\bar{A}$ are self-adjoint in $L^2(\mathbb{R}^d, \mathbb{C}^D)$ and essential spectra of A and $\bar{A}$ and spectrum of $\bar{A}$ coincide with

$$\bigcup_k \bigcup_{\xi \in Q'} \lambda_k(\xi) \tag{0.6}$$

where $\lambda_k(\xi)$ are eigenvalues of operator $\bar{A}$ restricted to the space $\mathbb{K}_{\xi,\{T\}}$ of the functions, *quasi-periodic with quasi-momentum* ξ *and transformation matrices* $\{T\} = \{T_1, \dots, T_d\}$,)[1] with the inner product as in $L^2(Q, \mathbb{C}^d)$ but multiplied by $(\operatorname{Vol} Q)^{-1}$ where $Q = [0,1]e_1 \oplus \dots [0,1]e_d$ is an *elementary cell*, $Q' = [0,1]e'_1 \oplus \dots \oplus [0,1]e'_d$ is a *dual cell*, $\langle e'_j, e_k\rangle = 2\pi\delta_{jk}\ \forall j,k$, $T^n = T_1^{n_1}\cdots T_d^{n_d}$ for $n = n_1e_1 + \dots + n_de_d$; without loss of generality we assume that $\lambda_k(\xi) \geq \lambda_{k-1}(\xi)\ \forall k$.

So, $\operatorname{Spec}\bar{A}$ has a zone character with possible overlapping of zones. Let us pick some energy level which is in the spectral gap: $E \notin \operatorname{Spec}\bar{A}$.

Later we consider also cases when E is either lower end or upper end of spectral gap: $[E, E+\epsilon) \cap \operatorname{Spec}\bar{A} = \{E\}$, $(E-\epsilon, E] \cap \operatorname{Spec}\bar{A} = \{E\}$.

Let $W(x) = W(x,x)$ where

(H2) $W(x,y)$ and $\bar{W}(x,y)$ are Hermitian and periodic with period lattice Γ and transformation matrices $\{T\} = \{T_1, \dots, T_d\}$ with respect to x,

and

$$|\nabla_x^\alpha \nabla_y^\beta W| \leq C\langle y\rangle^{-m\mu-|\beta|} \qquad \forall \alpha : |\alpha| \leq K\ \forall |\beta| : |\beta| \leq K, \qquad \mu > 0 \tag{0.7}$$

$$|\nabla_y^\beta (W - \bar{W})| = O(|y|^{-\delta-m\mu-|\beta|}) \qquad \text{as} |y| \to \infty \quad \forall \beta : |\beta| = 1 \tag{0.8}$$

$$\langle W(x,y)v, v\rangle \geq C|v|^2\langle y\rangle^{-m\mu} \qquad \forall x,y \in \mathbb{R}^d\ \forall v \in \mathbb{C}^d \tag{0.9}$$

where $\bar{W}$ is positive homogeneous with respect to y of degree $-m\mu$ and satisfies (0.9) as well.

Let us consider operator $A - tW$ with $t > 0$. For each t only finite number of eigenvalues of A belong to $(E-\epsilon, E+\epsilon)$ if E resides within spectral gap; further, if E is the lower (upper) end of the spectral gap then under appropriate conditions for each t only finite number of eigenvalues of A belong to $(E, E+\epsilon)$ ($(E-\epsilon, E)$ respectively). All the eigenvalues are monotone decreasing functions of t.

Let $\mathbf{N}(\tau)$ be a number of the eigenvalues of $A - tW$ (counting multiplicities) passing through E (reaching E, leaving E respectively) as t changes from $0+$ to $\tau-$. We are interested in asymptotics of $\mathbf{N}(\tau)$ as $\tau \to +\infty$.

[1] I. e. functions such that $u(x+n) = T^{-n}u(x)e^{i<n,\xi>} \quad \forall n \in \Gamma$

Theorem 0.1. *Let conditions* (H1), (H2), (0.2) – (0.5), (0.7) – (0.9) *be fulfilled. Let* $E \notin \operatorname{Spec} \bar{A}$.

(i) Further let us assume that there is an infinite number of λ_k *exceeding* E *(or equivalently* A *is not semibounded from above). Then*

$$|\mathbf{N}(\tau) - \mathcal{N}(\tau)| \le \begin{cases} \tau^{\frac{d-1}{m}} & \text{for } \mu > 1, \\ \tau^{\frac{d-1}{m}} \log \tau & \text{for } \mu = 1, \\ \tau^{\frac{d-1}{m\mu}} & \text{for } \mu < 1, \end{cases} \tag{0.10}$$

$$\mathcal{N}(\tau) = \int_{\mathbb{R}^d} \int_{Q'} \mathbf{n}(y, \xi, \tau) d\xi dy \tag{0.11}$$

where $\mathbf{n}(y, \xi, \tau)$ *is the number of eigenvalues of operator* $A(x, y, D) - tW(x, y)$ *crossing* E *as* t *runs from* $0+$ *to* $\tau-$*; this operator depends on* y *and is restricted to* $\mathbb{K}_{\xi.\{T\}}$. *Moreover*

$$\mathcal{N}(\tau) \asymp \begin{cases} \tau^{\frac{d}{m}} & \text{for } \mu > 1, \\ \tau^{\frac{d}{m}} \log \tau & \text{for } \mu = 1, \\ \tau^{\frac{d}{m}\mu}, & \text{for } \mu < 1. \end{cases} \tag{0.12}$$

(ii) Let us assume that there is finite number of λ_k *exceeding* E *(or equivalently* A *is semibounded from above) and that* $\mu < 1$. *Then*

$$|\mathbf{N}(\tau) - \mathcal{N}(\tau)| \le C\tau^{\frac{d-1}{m\mu}} \tag{0.13}$$

and

$$\mathcal{N}(\tau) \asymp \tau^{\frac{d}{m\mu}} \tag{0.14}$$

REMARK 0.2 (i) In assumptions of theorem 0.1(i) with $\mu > 1$ and under standard condition to periodic Hamilton trajectories fulfilled asymptotics (0.10) holds with the remainder estimate $o(\tau^{\frac{d-1}{m}})$.

(ii) In assumptions of theorem 0.1(i) with $\mu > \frac{d}{d-1}$ standard Weyl asymptotics holds with the remainder estimate $O(\tau^{\frac{d-1}{2m}})$ (and even $o(\tau^{\frac{d-1}{m}})$ under standard condition to Hamiltonian trajectories).

(iii) In assumptions of theorem 0.1(i) with $\mu > 1$ and $\delta > \delta_0 = 1 - (d-1)(\mu - 1)$ asymptotics (0.10) holds with

$$\begin{aligned} \mathcal{N}(\tau) = {} & \textstyle\int_{\mathbb{R}^d} \int_{Q'} \bar{\mathbf{n}}(y, \xi, \tau) d\xi dy \\ & + (2\pi)^{-d} \omega_d \textstyle\int_{\mathbb{R}^{2d}} \Big(\nu(x, \xi, \tau) - \nu(x, \xi, 0) - \bar{\nu}(x, \xi, \tau) + \bar{\nu}(x, \xi, 0) \Big) dx d\xi \end{aligned} \tag{0.15}$$

where $\nu(x, \xi, \tau)$ and $\bar{\nu}(x, \xi, \tau)$ are the numbers of the negative eigenvalues of matrices $A(x, x, \xi) - \tau W(x, x)$ and $\bar{A}(x, \xi) - \tau \bar{W}(x, x)$ respectively and $\bar{\mathbf{n}}(y, \xi, \lambda)$ is introduced in the manner similar to $\mathbf{n}(y, \xi, \tau)$ for operators $\bar{A}$ and $\bar{W}$.

(iv) In assumptions of theorem 0.1(i) with $\mu < (d-1)d^{-1}$, $\delta > (1-\mu)(d-1)(d-d\mu-1)^{-1}$ asyınptotics (0.10) holds with

$$\mathcal{N}(\tau) = \int_{\mathbb{R}^d}\int_{Q'} \bar{\mathbf{n}}(y,\xi,\tau)\,d\xi dy. \tag{0.16}$$

(v) In assumptions of theorem 0.1(ii) with $\delta > 1$ asymptotics (0.10) holds with $\mathcal{N}$ defined by (0.16). $\Diamond$

Now let us consider the case when E is a boundary of the spectral gap. First of all, we have

Theorem 0.3. *Let $d \geq 2$, $\bar{\mu} = \frac{m\mu}{2} > 1$, $A = \bar{A}$ and conditions $(H1), (H2), (0.2)-(0.5), (0.7)-(0.9)$ be fulfilled. Let us assume that either*

(a) $[E, E+\epsilon) \cap \operatorname{Spec}\bar{A} = \{E\}$ *or*

(b) $(E-\epsilon, E] \cap \operatorname{Spec}\bar{A} = \{E\}$.

Further, let us assume that

(H3) $\lambda_j(\xi) = E$ *implies that* $\lambda_j(\xi)$ *is the simple eigenvalue of* $\bar{A}(x, D_x)$ *on* $\mathbb{K}_{\xi,\{T\}}$ *and* $\mathrm{Hess}\lambda_j(\xi)$ *is non-degenerate.*

Then statements (i),(ii) of theorem 0.1 remain true.

REMARK 0.4 (i) Due to condition (H3) operator $A-E-tW$ "looks like" $\Lambda - tW$ in the case (b) with second order operator Λ (similar to $-\Delta$) and we need to assume that $\bar{\mu} > 1$ to avoid -0 being a point of accumulation of the spectra of both these operators. In the case (a) operator $A-E-tW$ "looks like" $-\Lambda - tW$ and condition $\bar{\mu} > 1$ seems to be "overkill".

(ii) It is known that condition (H3) holds for $A = -\Delta + V(x)$, $D = 1$ and the lowest λ_1; in this case $\xi_0 = 0$ is the only minimum point; the same is true for second order operator in divergent form.

Now we consider the case $\bar{\mu} = 1$ with extra-logarithmic factor.

Theorem 0.1′. *(i) Let all the conditions of theorem 0.1 be fulfilled excluding* $(0.7)-(0.9)$ *which are replaced by*

$$|\nabla_x^\alpha \nabla_y^\beta W| \leq C\rho^m \langle y\rangle^{-|\beta|} \qquad \forall\alpha : |\alpha| \leq K\ \forall\beta : |\beta| \leq K, \tag{0.7$'$}$$

$$|\nabla_y^\beta (W - \bar{W})| = o(\rho^m \langle y\rangle^{-|\beta|}) \qquad \text{as} |y| \to \infty \quad \forall\beta : |\beta| = 1 \tag{0.8$'$}$$

$$\langle W(x,y)v, v\rangle \geq C\rho^m |v|^2 \qquad \forall x, y \in \mathbb{R}^d \forall v \in \mathbb{C}^d \tag{0.9$'$}$$

with

$$\rho = |y|^{-\mu}(\log|y|)^{-\mu_1}(\log\log|y|)^{-\mu_2}\cdots(\underbrace{\log\ldots\log}_{l\text{times}}|y|)^{-\mu_l} \qquad (|y| > C) \tag{0.17}$$

and with $\bar{W}\rho^{-2}$ *positively homogeneous of degree* 0.

(i) Then statement (i) of theorem 0.1 remains true as soon as $\int \rho^{d-1}r^{d-2}dr < \infty$. *Moreover, if this assumption is violated, statement (i) modified in obvious way remains true (modification affects remainder estimate which is now* $\mathcal{R}(\tau) =$

$\tau^{\frac{d-1}{m}} \int_{r<r_0(\tau)} \rho^{d-1} r^{d-2} dr$ *with* $r_0(\tau)$ *defined from* $\tau^{\frac{1}{m}}\rho(r_0(\tau)) = 1$*). Moreover, as soon as* $\int \rho^d r^{d-1} dr < \infty$ *magnitude of the principal part does not change; if this assumption is violated, magnitude of the principal part will be* $\mathcal{M}(\tau) = \int_{r<r_0(\tau)} \rho^d r^{d-1} dr$.

(ii) Furthermore, statement (ii) remains true after modification of the magnitude of the principal part and remainder estimate (which are now $\mathcal{R}_0(\tau) = r_0(\tau)^{d-1}$ *and* $\mathcal{M}_0(\tau) = r_0(\tau)^d$ *respectively).*

Theorem 0.3′. *Let all the assumptions of theorem 0.1′ be fulfilled excluding condition* $E \notin \operatorname{Spec} \bar{A}$ *which is replaced by one of the assumptions (a) or (b) of theorem 0.3 and by* (H3). *Let* <u>*either*</u> $\bar{\mu} = \frac{m\mu}{2} > 1$ <u>*or*</u> $\bar{\mu} = 1$, $m\mu_1(d-1) > 2$ *and either* $d \geq 3$ *or* E *is the lower bound of the spectral gap (case* (a)*).*

Then both statements (i),(ii) of theorem 0.1′ remain true.

Theorem 0.3″. *Let conditions of theorem 0.3 be fulfilled excluding* (0.7) – (0.8) *which are replaced by* (0.7)′ – (0.8)′. *Moreover, let* $d = 2$ *and* $(E-\epsilon, E] \cap \operatorname{Spec} \bar{A} = \{E\}$. *Furthermore, let* <u>*either*</u> $\bar{\mu}_1 = \frac{m\mu_1}{2} > 1$ <u>*or*</u> $\bar{\mu}_1 = 1$, $\bar{\mu}_2 = \frac{m\mu_2}{2} > 1$.

Then statements (i) and statement (ii) of theorem 0.1′ remain true.

Amazingly, for $d = 2$, $\bar{\mu}_1 = \frac{m\mu_1}{2} = 1$, $\bar{\mu}_2 = \frac{\mu_2}{2} \in (0,1]$and E being the upper end of the spectral gap theorem 0.3' needs to be modified in the very significant way. Let us consider all the quasi-momentums ξ and all the eigenvalues λ_k such that $\lambda_k(\xi) = E$. Due to condition (H3) we can number them as ξ_p and λ_{k_p} with $p = 1, \dots, P$. Let $G_p = \mathrm{Hess}\lambda_{k_p}(\xi_p)$ and $|x|_p = |G_p^{-\frac{1}{2}} x|$. Let us assume that

$$|\langle y, \nabla_y\rangle^\alpha |y|_p^2 \mathcal{W}| \leq C(\log|y|)^{-2-\delta} \qquad \forall \alpha : 1 \leq \alpha \leq K \tag{0.18}$$

with $\delta > 0$.

Moreover, let us consider $\mathcal{W}_p(y) = (W(r\vartheta, y) w_p(x), w_p(x))$ where w_p is an eigenfunction corresponding to eigenvalue E and quasi-momentum ξ_p and $\bar{W}_p(r)$ a mean value of $\mathcal{W}(r\theta)$ over sphere $\{|y|_p = r\}$. Let us assume that

$$\nu_p(r) = r^2 (\log r)^2 \bar{W}_p(r)|_{r=\exp\exp z^{(1-\bar{\mu}_2)^{-1}}}$$

satisfy

$$|\partial_z^\alpha \nu_p(z)| \leq C z^{-\bar{\mu}_2(1-\bar{\mu}_2)^{-1} - |\alpha|} \qquad \forall \alpha \leq K. \tag{0.19}$$

Theorem 0.5. *Let conditions of theorem 0.3 be fulfilled excluding* (0.7)–(0.8) *which are replaced by* (0.7)′ – (0.8)′. *Moreover, let* $d = 2$ *and* $(E - \epsilon, E] \cap \operatorname{Spec} \bar{A} = \{E\}$. *Further, let* $\bar{\mu}_1 = \frac{m\mu_1}{2} = 1$ *but* $\bar{\mu}_2 = \frac{\mu_2}{2} \in (0,1)$. *Let conditions* (0.18), (0.19) *be fulfilled.*

Then statements (i) and statement (ii) of theorem 0.1′ remain true for $\mathcal{N}$ *replaced by* $\mathcal{N} + \sum_{1 \leq p \leq P} \mathrm{N}_p$ *where* $\mathrm{N}_p(\tau)$ *is the maximal dimension of the negative subspace of quadratic form*

$$(G_p \nabla u, \nabla u) - \tau(Wu, u), \qquad u \in \mathbb{H}_p \tag{0.20}$$

where $\mathbb{H}_p = \{u = v(|x|_p) w_p(x)\}$ *with the scalar function* $v \in L^2(\mathbb{R}^+, r dr)$, $\operatorname{supp} v \subset \{\log r (\log\log r)^{\bar{\mu}_2} > \tau^{\frac{1}{2}}\}$ *of one variable* r.

Substituting $z = (\log\log r)^{1-\bar\mu_2}$, $v = \left(\bar W r\frac{dr}{dz}\right)^{-\frac{1}{2}}\psi$, we see that $\mathbf{N}_p(\tau)$ is the dimension of the negative subspace of one-dimensional Schrödinger operator

$$D_z g D_z + \Phi - \tau$$

with $g \asymp 1$ and $\Phi \asymp z^{2(1-\bar\mu_2)}$ restricted to $z \geq z(\tau) = (\log\tau)^{1-\bar\mu_2^{-1}}$. Then in this statement one can replace $\mathbf{N}_p(\tau)$ by its Weyl approximation

$$\mathcal{N}_p(\tau) = \frac{1}{2}\int_{z\geq z(\tau)} g^{-\frac{1}{2}}(\tau-\Phi)_+^{\frac{1}{2}}dz \asymp \tau^{\frac{1}{2\bar\mu_2}} \tag{0.21}$$

Let us consider the operator treated in [BL]:

Theorem 0.6. *For operator $m = d = 2$, $A = -\Delta$ and $W = W(y)$ satisfying conditions of theorem 0.5 with $\mu_1 = 1$, $\mu_2 \in (0,1]$ asymptotics*

$$\mathbf{N}(\tau) = \mathcal{N}(\tau) + \mathcal{N}_1(\tau) + O(\tau^{\frac{1}{2}}) \tag{0.22}$$

holds.

Further, under standard condition to Hamiltonian trajectories this asymptotics holds with the remainder estimate $o(\tau^{\frac{1}{2}})$.

Remark 0.7 (i) Similar statement holds as $l = 2$, $\mu_2 = 1$ but in this case $z = \log\log\log r$, $\log\Phi(z) \asymp z$, $\mathcal{N}_p(\tau) \asymp \tau^{\frac{1}{2}}\log\tau$ and one can take $z(\tau) = 0$.

(ii) Similar statement holds as $l \geq 3$, $\mu = \mu_1 = \dots = \mu_{l-1} = 1$, $-\infty < \mu_l < 1$ but in this case $\mathcal{N}_p(\tau) \asymp \tau^{\frac{1}{2}}(\underbrace{\log\ldots\log}_{l-2\text{times}}\tau)^{(1-\mu_l)^{-1}}$.

Definitely $\mathcal{N}_1$ is not a principal part but still it is larger than the remainder estimate.

(iii) On the other hand, asymptotics (0.22) with remainder estimate $O(\tau^{\frac{d-1}{2}})$ and even $o(\tau^{\frac{d-1}{2}})$ under standard condition to Hamiltonian trajectories holds for $d \geq 3$ for operator $A = -\Delta - V$ where $V = -\kappa|x|^{-2}$ as $|x| \geq c$ with Hardy constant κ.

(iv) One can prove results similar to theorems 0.5, 0.6 in the case $l = 1$ with treschold at $\bar\mu_1 = d^{-1})^2$.

Remark 0.8 (i) The same asymptotics hold for operators not in $\mathbb{R}^d$ but in a domain X and with boundary condition $Bu|_{\partial X} = 0$ where X and B coincide with $\bar X$ and $\bar B$ as $|x| \geq c$ where $\bar X + \Gamma = \bar X$, condition $\bar B u = 0$ is periodic with transformation matrices $\{T_1, \dots, T_{d'}\}$ and we assume that

(Δ) there exist functions ϕ_j ($j = 1, \dots, d'$) such that $\phi_j(x + e_k) - \phi_j(x) = \delta_{jk}$ $\forall j,k = 1,\dots,d'$ and $\bar B e^{i\langle\phi(x),\xi\rangle}u|_{\partial\bar X} = \bar B u|_{\partial\bar X}$ $\forall\xi$ where $d' = d$ here.

(ii) Furthermore, the similar asymptotics hold for operators waveguides i.e. domains bounded in $x'' = (x_{d'+1}, \dots, x_d)$ and satisfying above conditions with $\Gamma \subset \mathbb{R}^{d'} \ni x' = (x_1, \dots, x_{d'})$ and $T_1, \dots, T_{d'}$; here $d' < d$.

(iii) On the other hand, the same asymptotics hold for operators with Weyl symbol $a(x, \xi - \alpha x)$ where α is a matrix such that $\alpha n \in \Gamma'$ $\forall n \in \Gamma$. In this case

[2] I realized it after discussion with M.Sh.Birman. I will elaborate in the detailed paper.

$\mathbb{K}_{\xi,\{T\}}$ is the space of functions such that $u(x+n) = T^{-n}u(x)e^{i\langle \alpha n, x\rangle >}$. As an example one can consider Schrödinger operators with periodic magnetic field with "admissible" but not necessary vanishing flows over elementary cell.

1. Sketch of the proofs.

1.1 Reformulation of the problem

Let us notice first that $\mathbf{N}(t)$ is a number of eigenvalues of operator $L = W^{-\frac{1}{2}}(A - E)W^{-\frac{1}{2}}$ belonging to $[0,t)$ and that L is a self-adjoint operator in $L^2(\mathbb{R}^d, \mathbb{C}^D)$: this is obvious for $A = \bar{A}$ and $\bar{L} = W^{-\frac{1}{2}}(\bar{A} - E)W^{-\frac{1}{2}}$ because $\bar{A} - E$ is invertible in frames of theorem 0.1 and this is true for perturbed operator since $W^{-\frac{1}{2}}(A - \bar{A})W^{-\frac{1}{2}}$ is relatively compact in the sense that $||Lu|| \le \epsilon||\bar{L}u|| + C_\epsilon||u|| \qquad \forall u \forall\epsilon > 0$. Furthermore, $Spec_{\text{ess}} L = \emptyset$.

Moreover, even if E is the extreme of a spectral gap, conditions of our theorems assure that $\bar{L}$ is self-adjoint unbounded operator with compact inverse and therefore all above statements (save that $\bar{A} - E$ is invertible) remain true.

1.2 Operator-valued operators

One can try to apply arguments of Chapters 8,9 of [Ivr1] directly but they lead to sharp remainder estimates only if $\rho^{d-1} \in L^1$ even in frames of theorem 0.1. We need to involve operator-valued theory (also developed in [Ivr1]) and we apply the following transformation (also engaged by Birman-Suslina):

Let $\mathbb{K} = \mathbb{K}_{0,\{T\}}$ introduced above and let us define an unitary operator $\mathcal{F}$ (*Gelfand's transformation*) from $L^2(\mathbb{R}^d, \mathbb{C}^D)$ to $L^2(Q', \mathbb{K})$:

$$\mathcal{F}u(\xi, x) = (2\pi)^{-\frac{d}{2}}(\operatorname{Vol} Q)\sum_{n\in\Gamma} T^n e^{-i\langle x-n,\xi\rangle}u(x-n). \tag{1.1}$$

One can see that

$$u(x) = (2\pi)^{-\frac{d}{2}}\int_{Q'} e^{i\langle x,\xi\rangle}\mathcal{F}u(\xi, x)d\xi. \tag{1.2}$$

Obviously $\mathcal{F}u(\xi,.) \in \mathbb{K}\ \forall\xi \in Q'$ and equality

$$||u||^2_{L(\mathbb{R}^d)} = ||\hat{u}||^2_{L(\mathbb{R}^d)} = \int_{Q'}\sum_{k\in\Gamma'}|\hat{u}(\xi + k)|^2 = \int_{Q'}||\mathcal{F}u(\xi,.)||^2_{\mathbb{K}}d\xi \tag{1.3}$$

together with (1.1),(1.2) prove that $\mathcal{F} : L^2(\mathbb{R}^d, \mathbb{C}^D) \mapsto L^2(Q', \mathbb{K})$ is an unitary operator.

Proposition 1.1. *Operator $\mathcal{F}$ transforms our operators in the following way:*

$$\begin{aligned}
\mathcal{F}A\mathcal{F}^* &= A^w(x - D_\xi, x, \xi + D_x),\\
\mathcal{F}\bar{A}\mathcal{F}^* &= \bar{A}^w(x, \xi + D_x),\\
\mathcal{F}W\mathcal{F}^* &= W^w(x - D_\xi, x, \xi + D_x).
\end{aligned} \tag{1.4}$$

PROOF. is easy by direct calculations from (1.2). The only thing to notice is that we can treat the fast copy of x in $\mathbb{R}^d$ in different ways than the slow one using periodicity with respect to fast x.

1.3 Proofs of theorems 0.1, 0.1′

Let $L = W^{-\frac{1}{2}}(A - E)W^{-1\frac{1}{2}}$ and let $e_L(x, x, 0)$ be the Schwartz kernel of spectral projector of L. Let conditions (H1),(H2), (0.1)-(0.5) and (0.7)-(0.9) be fulfilled.

Now theorems 0.1, 0.1′ are due to the following statement where here and below $r_0(\tau)$ is defined as above from equation $\rho\tau^{\frac{1}{m}} = 1$:

Proposition 1.2. *(i) Let ψ be r-admissible function supported in $B(\bar{x}, \frac{r}{2})$ with $r = \langle x\rangle \xrightarrow{\text{def}} = (1 + |\bar{x}|^2)^{\frac{1}{2}} \le Cr_0(\tau)$. Then*

$$\begin{gathered}\mathfrak{R}_1 = \Bigl|\int \psi(x)\Bigl(e_L(x, x, \tau) - e_L(x, x, \frac{\tau}{2}) - \\ (2\pi)^{-d}\int \bigl(\nu_L(x, \xi, \tau) - \nu_L(x, \xi, \frac{\tau}{2})\bigr)d\xi\Bigr)dx\Bigr| \le C\tau^{\frac{d-1}{m}}\rho^{d-1}r^d\end{gathered} \tag{1.5}$$

where ν_L is the eigenvalue counting function of $L(x, \xi) = W^{-\frac{1}{2}}(x)a(x, \xi)W^{-\frac{1}{2}}$.

Furthermore, for fixed r under standard bicharacteristic condition

$$\mathfrak{R}_1 = o(\tau^{\frac{d-1}{m}}). \tag{1.6}$$

(ii) Moreover, for $2 < r \le Cr_0(\tau)$

$$\begin{gathered}\Bigl|\int \psi(x)\bigl(e_L(x, x, \tau) - e_L(x, x, 0) - \\ (2\pi)^{-d}\int \bigl(\mathbf{n}_L(x, \xi, \tau) - \mathbf{n}_L(x, \xi, \frac{\tau}{2})\bigr)d\xi\bigr)dx\Bigr| \le C\tau^{\frac{d-1}{m}}r^{(1-\mu)(d-1)}\tau^{\frac{d-1}{m}}\rho^{d-1}r^{d-1}\end{gathered} \tag{1.7}$$

where $\mathbf{n}_L(x, \xi, \tau)$ is the number of eigenvalues between 0 and τ of operator L restricted to the space $\mathbb{K}_{\xi,\{T\}}$ for fixed "slow" argument x.

(iii) Let $E \notin \operatorname{Spec_{ess}} A$. Then for $r \ge C_0 r_0(\tau)$ with large enough constant C_0

$$\Bigl|\int \psi(x)\bigl(e_L(x, x, \tau) - e_L(x, x, \frac{\tau}{2})\bigr)dx\Bigr| \le C\tau^{-s}r^{-s} \tag{1.8}$$

with arbitrarily large s

(iv) Finally, let A be semibounded from above. Then estimate (1.8) holds for $r \le \epsilon_0 r_1(\tau)$ with small enough constant ϵ_0.

PROOF.

(i) Follows from standard local spectral asymptotics.

(ii) follows from spectral asymptotics for operators with operator-valued symbols (theorem 4.3.6-4.3.8 [Ivr1]) with $B = (I + \Delta)^{\frac{m}{2}}$ in $\mathbb{K}$ with the same arguments as in the proof of theorem 12.1.3 and similar statements in Chapter 12 [Ivr1].

(iii) follows from the fact that zones in question is classically forbidden for operators with operator-valued symbols (theorem 4.3.2 [Ivr1])

PROOF OF REMARK 0.2. Part (i) follows from proposition 1.2 including second part of its (i) assertion. Part (ii) is due to proposition 1.2(i) only.

Note that if $r^{-\delta} < h = \tau^{-\frac{1}{m}} r^{\mu-1}$ then one can replace in proposition 1.2 (ii) $\mathbf{n}$ by $\bar{\mathbf{n}}$. Combining with second part of proposition 1.3(iii) we get (v) and with proposition 1.2(i) we get (iv).

To prove statement (iii) note that if $\tau^{\frac{d}{m}} r^{-\delta+d(1-\mu)} < \tau^{\frac{d-1}{m}}$ then one can replace in proposition 1.2 (ii) $\mathbf{n}$ by $\bar{\mathbf{n}}$. On the hand, one can employ the theory of operators with operator-valued symbols for propagation of singularities only. This yields standard Weyl asymptotics as in proposition 1.2(i) but with the remainder estimate $O(\tau^{\frac{d-2}{m}} r^{(1-\mu)(d-2)+2}) + O(\tau^{\frac{d-1}{m}} r^{(1-\mu)(d-1)})$ where the first part comes from the third term in complete Weyl asymptotics. One can find r satisfying first condition and making last expression less than $O(\tau^{\frac{d-1}{m}})$ iff $\delta > \delta_0$.

1.4 Proofs of theorems 0.3, 0.3′

To prove these two theorems (as well as theorem 0.5) we need only to treat zone $\{|x| \ge C_0 r_0(\tau)\}$ where operator in question is no more elliptic (which was classically forbidden before) and it is not even microhyperbolic here. To overcome this difficulty one can use condition (H3) and applying arguments of the first paragraph of section 4.4 [Ivr1] one can basically reduce the operator in question to similar operator which is scalar, with $\bar{A}$ and W replaced by λ_{k_p} and $\mathcal{W}_p$ respectively while ξ is near ξ_p (zone is still classically forbidden if $\min_p |\xi - \xi_p| > \epsilon$). Now, since G_p is non-degenerate, we need no microhyperbolicity condition anymore - due to theorem 4.4.2 [Ivr1]. We can see then that the contributions of the ball $B(x, \frac{r}{2})$ $(r = |x|)$ to the remainder estimate and to the principal part do not exceed $C(\tau\rho^m r^2)^{\frac{d-1}{2}}$ and $C(\tau\rho^m r^2)^{\frac{d}{2}}$ respectively as soon as uncertainty condition $(\tau\rho^m)^{\frac{1}{2}} r \ge \epsilon$ holds. Moreover, in the case (a) after rescaling this zone is classically forbidden and these contributions are $O\big((\tau\rho^m r^2)^{-s}\big)$ and 0 respectively. Anyway, the total contribution of the zone $\{r_0(\tau) \le |x| \le r_1(\tau)\}$ to the remainder estimate doesn't exceed what we got for zone $\{|x| \le r_0(\tau)\}$ and the contribution to the principal part is described by Weyl formula. Here and below $r_1(\tau)$ is obtained from equality $(\tau\rho^m)^{\frac{1}{2}} r = \epsilon$.

Now we need to consider zone $\{|x| \ge r_1(\tau)\}$. Note, that in the case (b) $\lambda_{k_p} - E \ge -\epsilon\Delta$ and it is $\ge \epsilon_1 |x|^{-2}$ as $d \ge 3$ while in the case (a) $\lambda_{k_p} - E \le \epsilon\Delta$. This makes this zone classically forbidden but effective Plank constant is $\asymp 1$ there and to handle this one can apply arguments of chapter 8 of [Ivr1]. I leave details for the reader.

1.5 Case $d = 2$. I

To prove theorems 0.3, 0.3′ in the case (b) for $d = 2$ and to prove theorems 0.5, 0.6 we need to analyze again zone $\{r_0(\tau) \le |x| \le r_1(\tau)\}$ (and even $\{C_0 \le |x| \le r_1(\tau)\}$ and sharpen remainder estimate there. For this purpose we need to consider long-term propagation of singularities.

Introducing polar coordinates we obtain operator close to $-\partial_r^2 + r^{-1}\partial_r - r^{-2}\partial_\theta^2 - \tau\mathcal{W}$ which we consider as operator-valued 1-dimensional operator for a sake of propagation. Introducing $z = \log r$ we obtain the operator $-\partial_z^2 + \partial_\theta^2 - \tau\mathcal{W}_1$ where $\mathcal{W}_1 = r^2\mathcal{W}|_{r=e^z}$ (we also made a transaction to operator in $L^2(\mathbb{R}^+)$ instead of operator in $L^2(\mathbb{R}^+, e^2 z dz)$). Applying standard analysis like in [Ivr1], section 4.4 we get

$T_1 = 1 + \min(\log\frac{r_1(\tau)}{r}, \log\frac{r}{r_0(\tau)})$ for this operator and $T = rT_1$ for the original problem. Then we gain factor $rT(r,\tau)^{-1} =$ in remainder estimate integral $\int \tau^{\frac{1}{2}}\rho^{\frac{1}{2}}T^{-1}rdr$ which will be $O(\tau^{\frac{1}{2}})$ and even less than $\epsilon(r_0)\tau^{\frac{1}{2}}$ with $\epsilon \to 0$ as $r_0 \to \infty$. On the other hand, second (actually third) term in complete Weyl asymptotics is of magnitude $\int r^{-1}dr = \log r_1(\tau) = o(\tau^{\frac{1}{2}})$.

1.6 Case $d = 2$. II

To prove theorems 0.5, 0.6 we need to study zone $\{|x| \ge r_1(\tau)\}$ which is no more classically forbidden. In this zone we apply operator-valued theory completely. Let us decompose $L^2(\mathbb{S})$ into eigenspaces of D_θ^2 and notice that all the eigenspaces save the lowest one are classically forbidden and applying the same arguments as in [Ivr2] we basically can reduce operators $A - E$ and W to $D_r g D_r$ and $\bar{W}$. I leave details to the reader. Let us omit index p for simplicity.

Introducing coordinate z as in theorem 0.5 we get over $\{z \ge z(\tau)\}$ quadratic forms $\int g|D_z v|^2\kappa^{-1}dz$ and $\int r^2\bar{W}|v|^2\kappa dz$ where $\kappa = \frac{d\log r}{dz}$; this is equivalent to $\int G|D_z(r^2\bar{W})^{-\frac{1}{2}}\kappa^{-\frac{1}{2}}\psi|^2\kappa dz$ and $\int |\psi|^2dz$. Operator associated with these two forms is $D_z g D_z + \Phi$

$$
\begin{aligned}
&g = G_p\kappa^{-2}(r^2\bar{W}^{-1} \asymp 1,\\
&\Phi = G\kappa^{-1}S^2 - \partial_z(G\kappa^{-1}S),\\
&S = \partial_z\big((r^2\log^2 r\bar{W})^{-\frac{1}{2}}.\kappa_1^{-\frac{1}{2}}\log^2 r\big), \qquad \kappa_1 = \frac{d\log\log r}{dz}
\end{aligned}
$$

One can see easily that $\Phi \asymp z^{2(1-\bar{\mu}_2)^{-1}}$ and Φ is (Φ,γ) admissible with $\gamma = \frac{z}{2}$ as $\bar{\mu}_2 \in (0,1)$ and $(\underbrace{\log\ldots\log}_{l-2\text{times}}\Phi) \asymp z^{(1-\bar{\mu}_l)}$ and Φ is (Φ,γ) admissible with $\gamma = (\log\Phi)^{-M}$ as $\bar{\mu}_1 = \cdots = \bar{\mu}_{l-1} = 1$, $\bar{\mu}_l \in (-\infty, 1)$ which leads to proofs of theorems 0.5, 0.6 and remark 0.7.

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