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Abstract

Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex and open set with real analytic boundary. Let $T_\Omega \subset \mathbb{C}^2$ be the tube with base Ω, and let $\mathcal{B}$ be the Bergman kernel of T_Ω. If Ω is strongly convex, then $\mathcal{B}$ is analytic away from the boundary diagonal. In the weakly convex case this is no longer true. In this situation, we relate the off diagonal points where analyticity fails to the Trèves curves. These curves are symplectic invariants which are determined by the CR structure of the boundary of T_Ω. Note that Trèves curves exist only when Ω has at least one weakly convex boundary point.

1. Introduction.

Let $U \subset \mathbb{C}^n$ be open. Let $L^2(U)$ denote the Hilbert space of complex valued functions defined on U, which are square integrable with respect to Lebesgue measure. Let $H(U) = \{f \in L^2(U) : \bar{\partial} f = 0\}$, the closed subspace of holomorphic functions on U. We denote by $\mathcal{B}$

$$\mathcal{B} : L^2(U) \to H(U)$$

the orthogonal projection, which is known as the Bergman projection. If $\{\varphi_j\}$ denotes an orthonormal basis for $H(U)$, then it is well known that $\mathcal{B}$ has kernel, which we also denote by $\mathcal{B}$,

$$\mathcal{B}(z, w) = \sum \varphi_j(z)\overline{\varphi_j(w)}, \quad z, w \in U.$$

The above series is uniformly convergent on compact subsets of $U \times U$. $\mathcal{B}$ is holomorphic in z and anti-holomorphic in w. In particular $\mathcal{B}$ is real analytic on $U \times U$.

In case U is strictly pseudoconvex, the boundary behavior of $\mathcal{B}$ is well understood. If $z^0 \in \partial U$, then it follows that

$$\lim_{z \to z^0} \mathcal{B}(z, z) = +\infty.$$

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See Hörmander [16] for a more precise statement.

We now assume, in addition to strict pseudoconvexity, that the boundary of U is real analytic. If we have $z^0, w^0 \in \partial U$, with $z^0 \neq w^0$, then it follows that $\mathcal{B}$ extends to a full neighborhood of $(z^0, w^0) \in \mathbb{C}^n \times \mathbb{C}^n$ as a function holomorphic in z and anti-holomorphic in w. In particular $\mathcal{B}$ is real analytic near (z^0, w^0). This follows from the analytic hypoellipticity of $\Box_b$, a consequence of results of Trèves [27], Tartakoff [25]. Also see Kashiwara [17].

Our main interest here is the weakly pseudoconvex case where ∂U is real analytic. Here off-diagonal singularities may occur. For example, Christ and Geller [5] have shown that the Bergman kernel for the domain

$$U = \{z \in \mathbb{C}^2 : \Im z_2 > (\Re z_1)^m\}$$

is not analytic at certain points away from the boundary diagonal, when m is even and $m \geq 4$.

We now continue the study begun in [10]. Our goal here is to state recent results we have obtained concerning the Bergman kernel for tubes. We assume these tubes are convex with bounded base and analytic boundary. We show that off–diagonal singularities are described by the characteristic lines. These lines are contained in the boundary and are projections to the base of the Trèves curves. These curves are symplectic invariants which are determined by the CR structure of the boundary. Trèves curves exist exactly when the base of the tube has at least one weakly convex boundary point.

Trèves introduced these curves in [27], where he conjectured that the existence of such curves should prevent analytic hypoellipticity, for certain partial differential operators with double characteristics. Recently Trèves has extended his conjecture, [28]. The reader should note that in the case of tubes, the two conjectures are essentially the same.

We have been motivated by several important results on analytic regularity. These include, besides those already mentioned, Chen [2], Christ [3], [4], Derridj [6], Derridj–Tartakoff [7], Geller [11], Grigis–Sjöstrand [12], Hanges – Himonas [13], Helffer [14], Métivier [20], Sjöstrand [23], Tartakoff [24], Trepreau [26]. The reader may consult our survey, [9] for more references.

In section 2 we discuss the notions of convexity that we need. In section 3 we discuss the Trèves curves and characteristic lines for tubes. In section 4 we state recent results. In sections 5 we discuss the formula of Boutet de Monvel, [1]. In sections 6 and 7 we discuss some of the proofs.

2. Geometric Preliminaries.

In this section we recall the notions of convexity that we use. We begin by discussing Ω, the base of the tube T_Ω. Let $U \subset \mathbb{R}^n$ be open. Let $r : U \to \mathbb{R}$ be real analytic. The base Ω is defined as follows:

$$\Omega = \{y \in U : r(y) < 0\}.$$

We assume that $dr(y) \neq 0$ whenever $r(y) = 0$. Furthermore, we assume that

$$\bar{\Omega} \subset\subset U, \tag{1}$$

that is, we assume that the closure of Ω is a compact subset of U.

Throughout we will assume that Ω is *convex.* This means that for each $y \in \partial\Omega$, the boundary of Ω, we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial y_j \partial y_k}(y) a_j a_k \geq 0 \tag{2}$$

whenever $a_j \in \mathbb{R}$ and

$$\sum_{j=1}^{n} a_j \frac{\partial r}{\partial y_j}(y) = 0. \tag{3}$$

Note that it follows from (2) and (3) that Ω is *geometrically convex.* This means that if $y, y' \in \Omega$, then the segment connecting y to y' is contained in Ω. See for example [19], Proposition 3.1.8, page 102. If strict inequality holds in (2) whenever $(a_1, \ldots, a_n) \neq 0$ satisfies (3), we say that $y \in \partial\Omega$ is a *strongly convex* boundary point.

3. Symplectic geometry and Trèves curves.

Our goal in this section is the calculation of the Trèves curves for the tube T_Ω. These curves are determined by the symplectic geometry associated to the CR structure of ∂T_Ω. We begin with a general definition.

Let (M, ω) be an analytic symplectic manifold with symplectic form ω. If $\Sigma \subset M$ is a submanifold with $p \in \Sigma$, we denote by $T_p\Sigma$ the tangent space to Σ at p. We denote by $(T_p\Sigma)^\perp$ the orthogonal of $T_p\Sigma$ with respect to ω. Let $I \subset \mathbb{R}$ be an open interval containing 0. We have the following :

Definition 1. Let $\Sigma \subset M$ be an analytic submanifold and let $\gamma : I \to \Sigma$ be a non-constant analytic curve. We call γ a *Trèves curve* for Σ if

$$\frac{d\gamma}{dt}(t) \in (T_{\gamma(t)}\Sigma)^\perp \tag{4}$$

for all $t \in I$.

We now discuss the characteristic set of the CR structure of ∂T_Ω. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be natural coordinates. We think of $\mathbb{C}^n$ as the space $\mathbb{R}^n \times \mathbb{R}^n$ equipped with the complex structure generated by the functions $z_j = x_j + iy_j, j = 1, \ldots, n$. This then induces coordinates $(x, y, \xi, \eta) \in T^*\mathbb{C}^n$. Since $T_\Omega = \{z \in \mathbb{C}^n : r(y) < 0\}$, we have $T^*(\partial T_\Omega) \subset T^*(\mathbb{C}^n)$ is defined by two equations; that is we have

$$T^*(\partial T_\Omega) = \{(x, y, \xi, \eta) \in T^*(\mathbb{C}^n) : r(y) = 0 \text{ and } \sum_{j=1}^{n} \eta_j \frac{\partial r}{\partial y_j}(y) = 0\}. \tag{5}$$

We now study the CR structure on ∂T_Ω. We will work near a point $z = x + iy \in \mathbb{C}^n$ such that $r(y) = 0$ and $\frac{\partial r}{\partial y_k}(y) \neq 0$ for some k, $1 \leq k \leq n$. The following $n-1$ vector fields form a basis for the natural CR structure on the boundary of T_Ω near z

$$L_j \frac{\partial r}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{z}_j} - \frac{\partial r}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{z}_k}, j \neq k. \tag{6}$$

Then $\Sigma \subset T^*(\mathbb{C}^n)$, the characteristic set of the CR structure, is defined by $2n$ equations. Indeed, we have $(x, y, \xi, \eta) \in \Sigma$ if and only if (x, y, ξ, η) satisfies the two equations of (5) along with the following $2n - 2$ equations:

$$\xi_j \frac{\partial r}{\partial y_k} - \xi_k \frac{\partial r}{\partial y_j} = 0, \quad j \neq k \tag{7}$$

$$\eta_j \frac{\partial r}{\partial y_k} - \eta_k \frac{\partial r}{\partial y_j} = 0, \quad j \neq k. \tag{8}$$

It follows immediately from (7) and (8) that we have the following

Lemma 1. *Let Σ be the characteristic set for the natural CR structure induced on the boundary of T_Ω. Then we have*

$$\Sigma = \{(x, y, \xi, \eta) \in T^*(\mathbb{C}^n) : r(y) = 0, \eta = 0, \frac{\xi}{|\xi|} = \pm \frac{dr(y)}{|dr(y)|}\}.$$

We will now study the Trèves curves for Σ. Let $\rho^0 = (x^0, y^0, \xi^0, \eta^0) \in \Sigma$ and let $I \subset \mathbb{R}$ be an open interval containing the origin. Assume that

$$\gamma : I \to \Sigma$$

is a Trèves curve such that $\gamma(0) = \rho^0$. If $s \in I$, we write

$$\gamma(s) = (x(s), y(s), \xi(s), \eta(s)). \tag{9}$$

We have the following

Proposition 1. *Assume that $\partial\Omega$ is real analytic. Suppose that γ is a Trèves curve for Σ as described in* (9). *Then we have*

$$y(s) = y^0, \quad \xi(s) = \xi^0, \quad \eta(s) = 0 \tag{10}$$

for all $s \in I$. Furthermore we have

$$< \frac{dx}{ds}(s), dr(y^0) >= 0 \tag{11}$$

and

$$\sum_{j=1}^{n} \frac{\partial^2 r}{\partial y_j \partial y_l}(y^0) \frac{dx_j}{ds}(s) = 0 \tag{12}$$

for all $s \in I$ and $l = 1, \ldots, n$. Conversely, any non-constant curve $\gamma : I \to \Sigma$ satisfying (10), (11) *and* (12) *is a Trèves curve for Σ.*

Remark 1. If $(x^0, y^0, \xi^0, 0)$ and $(x^1, y^0, \xi^0, 0)$ lie on the same Trèves curve for Σ, then define $x(s) = sx^1 + (1-s)x^0$, $s \in \mathbb{R}$. It follows that the straight line $(x(s), y^0, \xi^0, 0)$, $s \in \mathbb{R}$ is a Trèves curve for Σ. This follows because of (11) and (12). We will call the projection to the base of this line a *characteristic line.* This is the set of points $x(s) + iy^0 \in \mathbb{C}^2$.

Remark 2. Note that by definition, Trèves curves are not constant. Hence it follows from the Proposition that if a Trèves curve passes through ρ^0, we must have $x(s)$ not constant. As a consequence we see that y^0 must be a weakly convex boundary point of Ω. So we see that if $\partial\Omega$ is strongly convex, it follows that Σ contains no Trèves curves. Indeed, if $\partial\Omega$ is strongly convex, it follows that T_Ω is strongly pseudoconvex, and hence Σ is symplectic.

4. Statements of Results.

Let $\Omega \subset \mathbb{R}^2$ be open, bounded and convex with real analytic boundary. Let $T_\Omega = \{z \in \mathbb{C}^2 : \Im z \in \Omega\}$.

Assume that $z^0 \in \partial T_\Omega$ and let L be a vector field defined near z^0 that generates the CR structure. Let $\mathfrak{g}$ be the Lie algebra generated by L and $\bar{L}$ under the commutator bracket. Since $\partial\Omega$ is bounded and analytic, it follows that there exists $X \in \mathfrak{g}$ such that $L, \bar{L}$ and X are linearly independent at z^0. We say that z^0 is a point of type m if the smallest possible commutator length for X is m. Note that m is even and $m \geq 2$; z^0 is a strictly pseudoconvex boundary point if and only if $m = 2$. Also observe that if z^0 and w^0 can be connected by a characteristic line, then z^0 and w^0 have the same type $m \geq 4$.

In the three theorems stated below we always assume that $z^0, w^0 \in \partial T_\Omega$, with $z^0 \neq w^0$.

Theorem 1. *Assume that z^0 and w^0 do not lie on the same characteristic line. Then $\mathcal{B}$, the Bergman kernel for T_Ω, extends as an analytic function to a full neighborhood of (z^0, w^0).*

Theorem 2. *Assume that z^0 and w^0 lie on the same characteristic line. Assume that the type of z^0 is m. Then $\mathcal{B}$, the Bergman kernel for T_Ω, extends as a smooth function, of Gevrey class m, to a full neighborhood of (z^0, w^0).*

The next result is of primary importance.

Theorem 3. *Assume that z^0 and w^0 lie on the same characteristic line. Let $y^0 = \Im z^0 = \Im w^0$. Assume that* (22) *is satisfied at y^0. Then $\mathcal{B}$, the Bergman kernel for T_Ω, cannot be extended as an analytic function to any neighborhood of (z^0, w^0).*

5. The Formula.

We give here a brief discussion of a result of Boutet de Monvel, [1]. See also Koranyi [18], Vinberg [29], Faraut and Koranyi [8]. Let $\Omega \subset \mathbb{R}^n$ be open, with $0 \in \Omega$. If Ω is bounded and convex, then we have the following formula for the Bergman kernel of T_Ω. Note that no smoothness assumptions on $\partial\Omega$ are necessary for the validity of this formula.

If we denote by $\mathcal{B}$ the Bergman kernel of T_Ω, then we have for $z, w \in T_\Omega$

$$\mathcal{B}(z,w) = \int_{\mathbb{R}^n} e^{i<z-\overline{w},\xi>} A(\xi)^{-1} \frac{d\xi}{(2\pi)^n} \tag{13}$$

where we define

$$A(\xi) = \int_\Omega e^{-2<\xi,y>} dy. \tag{14}$$

If the boundary of Ω is of class C^2 we may use Green's theorem to obtain

$$A(\xi) = \frac{1}{2|\xi|} \int_{\partial\Omega} e^{-2<\xi,y>} < -\frac{dr}{|dr|}, \frac{\xi}{|\xi|} > d\sigma(y) \tag{15}$$

where r is a defining function for Ω and $d\sigma(y)$ denotes the surface area on $\partial\Omega$.

6. Analytic singularities away from the boundary diagonal.

We now discuss some of the ideas used in the proof of Theorem 3. We use the notation of section 3. Assume that $z^0 \neq w^0$ with $z^0, w^0 \in \partial T_\Omega$. Also assume that z^0 and w^0 lie on the same characteristic line. It follows that there exists $y^0 \in \partial\Omega$, a weakly convex boundary point, such that $\Im z^0 = y^0 = \Im w^0$. Our assumption also guarantees the existence of a vector $a \in \mathbb{R}^n, |a| = 1$ and $t_0 \in \mathbb{R}, t_0 \neq 0$ such that

$$\Re z^0 = \Re w^0 + t_0 a.$$

The vector a also satisfies

$$< a, dr(y^0) >= 0 \text{ and } r''(y^0)a = 0.$$

All this follows from Proposition 1 and Remark 1. We now introduce the function $U(t)$ as follows :

$$U(t) = \mathcal{B}(z^0, w^0 + tdr(y^0)). \tag{16}$$

We will show that $\mathcal{B}$ has no analytic extension to any neighborhood of (z^0, w^0), by proving that U is not analytic near $\{t = 0\}$.

We will begin by choosing convenient coordinates. Note that the formula (13) is invariant under translations and real rotations. Hence we may assume that $y^0 = 0$. We may also assume that we have $\delta > 0$ and φ real valued and real analytic near $|y'| \leq \delta$ such that r has the form $r(y) = \varphi(y') - y_n$ with $d\varphi(0) = 0$. Here $y' = (y_1, \ldots, y_{n-1})$. Hence $dr(0) = (0, \ldots, 0, -1)$. We may also assume that $a = (a', 0), |a'| = 1, a' \in \mathbb{R}^{n-1}$ with $\varphi''(0)a' = 0$. So we see that if $\xi \in \mathbb{R}^n$ we have

$$< z^0 - (\overline{w}^0 + tdr(y^0)), \xi >= t_0 < a', \xi' > +t\xi_n.$$

Hence it follows that we may assume that $0 \in \partial\Omega$ and that for $t \in \mathbb{R}$, t near 0 we have

$$U(t) = \int_{\mathbb{R}^n} e^{i(t_0<a',\xi'>+t\xi_n)} A(\xi)^{-1} \frac{d\xi}{(2\pi)^n} \tag{17}$$

where $t_0 \in \mathbb{R}, t_0 \neq 0$ and $a' \in \mathbb{R}^{n-1}, |a'| = 1$.

Now that we have chosen convenient coordinates, we will discuss several localizing arguments. The main idea is that the important singularities of U arise near the interior normal of Ω. Given $M > 0$, we define Γ as follows:

$$\Gamma = \{\xi \in \mathbb{R}^n : \xi_n \geq M|\xi'|\}. \tag{18}$$

As usual $\xi' = (\xi_1, \ldots, \xi_{n-1})$. Note that Γ is a conic neighborhood of the vector $\mathbf{n} = (0, \ldots, 0, 1)$, which is the interior normal to Ω at the origin. We define

$$S_\delta = \{y \in \partial\Omega : y_n = \varphi(y'), |y'| \leq \delta\}.$$

and

$$A_\delta(\xi) = \frac{1}{2|\xi|} \int_{S_\delta} e^{-2<\xi,y>} < -\frac{dr}{|dr|}, \frac{\xi}{|\xi|} > d\sigma(y).$$

Now we introduce

$$U_{\Gamma,\delta}(t) = \int_{\Gamma} e^{i(t_0<a',\xi'>+t\xi_n)} A_\delta(\xi)^{-1} \frac{d\xi}{(2\pi)^n}.$$

We have the following

Lemma 2. *Let $\delta > 0$ be given. Then there exist $M > 0$ such that*

$$U - U_{\Gamma,\delta}$$

is an analytic function of t, for t near 0.

We now focus our attention on $U_{\Gamma,\delta}$. Our goal is to show that this function is not analytic near $t = 0$.

7. The two dimensional case.

We now begin our study of the two variable case. We assume that $\Omega \subset \mathbb{R}^2$ is open and convex with real analytic boundary. We may also assume that $0 \in \partial\Omega$ and that we have $\delta > 0$ and φ real valued and real analytic near $|y_1| \leq \delta$ such that r has the form $r(y) = \varphi(y_1) - y_2$ with $\varphi'(0) = 0$. Hence $dr(0) = (0, -1)$. Our previous work allows us to focus on $U_{\Gamma,\delta}(t)$. In the present case, we have

$$A_\delta(\xi) = \frac{1}{2|\xi|} \int_{-\delta}^{\delta} e^{-2(\xi_1 s+\xi_2\varphi(s))} a(s,\xi) ds \tag{19}$$

where

$$a(s,\xi) = \frac{\xi_2 - \varphi'(s)\xi_1}{|\xi|}. \tag{20}$$

Since we assume weak convexity, we have $\varphi''(0) = 0$. We may assume that we have, for $|s| \leq \delta$, a strictly positive analytic function u such that

$$\varphi(s) = u(s)s^m \tag{21}$$

where $m \geq 4$ is the type.

Now we make a technical assumption, which most likely can be removed. We assume that

$$u'(0) = 0. \tag{22}$$

We now introduce, for any constant $\alpha > 0$,

$$A^\alpha(\xi) = \frac{1}{2\xi_2}\int_{-\infty}^{+\infty} e^{-2(\xi_1 s+\xi_2\alpha s^m)}ds = \frac{1}{2\xi_2}\int_{-\infty}^{+\infty} e^{2(\xi_1 s-\xi_2\alpha s^m)}ds$$

and

$$V^\alpha(t) = \int_0^\infty \int_{-\infty}^{+\infty} e^{i(\xi_1 t_0+\xi_2 t)}\xi_2^p A^\alpha(\xi)^{-1}\frac{d\xi_1 d\xi_2}{(2\pi)^2}.$$

The first step in our study is to show that V^α is not analytic at $t = 0$ for a particular choice of the real number p. Indeed, we will obtain a precise estimate that precludes analyticity. This is crucial. Then we must show that V^α is a good approximation of $U_{\Gamma,\delta}$, for a particular choice of α.

The argument is based on that presented in [10]. In that paper, the study is based on the function

$$\mathcal{N}(\eta) = \int_{-\infty}^{+\infty} e^{2(\eta s-s^m)}ds.$$

A simple calculation shows that

$$A^\alpha(\xi) = \frac{1}{2\xi_2(\alpha\xi_2)^{1/m}}\mathcal{N}(\frac{\xi_1}{(\alpha\xi_2)^{1/m}}). \tag{23}$$

We know that $\mathcal{N}$ is entire, even and has zeroes only when $m \geq 4$ and even. Hence in this case, $\mathcal{N}^{-1}$ has finite radius of convergence at the origin, which we denote by R. We also know that $\mathcal{N}$ has zeroes only on the imaginary axis, see [21], hence the zeroes closest to the origin are $\pm iR$. We then have the following

Lemma 3. *Let $m \geq 4$ be even. Let $R > 0$ be the radius of convergence of $\mathcal{N}^{-1}$ at the origin. Let $\alpha > 0$ be arbitrary. Then there exists an even integer a depending only on m, such that $0 \leq a \leq m-2$ and a sequence of integers $k_j \to +\infty$ also depending only on m such that if $p = \frac{a-2m-1}{m}$ then we have*

$$|(D_t^{k_j}V^\alpha)(0)| \geq \frac{m}{2\pi\alpha^{\frac{a-1}{m}}|t_0|^{a+1}R^a}(\frac{1}{\alpha|t_0|^m R^m})^{k_j}(mk_j+a)!$$

We now must show that V^α is a good approximation of $U_{\Gamma,\delta}$. Rather than studying $U_{\Gamma,\delta}$, we introduce

$$V_{\Gamma,\delta}(t) = \int_\Gamma e^{i(t_0\xi_1+t\xi_2)}\frac{\xi_2^{p+2}}{|\xi|^2}A_\delta(\xi)^{-1}\frac{d\xi}{(2\pi)^2}. \tag{24}$$

Note that $V_{\Gamma,\delta}$ is the image of $U_{\Gamma,\delta}$ under the elliptic pseudodifferential operator $|D_t|^{p+2}\triangle^{-1}$. Here $\triangle$ is the Laplacian in the (t_0, t) variables. Hence it suffices to prove that $V_{\Gamma,\delta}$ is not analytic at $t = 0$. (We have assumed here that the vector a of section 6 has the form $a = (1, 0)$.)

We now must introduce

$$A_\delta^\alpha(\xi) = \frac{1}{2\xi_2}\int_{-\delta}^{+\delta} e^{-2(\xi_1 s + \xi_2 \alpha s^m)} ds = \frac{1}{2\xi_2}\int_{-\delta}^{+\delta} e^{2(\xi_1 s - \xi_2 \alpha s^m)} ds$$

and

$$V_{\Gamma,\delta}^\alpha(t) = \int_\Gamma e^{i(\xi_1 t_0 + \xi_2 t)} \xi_2^p A_\delta^\alpha(\xi)^{-1} \frac{d\xi_1 d\xi_2}{(2\pi)^2}.$$

It follows that we have

Proposition 2. *Let $\delta > 0$ and $\alpha > 0$ be given. Then there exists an $M > 0$ such that U is analytic near $t = 0$ if and only if*

$$V^\alpha + (V_{\Gamma,\delta} - V_{\Gamma,\delta}^\alpha)$$

is analytic near $t = 0$.

Since we already know that V^α is not analytic near $t = 0$, one would expect to proceed to prove that $V_{\Gamma,\delta} - V_{\Gamma,\delta}^\alpha$ is analytic near $t = 0$. We have not been able to do this. Instead we will estimate, from above, the derivatives of $V_{\Gamma,\delta} - V_{\Gamma,\delta}^\alpha$ and show directly that there exist $\alpha > 0, \delta > 0$ such that the function $V^\alpha + (V_{\Gamma,\delta} - V_{\Gamma,\delta}^\alpha)$ is not analytic near $t = 0$. We must study

$$D_t^k(V_{\Gamma,\delta} - V_{\Gamma,\delta}^\alpha)(0) \int_\Gamma e^{i\xi_1 t_0} \xi_2^{p+k} \left(\frac{\xi_2^2}{|\xi|^2} A_\delta(\xi)^{-1} - A_\delta^\alpha(\xi)^{-1}\right) \frac{d\xi}{(2\pi)^2}. \tag{25}$$

Note that the real number p has been fixed once and for all in Lemma 3. We must estimate $Q(\xi)$ which we define to be

$$Q(\xi) = \frac{\xi_2^2}{|\xi|^2} A_\delta^{-1} - (A_\delta^\alpha)^{-1} = \frac{\xi_2^2 A_\delta^\alpha - |\xi|^2 A_\delta}{|\xi|^2 A_\delta A_\delta^\alpha}. \tag{26}$$

To complete the proof, we must estimate $Q(\xi)$ for ξ complex, so that a very delicate contour deformation can be made in (25). The deformation depends on the number of derivatives k. It also must avoid the zeroes of the denominator, $A_\delta A_\delta^\alpha$. We approximate these zeroes by those of $\mathcal{N}$. However, to obtain the proper estimates we must know that $\pm iR$ are *simple zeroes* of $\mathcal{N}$. We have succeeded in proving this using a result of Pólya [21] written in 1923 and its sequel Pólya [22] written in 1968. Complete details will appear elsewhere.

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