JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1998), p. 1-9

<http://www.numdam.org/item?id=JEDP_1998____A13_0>

Journées *Équations aux dérivées partielles*
Saint-Jean-de-Monts, 2-5 juin 1998
GDR 1151 (CNRS)

Remarks on global existence and compactness for L^2 solutions of the critical Nonlinear Schrödinger equation in 2D

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Abstract

In the talk we shall present some recent results obtained with F. Merle about compactness of blow up solutions of the critical nonlinear Schrödinger equation for inital data in $L^2(\mathbf{R}^2)$. They are based on and are complementary to some previous work of J. Bourgain about the concentration of the solution when it approaches to the blow up time.

1. Introduction.

Let us consider the initial value problem for the nonlinear Schrödinger equation

$$\begin{cases} \partial_t u = \frac{i}{4\pi}(\Delta u \pm |u|^2 u), & x \in \mathbb{R}^2, \ t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

The positive sign represents the focusing case and solutions with large initial data can develop singularities. The negative sign (i.e. the defocusing situation) is expected to generate global wellposedness for arbitrary initial data. We want to review some recent results on these matters.

We shall start by explaining some of the rich structure of the space of solutions of (1.1). Let us talk first about the conservation laws. We have the following conserved quantities:

$$\int_{\mathbb{R}^2} \left|u(x,t)\right|^2 dx = \left\|u_0\right\|_{L^2}^2, \tag{1.2}$$

$$E\left(u(t)\right) = \frac{1}{2}\int_{\mathbb{R}^2}\left(\left|\nabla u\right|^2 \mp \frac{1}{2}|u|^4\right) dx = E(u_0), \tag{1.3}$$

Partially supported by grant of Ministerio de Cultura PB94-149
MSC 1991 : 35R30
Keywords : Nonlinear Schrödinger Equation, Blow-up, compactness

$$\int_{\mathbb{R}^2}\left(\left|(2\pi i x - t\nabla)u\right|^2 \mp \frac{t^2}{2\pi^2}|u|^4\right)dx = \|2\pi x u_0\|_{L^2}^2. \tag{1.4}$$

The first two are proved by straightforward calculations. The third one was obtained by Ginibre and Velo ([G-V]), see also [G] and [ZSS].

Next, let us consider the invariants of the equation. Assume u is a solution of (1.1). Then we have the following invariants :

i) Translations: given $x_0 \in \mathbb{R}^2, t_0 \in \mathbb{R}$

$$u_{x_0,t_0}(x,t) = U(x - x_0, t - t_0) \tag{1.5}$$

is also a solution and $\|u_{x_0,t_0}(0)\|_{L^2} = \|u(0)\|_{L^2}$.

ii) Dilations: given $\rho > 0$

$$u_\rho(x,t) = \frac{1}{\rho}u\left(\frac{x}{\rho}, \frac{t}{\rho^2}\right) \tag{1.6}$$

is also a solution and $\|u_\rho(0)\| = \|u(0)\|_{L^2}$

iii) Galilean invariance: given $\xi_0 \in \mathbb{R}^2$

$$u^{\xi_0}(x,t) = \exp\left\{2\pi i x\xi_0 - \pi i t|\xi_0|^2\right\}u(x - t\xi_0, t) \tag{1.7}$$

is also a solution with $u^{\xi_0}(0) = e^{2\pi i\xi_0}u(0)$. Hence $\|u^{\xi_0}(0)\|_{L^2} = \|u(0)\|_{L^2}$.

iv) Conformal transformation: take v as

$$v(x,t) = \frac{e^{\pi i\frac{|x|^2}{t}}}{it}\overline{u}\left(\frac{x}{t}, \frac{1}{t}\right). \tag{1.8}$$

Then v is also a solution of (1.1) and $\|v(0)\|_{L^2} = \|u(0)\|_{L^2}$.

We could also consider rotations, multiplication by complex numbers of unitary modulus, conjugation ($\overline{u}(-t)$) and other possibilities regarding (1.4). Although the above identities are obtained by straightforward calculations it is clarifying to understand them first in the linear setting.

Let us define $e^{it\Delta}u_0$ as the solution of

$$\begin{cases} \partial_t u = \frac{i}{4\pi}\Delta u, & x \in \mathbb{R}^2,\ t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.9}$$

We also have that

$$e^{it\Delta}u_0 = \frac{1}{it}\int_{\mathbb{R}^2} e^{\pi i\frac{|x-y|^2}{t}}u_0(y)dy \tag{1.10}$$

$$= \int_{\mathbb{R}^2} e^{-\pi i t|\xi|^2 + 2\pi i x\xi}\widehat{u_0}(\xi)d\xi \tag{1.11}$$

where

$$\widehat{u_0}(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i x\xi}u_0(x)dx.$$

Then for solutions of the linear problem (1.9) the galilean invariance follows by a simple change of variable ant the conformal transformation can be easily proved using (1.10) and (1.11) by completing the square. Moreover in this case we can identify the initial datum of the new solution. In fact if u is a solution of (1.9) and v is defined as in (1.8) then

$$v(0) = \overline{\hat{u}}(0). \tag{1.12}$$

With such a rich structure we can ask for explicit solutions of (1.1). Let us consider the negative sign and look them within the type $u_\omega(x,t) = e^{i\frac{t}{4\pi}\omega} Q_\omega(x)$. Hence Q_ω has to verify

$$\Delta Q_\omega - \omega Q_\omega + \left|Q_\omega\right|^2 Q_\omega = 0. \tag{1.13}$$

Berestycki and Lions (see [B-L]) proved the existence of a positive solution in L^2, that up to translations should be radial and also unique (see [K]). Let us take $\omega = 1$ and call it $Q(x)$. Moreover Q is in the Schwarzt class. Then Weinstein (see [W]) pointed out that if we construct the new solution obtained by the conformal transformation we get

$$v_1(x,t) = \frac{1}{it} e^{i\frac{|x|^2}{t}} e^{-i\frac{t}{4\pi}} Q\left(\frac{x}{t}\right). \tag{1.14}$$

In particular if t goes to zero,

$$\left|v(\cdot,t)\right|^2 \to \delta \int_{\mathbb{R}^2} Q^2\, dx, \tag{1.15}$$

and

$$\left\|v(\cdot,t)\right\|_{H^1} = \mathrm{O}\left(\frac{1}{|t|}\right). \tag{1.16}$$

Here δ denotes the Dirac mass.

It is interesting to compare u_1 and v_1 at time $t=1$. We have

$$u_1(1) = c_1 Q(x) \quad \text{and} \quad v_1(1) = c_2 Q(x) e^{i|x|^2} \tag{1.17}$$

with $|c_1| = |c_2| = 1$. Then although u_1 is a global solution, v_1 blows up at time $t=0$.

2. The local theory.

Let us consider now the question of solving (1.1). We have the following result due to Cazenave and Weissler (see [C-W] and [C] for general references of previous work due to Ginibre and Velo, Strauss, Tsutsumi and others).

Theorem 1. *(a) There is $\epsilon > 0$ such that if $\|u_0\|_{L^2} < \epsilon$ there exists a unique solution of (1.1) such that*

$$u \in C\left(\mathbb{R}\, L^2\left(\mathbb{R}^2\right)\right) \cap L^4\left(\mathbb{R}^2 \times \mathbb{R}\right). \tag{2.1}$$

(b) Given $u_0 \in L^2(\mathbb{R}^2)$ *there are* $T_* = T_*(u_0)$, $T^* = T^*(u_0)$ *and a unique solution of (1.1) such that for any* $T_* < T_1 < T_2 < T^*$

$$u \in \mathcal{C}\left([T_1, T_2]\ L^2\left(\mathbb{R}^2\right)\right) \cap L^4\left([T_1, T_2] \times \mathbb{R}\right). \tag{2.2}$$

Moreover if $T_* > -\infty$ *or* $T^* < +\infty$ *then*

$$\int_{T_*}^{T^*} \int_{\mathbb{R}^2} \left|u(x,t)\right|^4 dx dt = +\infty. \tag{2.3}$$

They also obtain results about continuous dependence which can be checked in the paper already mentioned. As we can see from the above statement the L^4 norm in space time seems to play a fundamental role. This is better understood if we recall some ofthe estimates available for the corresponding linear problem:

$$\begin{cases} \partial_t u = \frac{i}{4\pi}\left(\Delta u + F(x,t)\right), & x \in \mathbb{R}^2,\ t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2.4}$$

It was proved by Strichartz (see[St]) that if u solves (2.4) then

$$\left\|u\right\|_{L^4(\mathbb{R}^2 \times \mathbb{R})} \le C\left(\left\|u_0\right\|_{L^2} + \left\|F\right\|_{L^{4/3}(\mathbb{R}^2 \times \mathbb{R})}\right). \tag{2.5}$$

In order to solve the nonlinear problem (1.1), it is sufficient to prove the contraction in a suitable small ball of L^4 of the operator

$$\mathcal{T}(F) = e^{it\Delta}u_0 + \frac{i}{4\pi}\int_0^t e^{i(t-\tau)\Delta}|F|^2 F(\tau) d\tau. \tag{2.6}$$

Then from (2.5) we easily obtain

$$\left\|\mathcal{T}(F)\right\|_{L^4([T_1,T_2]\times\mathbb{R}^2)} \le C\left(\left\|u_0\right\|_{L^2} + \left\|F\right\|^3_{L^4([T_1,T_2]\times\mathbb{R}^2)}\right). \tag{2.7}$$

Notice that the size of the ball will depend on the size of $\left\|e^{it\Delta}u_0\right\|_{L^4([T_1,T_2]\times\mathbb{R}^2)}$. Cazenave and Weissler do it small by taking either $\|u_0\|_{L^2}$ small or $[T_1, T_2]$ small.

The example in (1.14) tells us that at least in the focussing case the smallness asumption is necessary. A more general argument is obtained from the convexity method of Glassey, and Zakharov, Sobolev and Synach which establishes that if $E(u_0) < 0$ the solution ceases to exist in finite time (see [G] and [ZSS]). In fact in this case both T_* and T^* are finite. The function Q obtained in (1.3) is such that $E(Q) = 0$.

Then a natural question is if there will be global existence in the repulsive case (i.e. negative sign in (1.1)). This seems to be quite a challenging question. From the theorem by Cazenave and Weissler we know that as long as the L^4 norm in space–time remains bounded the solution exists. In particular we can use (1.3) and (1.4), which tell us that if either $u_0 \in H^1(\mathbb{R}^2)$ or $xu_0 \in L^2(\mathbb{R}^2)$ the solution of (1.1) in the repulsive case is global.

Another possibility of obtaining global existence is by doing $\left\|e^{it\Delta}u_0\right\|_{L^4(\mathbb{R}\times\mathbb{R}^2)}$ small without u_0 being small. In other words by improving Strichartz's estimate (2.5) with $F \equiv 0$. This has been recently done in $\mathbb{R}^2$. Although the method can be extended to $\mathbb{R}$, similar results are not known in higher dimensions. Moyua, Vargas and Vega prove (see [MVV] and [B1] for related previous results) the following.

$$\left\|e^{it\Delta}u_0\right\|_{L^4(\mathbb{R}^2)} \leq C\left(\sum_{\delta}\sum_{\tau}\delta^4\left(\frac{1}{|\tau|}\int_{\tau}|\widehat{u_0}|^p\right)^{4/p}\right)^{1/4} \tag{2.8}$$

for $p \geq \frac{12}{7}$. Here δ is any diadic (i.e. $\delta = 2^j$ j an integer), τ a square belonging to a grid of squares of disjoint interiors of $\mathbb{R}^2$ of side length δ and $|\tau| = \delta^2$. From (2.8) we conclude

$$\left\|e^{it\Delta}u_0\right\|_{L^4(\mathbb{R}^2)} \leq C\left[\sup_{\delta,\tau}\delta\left(\frac{1}{|\tau|}\int|\widehat{u_0}|^p\right)^{1/p}\right]^{1-p/2}\left\|u_0\right\|_{L^2}^{p/2}, \tag{2.9}$$

which much be understood as a refinement of (2.5). It is also easy to generate examples that are not in L^2, but such that the right hand side of (2.8) is finite and as small as desired.

Finally in [MV] the authors use (1.8), (1.12) and the invariance of the L^4 norm by the conformal transformation to conclude that (2.8) still holds when in the right hand side $\widehat{u_0}$ is changed by u_0.

3. Concentration.

Another question, also related to the previous one about global existence in the repulsive setting, is to express the blow up in terms of a more natural norm. For example if we take the solution of (1.14) we have concentration of the L^2 norm in a hyperbolic rate and growth of the H^1 norm as $O\left(\frac{1}{|t|}\right)$. Is this a generic behaviour? When H^1 initial data is considered the situation is better understood. In particular in the case of critical mass (i.e. $\left\|u_0\right\|_{L^2} = \left\|Q\right\|_{L^2}$ with Q given by (1.13)), v_1 as in (1.14) is up to translation the only blow–up solution (see [M1]).

The situation for initial data in $L^2(\mathbb{R}^2)$ is more delicate. Recently Bourgain (see [B2]) has proved in this direction that if u blows up at time T^* then

$$\limsup_{t\to T^*}\sup_{|I|\leq(T^*-t)^{1/2}}\int_I\left|u(x,t)\right|^2 > \eta, \tag{3.1}$$

where $|I|$ is the side length of the square I and $\eta = \eta\left(\|u_0\|_{L^2}\right) > 0$. Notice that the concentration rate is parabolic and not hiperbolic as the exhibited in (1.14).

His argument is basically as follows. We know that $\|u(x,t)\|_{L^4([T_*,T^*]\times\mathbb{R}^2)} = \infty$. Hence by considering small intervals of time and using the integral equation as in (2.6) he concludes the existence of an increasing sequence of times $\{t_n\}$ going to T^* such that

$$\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}^2}\left|e^{i(t_n-t)\Delta}u(t_n)\right|^4 dx dt = C_0 \tag{3.2}$$

for some C_0 with $0 < C_0 < 1$. Now recall (2.9). We can find functions $\{f_{jn}\}_{j=1,\dots,N(C_0,\|u_0\|_{L^2})}$ supported in squares τ_{j_n} centerd at ξ_{j_n} and with sidelength ρ_{j_n} such that

$$(3.3) \qquad |f_{j_n}| \le \frac{A}{\rho_{j_n}} \quad , \ A = A(C_0)$$

and

$$(3.4) \qquad \|e^{it\Delta}\left(u(t_n) - \sum_{j=1}^{N} \widehat{f_{j_n}}\right)\|_{L^4} < C_0^2.$$

Next we have to analyze $e^{it\Delta}\widehat{f_j}$. Then using that there exists $p < 4$ (see [B1]) such that

$$\|e^{it\Delta}\widehat{f_{nj}}\|_{L^p(\mathbb{R}^3)} < C\|f_{j_n}\|_{L^\infty(\tau_j)},$$

we can find tubes Q_{kj_n} in $\mathbb{R}^3$ centered at (x_{kj_n}, t_{kj_n}) $\ 1 \le k \le \tilde{N}(C_0, N)$ of the type

$$(3.5) \qquad Q_{kj_n} = \{(x,t)\ \rho_{j_n}|x - x_{kj_n} - t\xi_{j_n}| + \rho_{j_n}^2|t - t_{kj_n}| < 1\}$$

such that

$$(3.6) \qquad \|e^{it\Delta}\widehat{f_{j_n}}\|_{L^4\left(\mathbb{R}^3 \setminus \cup_{k=1}^{\tilde{N}} Q_{kj_n}\right)} < \epsilon(C_0, N).$$

Finally using the integral equation again, (3.1) is obtained (see [B2]).

As we see a large part of the argument is based on the analysis of sequences of solutions of the linear problem with large L^4_{xt} norm. Also notice that the four parameters in (1.5), (1.6) and (1.7) have already appeared.

4. Compactness.

Our starting point in this section is the sequence of solutions of the free problem (1.9) which have large L^4 norm as in (3.2). We want to prove compactness. It is clear that this compactness will have to be up the invariants of the equation that were analyzed in the introduction. In a joint work with F. Merle (see [MV]) we have obtained the following result.

Theorem 2. *Let $\{u_n(x,t)\}$ be any sequence of solutions of the free problem (1.9) with*

$$(4.1) \qquad \int_{\mathbb{R}^2} |u_n|^2 dx \le M.$$

Then given $\epsilon > 0$ there is a subsequence also called u_n, and there are functions $\mathcal{U}_1, \dots, \mathcal{U}_{N(\epsilon)}$, and $\beta = \beta(M, \epsilon) > 0$ with

$$(4.2) \qquad \sum_{j=1}^{N(\epsilon)} \|\mathcal{U}_j\|_{L^2}^2 \le M,$$

and ξ_{j_n}, $x_{j_n} \in \mathbb{R}^2$, $\rho_{j_n} > 0$, $t_{j_n} \in \mathbb{R}$, $\left\| \mathcal{U}_j \right\|_{L^2} > \beta$ *such that*

$$u_n(t,x) = \sum_{j=1}^{N(\epsilon)} H_{j_n}(t,x) + v_n(t,x), \tag{4.3}$$

where,

$$H_{j_n} = e^{it\Delta}\left(e^{2\pi i(\cdot)\xi_{j_n}}(\rho_{j_n}^{-1})\left(e^{it_{j_n}\Delta}\left[\mathcal{U}_j(\cdot + x_{j_n})\right]\right)\right),$$

with $\rho(f)(x) = \frac{1}{\rho} f(\frac{x}{\rho})$, *and* v_n *solutions of (1.9) such that*

$$\|v_n\|_{L^4(\mathbb{R}\times\mathbb{R}^2)} \leq \epsilon.$$

Moreover H_{j_n} *and* v_n *are asymptotically decoupled in the sense that*

$$\lim_n \int_{\mathbb{R}^2} H_{j_n}(t) H_{k_n}(t)\, dx = 0 \qquad 1 \leq j \leq N(\epsilon), \qquad j_n \neq k_n \tag{4.4}$$

and

$$\lim_n \int_{\mathbb{R}^2} H_{j_n}(t) v_n(t)\, dx = 0 \qquad 1 \leq j \leq N(\epsilon). \tag{4.5}$$

In fact (4.4) is a consequence of the following fact:

$$\begin{aligned} &\Bigg(\frac{\rho_{j_n}}{\rho_{k_n}} + \frac{\rho_{k_n}}{\rho_{j_n}} + \frac{|\xi_{j_n} - \xi_{k_n}|}{\rho_{j_n}} \\ &\left|t_{j_n}\frac{\rho_{k_n}}{\rho_{j_n}} - t_{k_n}\frac{\rho_{j_n}}{\rho_{k_n}}\right| + \rho_{k_n}\left|\left(\frac{\xi_{k_n} - \xi_{j_n}}{\rho_{j_n}}\frac{t_{j_n}}{\rho_{j_n}}\right) + \frac{x_{j_n}}{\rho_{j_n}} - \frac{x_{k_n}}{\rho_{k_n}}\right|\Bigg) \to +\infty \end{aligned} \tag{4.6}$$

as n goes to infinity. Notice that the above condition is symmetric in (j_n, k_n).

As we see the invariants (1.5)–(1.7) appear in the statement of the theorem. We can use however the conformal transformation (1.8) to avoid the dependence of t_{j_n} which appears in (4.3) in the definition of H_{j_n}. We have the following result also proved in [MV].

Theorem 3. *Take* $u_n = u(t_n)$ *the solution of (1.1) which blows up at* T *and such that (3.2) hold.*

Then given $\epsilon > 0$ *there exist -up to a subsequence,* $\tilde{\mathcal{U}}_1, \ldots, \tilde{\mathcal{U}}_{\tilde{N}(\epsilon)}$, $\beta = \beta(M,\epsilon) > 0$, $\alpha > 0$, ξ_{j_n}, $x_{j_n} \in \mathbb{R}^2$, $\rho_{j_n} > 0$ *and* $c_{j_n} \in \mathbb{R}$ *such that*

$$\begin{aligned} u_n &= u(t_n) = \left(\sum_{j=1}^{\tilde{N}(\epsilon)} e^{i\xi_{j_n}(\cdot)}(\rho_{j_n})\left(e^{ic_{j_n}|\cdot|^2}\tilde{\mathcal{U}}_j(\cdot - x_{j_n})\right)\right) + \tilde{v}_n \\ &=: \sum_{j=1}^{\tilde{N}(\epsilon)} \tilde{H}_{j_n} + \tilde{v}_n, \end{aligned} \tag{4.7}$$

with $\|e^{it\Delta}\tilde{v}_n\|_{L^4([t_n,T]\times\mathbb{R}^2)} \leq \epsilon$, $\left\|\tilde{\mathcal{U}}_j\right\|_{L^2} > \beta$, $\rho(f)(x) = \frac{1}{\rho}f(\frac{x}{\rho})$, *and*

$$\lim_n \frac{(T - t_n)}{(\rho_{n_j})^2} \geq \alpha \qquad for \qquad 0 \leq c_{j_n}.$$

Moreover $\tilde{H}_{j_n}$ *and* $\tilde{v}_n$ *are asymptotically decoupled in the sense that*

$$\lim_n \int_{\mathbb{R}^2} \tilde{H}_{j_n}(t)\tilde{H}_{k_n}(t)\,dx = 0 \qquad 1 \le j \le \tilde{N}(\epsilon), \qquad j_n \ne k_n \tag{4.8}$$

and

$$\lim_n \int_{\mathbb{R}^2} \tilde{H}_{j_n}(t)\tilde{v}_n(t)\,dx = 0 \qquad 1 \le j \le \tilde{N}(\epsilon) \tag{4.9}$$

Theorem 2 says in particular that at time t_n either we have already the concentration or we have the oscilating term $e^{ic_{j_n}|x|^2}$ with $c_{j_n} < 0$ which should cause the concentration in a later time as in (1.17). Notice that in this case the concentration is backward in time while in (4.7) we are assuming blow up for a forward time.

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