JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1993), p. 1-5
<http://www.numdam.org/item?id=JEDP_1993____A14_0>

Smoothing of Dispersive Waves

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Given the classical Schrödinger equation $i\partial_t u = \Delta u$, $(x \in \mathbf{R}^n)$ with initial condition $u(x,t) = \varphi(x)$, we know that the mapping $\varphi \mapsto u(t)$ is unitary on $L^2(\mathbf{R}^n)$ and hence the solution is no smoother than the initial data. However, if $\varphi \in L^1$ with compact support, then the solution is analytic for $t \neq 0$, as we see immediately from the formula $u(t) = c_n t^{-n/2} \exp(i|x|^2/4t) * \varphi$. A way to see this smoothing property one derivative at a time is to use the Ginibre-Velo identity $\|xu + 2it u_x\|_{L^2} = \|x\varphi\|_{L^2}$. In a remarkable paper in 1983 Kato proved a similar property for the KdV equation $u_t + u_{xxx} + uu_x = 0$; namely, the solution is C^∞ for $t > 0$ if $\int_{-\infty}^{\infty} |\varphi|^2(1+e^x)\,dx < \infty$.

In this talk I report on joint work with W. Craig and T. Kappeler generalizing and sharpening such smoothing results. We consider a general class of linear dispersive equations with variable coefficients, for which we prove "micro semi-local" smoothing estimates with asymptotic conditions at infinity. Our method works for certain classes of nonlinear equations as well but I will not consider them here. Using different methods, some related problems have been considered by Ponce, Sjölin, Vega [7], Constantin-Saut, Hayashi-Nakamitsu-Tsutsumi, Yamazaki [8], and Kapitanski-Safarov [5]. (Other references may be found in [2].)

There is some nice intuition which explains the smoothing property. Singularities travel along bicharacteristic rays and they travel at infinite speed, as first proved by Lascar [6] and Boutet de Monvel [1] in the 1970's. The rays, if they are not trapped, travel in from spatial infinity. Assuming that the initial datum φ vanishes at infinity backwards along the rays, we deduce that in infinitesimal time the singularities from φ have passed by any given point and the solution is smooth. So where has the wave front set $WF(u)$ gone? It resides in the set of points which are trapped in the past. In particular any periodic orbit of the bicharacteristic flow could be in $WF(u)$.

The class of linear equations we treat is the following. Consider the evolution equation

$$i\frac{\partial u}{\partial t} = a(x,D)\,u\,, \qquad \left(D_j = \frac{1}{i}\frac{\partial}{\partial x_j}\right) \tag{1}$$

Lecture presented at St Jean-de-Monts, June 4, 1993. Research supported in part by NSF Grant 90-23864 and ARO Grant DAAL-3-90-G0012.

with a real C^∞ symbol $a(x,\xi)$. Assume it is

(i) Dispersive : $\left|\dfrac{\partial a}{\partial \xi}\right| \longrightarrow \infty$ uniformly in x as $|\xi| \to \infty$.

(ii) Flat at infinity :

$$\left|\partial_x^\alpha \partial_\xi^\beta [a(x,\xi) - a(\infty,\xi)]\right| \le C_{\alpha\beta} \frac{\langle \partial a/\partial \xi \rangle}{\langle \xi \rangle^{|\beta|-1} \langle x \rangle^{1+\delta+|\alpha|}} \tag{2}$$

where $\delta > 0$ and $\langle \eta \rangle = (1+|\eta|^2)^{1/2}$.

(iii) Non-trapping : $|x(t)| \to \infty$ as $t \to \infty$ uniformly for $|x(0)|$ and $|\xi(0)|^{-1}$ bounded, where $(x(t), \xi(t))$ denotes a solution of the bicharacteristic equations

$$\frac{dx}{dt} = \frac{\partial a}{\partial \xi}, \quad \frac{d\xi}{dt} = -\frac{\partial a}{\partial x} . \tag{3}$$

We use the energy method in the following simple form. Write the equation (1) as $i\,\partial_t u = Au$ and write $\langle u, v \rangle = \mathrm{Re} \int u \bar{v}\, dx$. Assuming $A^* = A$ and choosing any linear operator B, we take the inner product of (1) with $(B - B^*)u$ to obtain

$$\frac{d}{dt} \langle Bu, iu \rangle = \langle [A,B]u, u \rangle , \tag{4}$$

$[A,B] = AB - BA$. If $\eta(t)$ is any scalar function for which (for simplicity) $\eta(0) = \eta(T) = 0$, then

$$-\int_0^T \eta'(t) \langle Bu, iu \rangle \, dt = \int_0^T \eta(t) \langle [A,B]u, u \rangle \, dt . \tag{5}$$

If A has order m_A, then $[A,B]$ has a gain of $m_A - 1$ derivatives relative to B. Thus we deduce a gain of regularity if $\eta \ge 0$ and $[A,B] = C$ is an elliptic operator in some region.

On the symbol level the last equation takes the form

$$\{a,b\} = ic, \quad \{a,b\} \equiv \sum_j \partial_{\xi_j} a \, \partial_{x_j} b - \partial_{x_j} a \, \partial_{\xi_j} b .$$

Using the flow $(X(t;x,\xi), \Xi(t;x,\xi))$ defined by (3), this is the same as

$$\frac{d}{dt} b(X, \Xi) = ic(X, \Xi) .$$

Thus, given a symbol $c(x,\xi) \ge 0$, we define

$$b(x,\xi) = -i \int_0^{+\infty} c(X(t;x,\xi), \Xi(t;x,\xi)) \, dt , \tag{6}$$

which is justified if the integral converges and $b(X,\Xi) \to 0$ as $t \to +\infty$. Thus b is the integral of c over the forward flow defined by a. Typically, the supports of b and c will be in a "backward domain" Ω which has the properties that the trajectory backwards from any point $(x,\xi) \in \Omega$ remains within Ω and tends to spatial infinity and the trajectory forwards from $(x,\xi) \in \Omega$ exits from Ω within the time $\{\langle x\rangle + D\}/|a_\xi|$ where D is fixed. A typical backward domain is a neighborhood of a fixed backward trajectory.

The symbols must grow as $|x| \to \infty$ in the backward directions. For example, in the simple case $a(x,D) = D^2$ and $b(x,D) = b(x,D)$ in one dimension, we have $C = [D^2, bD] = b'(x)D^2+$ (lower order terms). We require $b'(x) \geq 0$ and $\not\equiv 0$. Thus $b(x)$ tends to two different constants at $\pm\infty$. In the second induction step $c(x,\xi)$ behaves like a non-zero constant as $|x| \to \infty$ in one direction. Hence $b(x,\xi)$ grows like $O(|x|)$. Repeated induction leads to the requirement that the symbols behave like $|x|^k$ as $|x| \to \infty$ in one direction. Thus we are led to the following definition.

b belongs to the symbol class $S^\Omega_\omega(m,k,\rho,\delta)$ if

(i) $\operatorname{supp}(b) \subset \Omega$

(ii) $|\partial_x^\alpha \partial_\xi^\beta b(x,\xi)| \leq C_{\alpha\beta} \langle\xi\rangle^{m-|\beta|} \langle x\rangle^{k+\rho|\beta|-\delta|\alpha|}$

(iii) $b(x,\xi) \geq b_0 \langle\xi\rangle^m \langle x\rangle^k$ $(b_0 > 0)$ in ω where $\omega \subset \bar\omega \subset \Omega$.

For simplicity we now limit ourselves to the second-order case.

Theorem 1. (Regularity) *Let $a(x,D)$ have order 2 and satisfy the conditions (dispersive, flat at ∞, non-trapping backwards) stated earlier. Let K be a positive integer and $0 \leq \rho < \delta \leq 1$, $\rho + \delta \geq 1$. Given initial data $\varphi \in L^2(\mathbf{R}^n)$ such that*

(7) $$|\langle B^0\varphi,\varphi\rangle| < \infty \text{ for some } B^0 = b^0(x,D),\, b^0 \in S^\Omega_{\Omega_0}(0,K,\rho,\delta)$$

for some backward domains $\Omega_0 \subset \bar\Omega_0 \subset \Omega$. Then there exist operators

(8) $$B_k = b_k(x,D), \quad b_k \in S^\Omega_\omega(k, K-k, \rho, \delta)$$

and a backward domain $\omega \subset \Omega_0$ such that for all $0 < T < \infty$ the following estimates hold :

(9) $$\int_0^T t^k\,|\langle B_{k+1}u,u\rangle|\,dt < \infty$$

(10) $$\sup_{[0,T]} t^k\,|\langle B_k u,u\rangle| < \infty$$

for $k = 0,1\ldots,K$. (In the special case $k = K$, the symbol class $S(K+1,-1,\rho,\delta)$ is redefined to replace $\langle x\rangle^{-1}$ by a fixed integrable function of $|x|$.) (The construction provides a set ω only slightly smaller than Ω_0.)

Thus we gain one derivative of regularity of the solution for each power of decay of the initial data.

Corollary. *If there are no trapped rays at all, and $|\varphi|^2$ possesses finite moments of all orders, then $u \in C^\infty$ for all $t \neq 0$.*

What about the behavior forward along a ray? If a ray is not trapped backwards but is trapped forwards, the solution still is microlocally smooth all along the forward ray in an appropriately narrow microlocal neighborhood. On the other hand, if it is not trapped forwards then it tends to infinity in the forward direction. We now address the question of the asymptotic behavior of the derivatives of the solution u in the forward direction.

For that purpose we need symbols supported in a "full domain" $\Omega = \Omega^+ \cup \Omega^-$, where Ω^- is a backward domain, $\mathcal{O}$ is an open set contained in Ω^-, and Ω^+ is the set $\mathcal{O}$ carried forward by the flow. In the relatively narrow forward set Ω^+ the symbols are of the "bad" type $\rho = 1$, $\delta = 0$. We consider two full domains $\omega \subset \bar{\omega} \subset \Omega$ and define $b \in \tilde{S}^{\Omega}_{\omega}(m, k_+, 1, 0)(m, k_-, \rho, \delta)$ if

(i) $\operatorname{supp}(b) \subset \Omega$.

(ii) In Ω^-, b belongs to $S^{\Omega^-}_{\omega^-}(m, k_-, \rho, \delta)$ as defined earlier.

(iii) In Ω^+, b satisfies

$$|\partial_x^\alpha \partial_\xi^\beta (\xi \cdot \partial_\xi)^\gamma b(x,\xi)| \leq C_{\alpha\beta\gamma} \langle \xi \rangle^{m-|\beta|} \langle x \rangle^{k_+ + |\beta|}$$

and in ω^+ it satisfies

$$b(x,\xi) \geq b_0 \langle \xi \rangle^m \langle x \rangle^{k_+} \quad (b_0 > 0) \ .$$

Theorem 2. (Forward behavior) *Let $a(x,D)$, K, δ, ρ, φ and B^0 satisfy the conditions of Theorem 1 with $b^0 \in S^{\Omega^-}_{\Omega_0^-}(0, K, \rho, \delta)$ where $\Omega = \Omega^+ \cup \Omega^-$ is a full domain and Ω_0^- is a backward domain with closure in Ω^-. Then there exist operators $B_k = b_k(x,D)$, $b_k \in \tilde{S}^{\Omega}_{\omega}(k, -k-1, 1, 0)(k, K-k, \rho, \delta)$ and a full domain ω such that for all $0 < T < \infty$ the estimates (9) and (10) hold. (Here $\omega \subset \Omega$ and $\omega^- \subset \Omega_0^- \subset \Omega^-$ are only slightly separated.)*

Thus there are bounds on the growth rates of the derivatives of u in the forward direction.

The symbols that appear in Theorem 1 satisfy $\rho < \delta$ and $1 \leq \rho + \delta$ and can be treated as a special case of the Weyl calculus [4]. On the other hand, the symbols that appear in the forward direction have type $(m, k, 1, 0)$ with $k \leq -m$ and two additional properties. They have support in the forward flow, and the derivatives $\xi \cdot \partial_\xi$ in the direction of the flow have $\rho = 0$. These properties allow us to prove that such a "forward" operator

$B = b(x, D)$ is bounded on $L^2(\mathbb{R}^n)$ if $m = k = 0$. Moreover, if $k < -m$ we have the generalized Gårding inequality $\langle Bu, u\rangle \geq \text{const}\,\langle Cu, u\rangle - \text{const}\,\langle u, u\rangle$, for some operator C of type $(m-1, k+1, 1, 0)$.

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