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UNIFORM ESTIMATES FOR A CLASS OF EVOLUTION EQUATIONS

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Abstract

L^∞ estimates are derived for the oscillatory integral $\int_0^{+\infty} e^{-\imath(x\lambda+\frac{1}{m}t\lambda^m)}a(\lambda)d\lambda$, where $2 \leq m \in \mathbb{R}$ and $(x,t) \in \mathbb{R}\times\mathbb{R}_+$. The amplitude $a(\lambda)$ can be oscillatory, e.g. $a(\lambda) = e^{\imath t\mathfrak{p}(\lambda)}$ with $\mathfrak{p}(\lambda)$ a polynomial of degree $\leq m-1$, or it can be of polynomial type, e.g. $a(\lambda) = (1+\lambda)^k$ with $0 \leq k \leq \frac{1}{2}(m-2)$. The estimates are applied to the study of solutions of certain linear pseudodifferential equations, of the generalized Schrödinger or Airy type, and of associated semilinear equations.

The purpose of the work described here is to derive precise "uniform" estimates for the solutions of initial value problems

$$(1)\qquad \begin{gathered} \imath^{-1}\partial u/\partial t = P(D_x)u + V(x)u + \gamma|u|^{p-1}u, \ in\ \mathbb{R}\times(0,T), \\ u(x,0) = u_0(x). \end{gathered}$$

In (1), $\imath = \sqrt{-1}$, $D = \imath^{-1}\partial/\partial x$; p and γ are real numbers, $p \geq 1$. The convolution operator $P(D)$ and the potential $V(x)$ will be submitted to various conditions. Naturally enough, our approach is to handle first the case $\gamma = 0$, $V \equiv 0$, then the case $\gamma = 0$, and finally the "general" case.

The initial value problem

$$(2)\qquad \imath^{-1}\partial u/\partial t = P(D)u$$

$$(3)\qquad u\big|_{t=0} = u_0,$$

has the evident solution $u = e^{\imath tP(D)}u_0$, ie.,

$$(4)\qquad u(x,t) = (2\pi)^{-1}\int_{\mathbb{R}} e^{\imath(x\xi + tP(\xi))}\hat{u}_0(\xi)d\xi.$$

Actually (4) is the unique solution of (1)–(2) which is a smooth function of $t \in \mathbb{R}$ valued in the space of tempered distributions of $x \in \mathbb{R}$. Our basic assumption is that the symbol $P(\xi)$ has the following expression:

(5) $$P(\xi) = [c_0^+ H(\xi) + c_0^- H(-\xi)]|\xi|^{\mathrm{m}} + R(\xi),$$

where H is the Heaviside function, ie., $H(\xi) = 1$ if $\xi > 0$, $H(\xi) = 0$ if $\xi < 0$, c_0^+ and c_0^- are real constants $\neq 0$, m is a real number ≥ 2 and $R(\xi)$ is a real–valued C^2 function in $\mathbb{R}\backslash\{0\}$ such that, for some constant $A > 0$ and all $\xi \in \mathbb{R}\backslash\{0\}$,

(6) $$|R| \leq A(1+|\xi|^{\mathrm{m}-1}),\ |R_\xi| \leq A(1+|\xi|^{\mathrm{m}-2}),\ |R_{\xi\xi}| \leq A(1+|\xi|^{\mathrm{m}-3}).$$

Our guiding philosophy has been to trade off freedom in the choice of the phase function for greater flexibility in that of the amplitudes. It has seemed convenient to incorporate into the amplitude the exponential of the lower order terms in the phase function. To compare our results to those of [KENIG–PONCE–VEGA, 1991] one may rightly say that our phase function is quite special, but our amplitudes are more general. Thus our approach to the Fourier integral (4) is to regard

$$x\xi + t[c_0^+ H(\xi) + c_0^- H(-\xi)]|\xi|^{\mathrm{m}}$$

as the phase function, whereas $\exp(\imath t R(\xi))$ is regarded as an amplitude. This leads to the study of oscillatory integrals

$$K_{\mathrm{m}}(x,t) = \int_0^{+\infty} \mathrm{e}^{-\imath(x\lambda + \frac{1}{\mathrm{m}} t\lambda^{\mathrm{m}})} a(\lambda) \mathrm{d}\lambda,$$

in which $x \in \mathbb{R}$, t is a number > 0 and $2 \leq \mathrm{m} \in \mathbb{R}$. The amplitude $a(\lambda)$ will always be of class C^2, with its derivatives of order ≤ 2 submitted to growth conditions. These allow $a(\lambda)$ to be "oscillatory", ie., of the kind $\mathrm{e}^{\imath t \mathfrak{p}(\lambda)}$ with $\mathfrak{p}(\lambda)$ a real polynomial of degree $\leq$ m–1. Alternatively, $a(\lambda)$ could be a constant coefficient "symbol" of order $\leq \frac{1}{2}$m–1. Here are the precise admissibility requirements:

($\star$) *a is a complex–valued C^2 function in $\mathbb{R}_+$ that satisfies the following condition:*

$\forall$ j, k $\in \mathbb{Z}_+$, $0 \leq j \leq 2$, $0 \leq k \leq 3$, $\exists$ *constants* $M_{jk} \geq 0$ *such that, in* $\mathbb{R}_+$,

(*i*) $$|a| \leq M_{00} + M_{01}\lambda^{(m-2)/2}, \quad M_{01} = 0 \text{ if } m = 2;$$

(*ii*) $$|a_\lambda| \leq M_{10} + M_{11}t\lambda^{m-2} + M_{12}\lambda^{(m-4)/2}, \quad M_{12} = 0 \text{ if } m \leq 4;$$

(*iii*) $$|a_{\lambda\lambda}| \leq M_{20} + M_{21}t\lambda^{m-3} + M_{22}t^2\lambda^{2(m-2)} + M_{23}\lambda^{(m-6)/2},$$
$$M_{21} = 0 \text{ if } m < 3, \; M_{23} = 0 \text{ if } m \leq 6.$$

*The amplitude a is said to be of **type** ($\mathcal{O}$) if* $M_{01} = M_{12} = M_{23} = 0$; *of **type** ($\mathcal{P}$) if* $M_{01} + M_{12} + M_{23} \neq 0$ *and if there are finitely many points* $0 < \lambda_1 < \cdots < \lambda_N < +\infty$ *such that neither the real part nor the imaginary part of* a_λ *changes sign in any one of the intervals* $(0,\lambda_1]$, $(\lambda_j,\lambda_{j+1}]$ $(1 \leq j < N)$, $[\lambda_N,+\infty)$.

The method used to estimate $|K_m(x,t)|$ for $t > 0$ is a special case of the stationary phase method. It differs, depending on whether x is ≥ 0, in which case the phase has no critical point, or < 0, in which case the phase has the critical point $(|x|/t)^{1/(m-1)}$. (See [PHONG–STEIN, 1992] for applications of the stationary phase method to Fourier integrals whose phase functions are homogeneous polynomials). The following estimate is valid in both half–lines $x \geq 0$ or $x \leq 0$:

THEOREM 1.– *Suppose a is an admissible amplitude, either of type ($\mathcal{O}$) or of type ($\mathcal{P}$).*

Then, for all $x \in \mathbb{R}$ *and all* $t > 0$,

$$|K_{\mathrm{m}}(x,t)| \leq C_{\mathrm{m}}[M_{00}t^{-1/\mathrm{m}} + M_{10}t^{-2/\mathrm{m}} + M_{20}t^{-3/\mathrm{m}} + M_{11} + M_{12} + M_{22}t^{1/\mathrm{m}} + (M_{01}+M_{12}+M_{23})t^{-1/2}].$$

When the amplitude a *is of type* $(\mathcal{O})$ *the constant* C_{m} *depends only on* m; *when* a *is of type* $(\mathcal{P})$ *it also depends on the number* N *of changes of sign of* $\mathcal{R}e\, a_\lambda$ *and* $\mathcal{I}m\, a_\lambda$ *in* $(0,+\infty)$.

It is important to note that the constants M_{jk} in $(\star)$ are allowed to depend on x and t. In our treatement the latter are simply real numbers $(t > 0)$ and the dependence of the constants is easily tracked. As a result the constants $c_0^{\pm}$ in (5), as well as the "remainder R, are allowed to depend on (x,t). Of course, in this case the integral (4) does not represent any more the "fundamental solution" of the initial value problem (2)–(3). One must make use of the eikonal and transport equations; but under suitable hypotheses these will define phase and amplitude functions with the desired properties – also in the variable coefficients situation.

Applying estimates such as that in Theorem 1 to linear pseudodifferential initial value problems, yields various global growth (or decay) and regularity properties of their solutions. By adapting ideas of [TOMAS, 1975] and [KATO, 1989] to derive global (in space–time) estimates of solutions of homogeneous and inhomogeneous initial value problems:

$$(7) \qquad \imath^{-1}\partial u/\partial t = P(\mathrm{D}_x)u + f \;\; in\, \mathbb{R}\times\mathbb{R}_+,\;\; u\big|_{t=0} = u_0 \;\; in\, \mathbb{R}.$$

THEOREM 2.— *Suppose* $P(\xi) = [c_0^+ H(\xi) + c_0^- H(-\xi)]|\xi|^{\mathrm{m}}$ $(2 \leq \mathrm{m} \in \mathbb{R})$. *Let* p, r *be positive numbers such that* $2 \leq \mathrm{p} \leq \infty$, $1/\mathrm{p} + \mathrm{m}/\mathrm{r} = 1/2$.

I. *Let* u *denote the solution of* (7) *when* $f \equiv 0$. *If* $u_0 \in L^{p'}$ $(1/p + 1/p' = 1)$ *then, for all* $t > 0$, $u(\cdot,t) \in L^p$ *and*

$$\|u(\cdot,t)\|_{L^p(\mathbb{R})} \leq C^{1-2/p} t^{-2/r} \|u_0\|_{L^{p'}(\mathbb{R})},$$

with a positive constant C *that solely depends on* m *and on* $c_0^{\pm}$.

II. *If* $u_0 \in L^2(\mathbb{R})$ *the solution* u *of* (7) *when* $f \equiv 0$ *belongs to* $L^r(\mathbb{R}_+;L^p(\mathbb{R}))$. *Moreover,*

$$\|u\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}))} \leq C_1 \|u_0\|_{L^2(\mathbb{R})},$$

with a positive constant C_1 *that solely depends on* m, $c_0^{\pm}$.

III. *Let* a, b *be two numbers,* $1 \leq \mathrm{a} \leq 2$, $1/\mathrm{a} + \mathrm{m}/\mathrm{b} = \mathrm{m} + \frac{1}{2}$. *The solution* v *of* (7) *when* $u_0 \equiv 0$ *belongs to* $L^r(\mathbb{R}_+;L^p(\mathbb{R}))$ *iff* $f \in L^b(\mathbb{R}_+;L^a(\mathbb{R}))$. *Moreover,*

$$\|v\|_{L^r(\mathbb{R}_+;L^p(\mathbb{R}))} \leq C_2 \|f\|_{L^b(\mathbb{R}_+;L^a(\mathbb{R}))}$$

with a positive constant C_2 *that solely depends on* m, $c_0^{\pm}$.

COROLLARY.— *Let* P *be as in Theorem 2. There is a constant* $C > 0$ (depending only on m and on $c_0^{\pm}$) *such that, for all* $u_0 \in L^2(\mathbb{R})$,

$$\left[\int_{\mathbb{R}^2} |u(x,t)|^{\mathrm{p}} \mathrm{d}x\mathrm{d}t\right]^{1/\mathrm{p}} \leq C\left[\int_{\mathbb{R}} |u_0(x)|^2 \mathrm{d}x\right]^{1/2},$$

where u is the solution (7) *when* $f \equiv 0$ *and* $\mathrm{p} = 2(\mathrm{m}+1)$.

This generalizes the result in [STRICHARTZ, 1977] in the case $P(\xi) = -\xi^2$ (then (2) is the Schrödinger equation) and the result in [KENIG–PONCE–VEGA, 1991], in the case $P(\xi) = \xi^3$, ie., (2) is the Airy equation.

When acting on the Sobolev space $\mathrm{H}^s(\mathbb{R})$ (with $s \in \mathbb{R}$ arbitrary) the operator $\mathrm{e}^{itP(\mathrm{D})}$ is unitary. Thus one cannot expect any gain of L^2 differentiability in x–space. Nevertheless, one may regard the corollary to Theorem 2 as not only a result on the global decay of the solution $u(x,t)$ but also on its increased "regularity" in comparison to u_0 – in so far that the elements of $\mathrm{L}^{2(\mathrm{m}+1)}$ can be said to be "more regular" than those of L^2 (recall that $2(\mathrm{m}+1) \geq 6$). The remarkable fact, however, is that, for $t > 0$, the solution $u(x,t)$ is truly more regular (with respect to the variable x) than the initial value u_0 – in the customary sense of the term "regular". Indeed, the following is true (cf. [KENIG–PONCE–VEGA, 1991]).

THEOREM 3.– *Let P be as in Theorem 2. Let* $u_0 \in \mathrm{L}^2(\mathbb{R})$ *and u be the solution of* (7) *with* $f \equiv 0$. *If* $0 \leq \beta \leq (\mathrm{m}-2)/4$, *then* $|\mathrm{D}|^{\beta} u(\cdot,t) \in \mathrm{L}^{\infty}(\mathbb{R})$ *for a. e.* $t > 0$, *and there is a constant* $C > 0$, *independent of* u_0, *such that*

$$\left[\int_{\mathbb{R}} \|u(\cdot,t)\|^{2\mathrm{m}}_{\mathrm{L}^\infty(\mathbb{R})} \mathrm{d}t\right]^{1/\mathrm{m}} + \left[\int_{\mathbb{R}} \||\mathrm{D}|^{(\mathrm{m}-2)/4} u(\cdot,t)\|^{4}_{\mathrm{L}^\infty(\mathbb{R})} \mathrm{d}t\right]^{1/4} \leq C\,\|u_0\|_{\mathrm{L}^2(\mathbb{R})}.$$

Thus the solution $u(x,t)$ is differentiable with respect to x, in the L^∞ sense, up to order $(m-2)/4$, for a.e. $t > 0$. Compare this with the local result that $u_0 \in L^2 \Rightarrow u(\cdot,t) \in H^{(m-1)/2}_{loc}(\mathbb{R})$ for a.e. $t > 0$. We refer the reader to [BEN–ARTZI–DEVINATZ, 1991], [BEN–ARTZI–KLAINERMAN, 1992] and [CONSTANTIN–SAUT, 1988] for a discussion of local smoothing in the L^2 sense (and in any number of dimensions).

When $P(\lambda) = c\lambda^m$ $(0 \neq c \in \mathbb{R})$, a homogeneity argument shows that the right–hand side, in the estimate in Theorem 1, can be estimated by $Ct^{-1/m}$. Inserting derivatives or, which amounts to the same, an amplitude of the kind λ^β, leads to the following statement:

THEOREM 4.– *Let P be as in Theorems 2 and 3. Assume that the initial value u_0 in* (3) *belongs to* $L^1(\mathbb{R})$. *Then, if* $0 \leq \beta \leq \frac{1}{2}(m-2)$, $|D|^\beta u(\cdot,t) \in L^\infty(\mathbb{R})$ *for every $t \neq 0$. More– over, there is a constant $C > 0$, depending only on P, such that, for all $t > 0$,*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C\, t^{-1/m} \|u_0\|_{L^1(\mathbb{R})}, \tag{8}$$

$$\|\,|D|^{(m-2)/2} u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq C\, t^{-1/2} \|u_0\|_{L^1(\mathbb{R})}. \tag{9}$$

The inequality (8) generalizes the well known decay estimates for the Schrö– dinger and Airy equations. For the "generalized" Airy equation, ie., $P(\xi) = |\xi|^{2\alpha}\xi$ $(\frac{1}{2} \leq a \in \mathbb{R})$, the decay estimate has been proved in [SIDI–SULEM–SULEM, 1986]. The estimate (9) is also proved in [KENIG–PONCE–VEGA, 1991] . It shows another aspect of the smoothing effect of the differential equation in (7).

The preceding statements are valid for a homogeneous symbol P of degree m.

Adding lower order terms R (satisfying (6)) limits the validity of the estimates to finite intervals $0 < t \leq T < +\infty$. Such estimates make it possible to handle the presence of a potential $V \in L^1(\mathbb{R})$, by a simple argument of the Picard–Ovcyannikov type. Here one seeks a solution of the initial value problem

$$(10) \qquad i^{-1}\partial u/\partial t = P(D_x)u + V(x)u \text{ in } \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+ , \ u(x,0) = u_0(x).$$

THEOREM 4.– *Let P be given by* (5) *and* (6), *and suppose* $V \in L^1(\mathbb{R})$. *If* $u_0 \in L^1(\mathbb{R})$, *the initial value problem* (10) *has a unique solution u whose trace $u(\cdot,t)$ belongs to* $L^\infty(\mathbb{R})$ *for each $t > 0$ and satisfies an estimate* $\|u(\cdot,t)\|_{L^\infty} \leq C(T)t^{-1/m}$ *if* $0 < t \leq T < +\infty$.

The solution u is a continuous function in $\mathbb{R}\times\mathbb{R}_+$; it satisfies the following estimate, for some constant $C > 0$, independent of $\|V\|_{L^1}$, $\|u_0\|_{L^1}$ *and of* $t > 0$,

$$\|u(\cdot,t)\|_{L^\infty} \leq C\|u_0\|_{L^1}t^{-1/m}(1+t^{2/m})\exp[C\|V\|^{\gamma}_{L^1}t(1+t^{2/m})^{\gamma}],$$

with $\gamma = m/(m-1)$.

The preceding results enable us to apply to the semilinear problem (1) the methods of [KATO, 1989] and of [TSUTSUMI, 1987*a*, 1987*b*]. We must make the hypotheses that the potential V is real-valued and belongs to $L^1(\mathbb{R})$, and that $P(D_x) + V(x)$, which is formally self-adjoint on $\mathcal{S}(\mathbb{R})$, generates a unitary group of operators on $L^2(\mathbb{R})$.

THEOREM 5.– *If* $1 \leq p < 2m+1$, *then to each* $u_0 \in L^2(\mathbb{R})$ *there are constants* T, $C > 0$, *depending only on the symbol P, on $\|V\|_{L^1(\mathbb{R})}$ and on $\|u_0\|_{L^2(\mathbb{R})}$, such that* (1) *has a unique weak solution $u(x,t)$ in $\mathbb{R}\times[0,T)$, having the following properties:*

(*i*) *The map $t \to u(\cdot,t) \in L^2(\mathbb{R})$ is continuous on $[0,T)$ and $u(x,0) = u_0(x)$.*

(*ii*) *If* a, b *are positive numbers such that* $2 \leq a \leq \infty$, $1/ma + 1/b = 1/2m$, *then $u \in L^b(0,T;L^a(\mathbb{R}))$ and*

$$\left[\int_0^T \|u(\cdot,t)\|^b_{L^a(\mathbb{R})} dt\right]^{1/b} \leq C\,\|u_0\|_{L^2(\mathbb{R})}.$$

It follows from (*ii*) that $u(\cdot,t) \in L^\infty(\mathbb{R})$ for a. e. $t \in (0,T)$.

In order to obtain strong solutions one needs to impose further restrictions on the potential V. On the other hand, one can then remove the upper bound on the power $p \geq 1$:

THEOREM 6.— *Let* $k \geq 1$ *be an integer and assume $V \in C^k(\mathbb{R})$, and moreover that every derivative $V^{(j)}$ of V of order $j \leq k$ belongs to* $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Then, to each $u_0 \in H^k(\mathbb{R})$ there are constants T, $C > 0$, depending only on P, on $\max\limits_{0 \leq j \leq k} (\|V^{(j)}\|_{L^1(\mathbb{R})}, \|V^{(j)}\|_{L^\infty(\mathbb{R})})$ and on $\|u_0\|_{H^k(\mathbb{R})}$, such that (1) *has a unique strong solution $u(x,t)$ in $\mathbb{R}\times[0,T)$ with the following properties:*

The map $t \to u(\cdot,t) \in H^k(\mathbb{R})$ is continuous on $[0,T)$ and $u(x,0) = u_0(x)$. Moreover,

$$\sup_{0 \leq t \leq T} \|u(\cdot,t)\|_{H^k(\mathbb{R})} \leq C\,\|u_0\|_{H^k(\mathbb{R})}.$$

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