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INVERSE PROBLEMS AND MICROLOCAL ANALYSIS

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0. Introduction.

In this lecture we shall discuss some recent problems in inverse scattering for the two-body Schrödinger operator $H_v = H_0 + v$ in $\mathbf{R}^n$ where $H_0 = -\Delta$. The main part of the presentation will be devoted to the definition of exceptional points for H_v and a study of the geometrical properties of the set $\mathcal{E}$ of such points. At the end of the lecture we explain briefly why the investigation of the set $\mathcal{E}$ is important in inverse scattering.

The exceptional points were considered already many years ago by Faddeev [1, 2]. They appear as the zeros of a Fredholm determinant of the generalized Lippmann – Schwinger equation obtained when the generalized direction dependent Green's function is introduced. In the $\overline{\partial}$ -approach to inverse scattering and the characterization problem for scattering matrices the set $\mathcal{E}$ is again important (see Henkin-Novikov [3] and Lavine-Nachman [5], Newton [7] and Weder [8]). In our approach [6] to inverse scattering, which is similar to Faddeev's, we reduce the Schrödinger operator to a direction dependent family of finite rank perturbations of the Laplacian. The set $\mathcal{E}$ is then the set of points ζ where a certain determinant of a finite-dimensional matrix $M(\zeta)$ vanishes. The parameter ζ is the generalized complex momentum variable which lies in manifold with boundary and interior equal to $\mathbf{C}^n \setminus \mathbf{R}^n$.

It was proved by Henkin-Novikov and Nachman-Lavine that $\mathcal{E}$ has real points as soon as it is non-empty. This result is also obvious from our approach, and we shall also state some other already known results about $\mathcal{E}$ and indicate how they are proved by our methods. Probably much more can be said about $\mathcal{E}$ and some open problems about that set are stated in [7]. At the end of our lecture we also sketch briefly how the set $\mathcal{E}$ may prevent the scattering matrix from being factorized into a product of upper and lower triangular matrices for some values of the energy and some directions in space.

It will be assumed that n is odd and that $n \geq 3$. Also we assume that the potential v is real-valued and that

$$\int (1+|x|)^{2-n+|\alpha|} |v^{(\alpha)}(x)|\, dx < \infty \tag{0.1}$$

for any α. However most results stated here (except those making use of intertwining operators) are independent of the parity of the dimension and extend to more general classes of short range potentials. This will be quite clear from the definition of $\mathcal{E}$, and we refer to Weder [8] for more details. We also remark here that in the proof of some of our results we have made use of some spaces of functions constructed in Hörmander [4, Chapter 14], where the limiting absorbtion principle for general short range potentials is discussed.

1. Fundamental solutions of Helmholtz' equations.

We first recall the following fact:

Lemma 1.1. *Assume that* $\operatorname{Im}\lambda > 0$. *Then there exists a unique* $g = g_\lambda \in \mathcal{S}'(\mathbf{R}^n)$ *such that* $(H_0 - \lambda^2)g(x) = \delta(x)$. *There is a polynomial* p_n *of degree* $(n-3)/2$ *with real coefficients such that*

$$g_\lambda(x) = e^{i\lambda|x|}|x|^{2-n}p_n(i\lambda|x|). \tag{1.1}$$

We observe that the limits

$$g_\lambda^{\pm}(x) = \lim_{\epsilon\to+0} g_{\pm|\lambda|+i\epsilon}(x)$$

exist in $\mathcal{S}'$ when $\lambda \in \mathbf{R}$, and the Fourier transform of $g_\lambda^{\pm}$ equals $(\xi^2 - \lambda^2 \mp i0)^{-1}$, where $\xi^2 = \langle \xi, \xi \rangle$.

Next we introduce a large class of generalized fundamental solutions for Helmholtz' operator.

Lemma 1.2. *Let* $\zeta \in \mathbf{C}^n$ *and assume that* $\operatorname{Re}\zeta$ *and* $\operatorname{Im}\zeta$ *are linearly independent. Then there exists a unique distribution* $g_\zeta \in \mathcal{D}'(\mathbf{R}^n)$ *such that*
(i) $(H_0 - \zeta^2)g_\zeta(x) = \delta(x)$,
(ii) $f_\zeta(x) \equiv e^{-i\langle x,\zeta\rangle}g_\zeta(x) \in \cap_{2<p\le\infty} L^p(\mathbf{R}^n \setminus 0)$.

PROOF: We set $g_\zeta(x) = e^{i\langle x,\zeta\rangle}f_\zeta(x)$, where f_ζ is the inverse Fourier transform of $1/p_\zeta$ and $p_\zeta(\xi) = (\xi+\zeta)^2 - \zeta^2 = \xi^2 + 2\langle\xi,\zeta\rangle$. Then (i) holds, and since $\operatorname{Re} dp_\zeta(\xi), \operatorname{Im} dp_\zeta(\xi)$ are linearly independent at the real zeros of p_ζ it follows that $1/p_\zeta \in \cap_{1\le q<2} L^q_{\mathrm{loc}}$. Hence (ii) follows since p_ζ is an elliptic polynomial. Finally, the uniqueness assertion follows from (ii), since the real zeros of p_ζ are contained in a compact subset of a hypersurface.

Next we want to extend g_ζ to a larger set of parameters ζ. The map

$$\mathbf{R}^n \times \overline{\mathbf{R}_+} \times S^{n-1} \ni (x, r, \theta) \mapsto x + ir\theta \in \mathbf{C}^n$$

allows us to consider $\mathbf{C}^n \setminus \mathbf{R}^n$ as the interior of a smooth manifold $\widetilde{\mathbf{C}^n}$ with boundary $\partial\widetilde{\mathbf{C}^n} = \mathbf{R}^n \times S^{n-1}$ so that the map

$$\mathbf{C}^n \setminus \mathbf{R}^n \ni \zeta \mapsto \theta(\zeta) = \operatorname{Im}\zeta/|\operatorname{Im}\zeta| \in S^{n-1}$$

extends to a smooth map $\widetilde{\mathbf{C}^n} \mapsto S^{n-1}$. We observe also that the inclusion map $\mathbf{C}^n \setminus \mathbf{R}^n \mapsto \mathbf{C}^n$ extends to a smooth map $\pi : \widetilde{\mathbf{C}^n} \mapsto \mathbf{C}^n$, and if $\zeta, w \in \widetilde{\mathbf{C}^n}$ we set

$$\langle \zeta, w \rangle = \zeta w = \langle \pi(\zeta), \pi(w) \rangle.$$

Let

$$M = \{\zeta \in \widetilde{\mathbf{C}^n}; \operatorname{Re}\pi(\zeta), \theta(\zeta) \text{ are linearly independent }\}.$$

If $\zeta, w \in \widetilde{\mathbf{C}^n}$ we write $\zeta \sim w$ if
<u>either</u> $\zeta, w \in M$, $\zeta^2 = w^2$, $\theta(\zeta) = \theta(w)$, $\langle \zeta, \theta(\zeta)\rangle = \langle w, \theta(w)\rangle$
<u>or</u> $\zeta, w \notin M$, $\zeta^2 = w^2$, $\langle \zeta, \theta(\zeta)\rangle = \langle w, \theta(w)\rangle$.

This gives us an equivalence relation on $\widetilde{\mathbf{C}^n}$.

Proposition 1.3. *The map* $(\mathbf{C}^n \setminus \mathbf{R}^n) \cap M \ni \zeta \mapsto g_\zeta$ *extends by continuity to a continuous mapping from* $\widetilde{\mathbf{C}^n}$ *to* $\mathcal{D}'(\mathbf{R}^n)$ *and the following conditions are fulfilled:*

(i) $(H_0 - \zeta^2) g_\zeta(x) = \delta(x)$.

(ii) The maps $\mathbf{C} \setminus \mathbf{R} \ni z \mapsto g_{\alpha + z\beta} \in \mathcal{D}'$ *are analytic when* $\alpha, \beta \in \mathbf{R}^n$.

(iii) The set $\{f_\zeta(x) = e^{-i\langle x, \zeta\rangle} g_\zeta(x); \zeta \in \widetilde{\mathbf{C}^n}\}$ *is a bounded set in* $\mathcal{S}'$.

(iv) $g_{-\zeta}(x) = g_\zeta(-x)$ *and* $\overline{g_\zeta(x)} = g_w(x)$, *where* $w = -\overline{\zeta}$.

(v) $g_\zeta = g_w \Leftrightarrow \zeta \sim w$.

In the proof of this proposition one needs some observations.

Lemma 1.4. *Let* $\theta \in S^{n-1}$. *Then there exists a unique distribution* $E_\theta \in \mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^n)$ *with the following properties:*

(i) $(\Delta_x - \Delta_y) E_\theta(x, y) = \delta(x, y)$.

(ii) $|x| = |y|$ *and* $\langle y - x, \theta\rangle \geq 0$ *in the support of* E_θ.

(iii) There are functions $f_{\theta,\alpha} \in L^\infty$ *with support in the set where* $|x| \geq |y|$ *so that*

$$E_\theta(x, y) = \sum_{|\alpha| = n-1} (\partial_x + \partial_y)^\alpha \big(|x - y|^{1-n} f_{\theta,\alpha}(x, y)\big).$$

(iv) $E_\theta(x + t\theta, y + t\theta) \to 0$ *in* $\mathcal{D}'$ *when* $|t| \to \infty$.

The distributions E_θ depend continuously on θ. It follows from the uniqueness assertions that $E_\theta(x, y)$ is real, and

$$E_{T\theta}(Tx, Ty) = E(x, y), \quad E_\theta(T'x, T''y) = E_\theta(x, y) \tag{1.2}$$

if T, T', T'' are orthogonal linear transformation on $\mathbf{R}^n$ and $T'\theta = T''\theta = \theta$.

One can prove that

$$g_\zeta(x) = -\int E_{\theta(\zeta)}(x, y) e^{i\langle y, \zeta\rangle}\, dy, \tag{1.3}$$

when $\zeta \in (\mathbf{C}^n \setminus \mathbf{R}^n) \cap M$. This formula allows us therefore to define g_ζ for any $\zeta \in \widetilde{\mathbf{C}^n}$ and all statements except the last of Proposition 1.3 follow easily. In order to prove that assertion we need to describe the behaviour at infinity of $g_\zeta(x)$ when ζ is real.

Lemma 1.5. *Assume that* $\zeta \in M$ *and that* $\operatorname{Im} \pi(\zeta) = 0$. *Then* $g_\zeta(x) = O(|x|^{-1})$ *at infinity. Moreover, if (for example)* $\langle \zeta, \theta(\zeta)\rangle \geq 0$ *and* Γ_+, Γ_0 *and* Γ_- *are closed conic sets in* $\mathbf{R}^n \setminus 0$ *such that for some* $\varepsilon > 0$

$$\begin{aligned} x \in \Gamma_+ &\Rightarrow 1 - \varepsilon > \langle \widehat{x}, \theta\rangle > \langle \widehat{\eta}, \theta\rangle + \varepsilon \\ x \in \Gamma_0 &\Rightarrow \langle \widehat{\eta}, \theta\rangle - \varepsilon > \langle \widehat{x}, \theta\rangle > \varepsilon - \langle \widehat{\eta}, \theta\rangle \\ x \in \Gamma_- &\Rightarrow -\langle \widehat{\eta}, \theta\rangle - \varepsilon > \langle \widehat{x}, \theta\rangle > -1 + \varepsilon, \end{aligned}$$

where $\eta = \pi(\eta)$, $\theta = \theta(\zeta)$ *and* $\widehat{x} = x/|x|$, *then*

$$\begin{aligned} g_\zeta(x) &= O(|x|^{-n/2}) \text{ in } \Gamma_+ \\ g_\zeta(x) - g^+_{|\eta|}(x) &= O(|x|^{-n/2}) \text{ in } \Gamma_0 \\ g_\zeta(x) - g^+_{|\eta|} - g^-_{|\eta|}(x) &= O(|x|^{-n/2}) \text{ in } \Gamma_- \end{aligned}$$

Lemma 1.6. *Assume that $\zeta \in \widetilde{\mathbf{C}^n}$ is the limit as $\varepsilon \to +0$ of $(\lambda + i\varepsilon)\theta$, where $\operatorname{Im}\lambda \geq 0$ and $\theta \in S^{n-1}$. Then $g_\zeta(x) = g_\lambda(x)$.*

Since the wave front set of the Fourier transform of f_ζ when $\zeta \in (\mathbf{C}^n \setminus \mathbf{R}^n) \cap M$ equals the set of all non-zero linear combinations of $\operatorname{Re} dp_\zeta(\xi)$ and $\operatorname{Im} dp_\zeta(\xi)$ where ξ is a real zero of $p_\zeta(\xi)$, it is clear that there is no half-plane such that $f_\zeta(x)$ is rapidly decreasing in any closed cone contained in the interior of that half-plane. Hence if $g_\zeta = g_w$ and $\operatorname{Im}\pi(\zeta) \neq 0$ then $\operatorname{Im}\pi(\zeta) = \operatorname{Im}\pi(w)$ and since $\zeta^2 = w^2$ we conclude then that $\zeta \sim w$. On the other hand if $\zeta \in M$, ζ, w are both real and $g_\zeta = g_w$, then it follows from Lemma 1.5 that $\zeta \sim w$. If $g_\zeta = g_w$ and $\zeta, w \notin M$, then it follows from Lemma 1.6 that $\zeta \sim w$. This completes the proof of Proposition 1.3 since (1.3) implies that $g_\zeta = g_w$ when $\zeta \sim w$.

2. Exceptional points.

By using some interpolation theory one can prove the following result.

Lemma 2.1. *Assume that $0 \leq \sigma \leq 1$. Then there is a Hilbert space $\mathcal{L}_\sigma \subset L^2(\mathbf{R}^n)$ with the following properties*

(i) If $\chi \in C_0^\infty(\mathbf{R}^n)$ and $\chi(0) = 1$ then $(1 + |x|)^\sigma(1 - \chi(D))u(x) \in L^2$ when $u \in \mathcal{L}_\sigma$.

(ii) Assume that A, $[A, T]$ and $[[A, \partial/\partial x_j], T]$ are continuous on L^2 when $T = \partial/\partial x_k$, $T = \sum x_k \partial/\partial x_k$ or $T = x_k\partial/\partial x_\ell - x_\ell \partial/\partial x_k$. Then A is continuous on $\mathcal{L}_\sigma$.

Since the adjoint of A satisfies the same conditions it follows that the operator A above is also continuous on the dual $\mathcal{L}_{-\sigma}$ of $\mathcal{L}_\sigma$.

We now define the spaces $\mathcal{L}_{\sigma,\zeta}$ when $|\sigma| \leq 1$ and $\zeta \in \widetilde{\mathbf{C}^n}$:

$$\begin{aligned} \zeta \in (\mathbf{C}^n \setminus \mathbf{R}^n) \cap M &\Rightarrow \mathcal{L}_{\sigma,\zeta} = \{u : e^{-i\langle x,\zeta\rangle}(1 + |\langle x, \operatorname{Im}\zeta\rangle|)^\sigma u \in L^2\} \\ \zeta \in \mathbf{C}^n \setminus \mathbf{R}^n, \zeta \notin M &\Rightarrow \mathcal{L}_{\sigma,\zeta} = L^2 \\ \zeta \in \partial\widetilde{\mathbf{C}^n} &\Rightarrow \mathcal{L}_{\sigma,\zeta} = \mathcal{L}_\sigma. \end{aligned}$$

We observe that $\mathcal{L}_{-\sigma,-\zeta}$ is the dual of $\mathcal{L}_{\sigma,\zeta}$.

Set

$$G_\zeta u(x) = \int g_\zeta(x - y)u(y)\, dy, \quad u \in C_0(\mathbf{R}^n).$$

This maps extends to a continuous map from $\mathcal{L}_{\sigma,\zeta}$ to the dual of this space when $1/2 < \sigma \leq 1$. We set

$$X_\zeta = \cup_{1/2 < \sigma \leq 1} G_\zeta \mathcal{L}_{\sigma,\zeta}.$$

We can now introduce the notion of exceptional point.

Definition 2.2. *We say that $\zeta \in \widetilde{\mathbf{C}^n}$ is an exceptional point for H_v if $(H_v - \zeta^2)u = 0$ for some $0 \neq u \in X_\zeta$.*

We let $\mathcal{E}$ denote the set of exceptional points.

Theorem 2.3. *There exists a continuous function $r(\zeta)$ on $\widetilde{\mathbf{C}^n}$ which is rapidly decreasing at infinity so that*

$$\mathcal{E} = \{\zeta; r(\zeta) = 1\}.$$

The set $\mathcal{E}$ and the function r have the following properties:

(i) $r(\alpha + z\beta)$ is analytic on $\mathbf{C} \setminus \mathbf{R}$ when $\alpha, \beta \in \mathbf{R}^n$.

(ii) $r(\zeta) \in \mathbf{R}$ when $\zeta^2 \in \mathbf{R}$ and $\zeta \notin \partial\widetilde{\mathbf{C}^n}$.

(iii) $\zeta \in \mathcal{E} \Rightarrow -\zeta, \overline{\zeta} \in \mathcal{E}$.

(iv) If $\zeta \notin M$ and $\pi(\zeta) \neq 0$ then $\zeta \in \mathcal{E}$ if and only if ζ^2 is an eigenvalue.

(v) If $\mathcal{E}$ is non-empty then $\mathcal{E}_{\mathbf{R}} = \mathcal{E} \cap \partial\widetilde{\mathbf{C}^n}$ is non-empty. In particular, $\mathcal{E}_{\mathbf{R}}$ is not empty if H_v has a negative eigenvalue.

(vi) $\mathcal{E}$ is a union of equivalence classes for $\sim$.

We shall make some remarks on the proof of this result. One can construct a continuous family of isomorphisms A_θ, $\theta \in S^{n-1}$, on L^2 such that the conditions (ii) of Lemma 2.1 are fulfilled and

$$H_v A_\theta = A_\theta(H_0 + K_\theta) \tag{2.1}$$

where

$$K_\theta = \sum_1^N f_{j,\theta} \otimes g_{j,\theta} \tag{2.2}$$

and $f_{j,\theta}^{(\alpha)}\, g_{j,\theta}^{(\alpha)} \in L^1(\mathbf{R}^n)$ for any α. Moreover, these functions are real valued, depend continuously on θ and $\langle y - x, \theta\rangle \geq 0$ in the support of the kernels of A_θ and K_θ. The operators A_θ are isomorphisms on the spaces X_ζ when $\theta = \theta(\zeta)$.

It follows from these properties that ζ is exceptional for H_v if and only if it is exceptional for $H_0 + K_\theta$ where $\theta = \theta(\zeta)$. Some simple computations show then that $\zeta \in \mathcal{E}$ if and only if $\det(I - R(\zeta)) = 0$ where $R(\zeta)$ is the $N \times N$ matrix with entries

$$R_{jk}(\zeta) = \langle G_\zeta f_{k,\theta(\zeta)}, g_{j,\theta(\zeta)}\rangle.$$

We now set $r(\zeta) = 1 - \det(I - R(\zeta)$. Then the first assertion of the theorem holds and $r(\zeta)$ is a continuous function on $\widetilde{\mathbf{C}^n}$. It can be proved that it vanishes rapidly at infinity. We observe also that

$$R_{jk}(\zeta) = \int\int E_{\theta(\zeta)}(x,y) g_{j,k,\theta(\zeta)}(x) e^{i\langle y,\zeta\rangle}\, dy \tag{2.3}$$

where $g_{j,k,\theta} = \check{f}_{k,\theta} * g_{j,\theta}$. Combining this with the fact that $\theta(-\overline{\zeta}) = \theta(\zeta)$ one finds that $\overline{r(\zeta)} = r(-\overline{\zeta})$. Hence $\zeta \in \mathcal{E} \Rightarrow -\overline{\zeta} \in \mathcal{E}$. When $\zeta \in \widetilde{\mathbf{C}^n}$ is given we can always choose $A_{\theta(-\zeta)}$ as the adjoint of the inverse of $A_{\theta(\zeta)}$. A simple calculation, which makes use of the fact that $E_{-\theta}(x,y) = E_\theta(-x,-y)$ and that the A_θ are real, shows then that $R(\overline{\zeta}) = R(\zeta)^*$ for such ζ. Hence $\mathcal{E}$ is invariant under conjugation and we have verified (iii). The conditions (i), (iv) and (vi) are easy to verify, so we finish this section by discussing (ii) and (v).

When proving (ii) we may in view of (vi) assume that $\zeta = se_{n-1} + ite_n$, where $s \in \mathbf{R}$ and $t > 0$. Then $\theta(\zeta) = \theta = e_n$ and it follows from (1.2) that $E_\theta(x,y) = E_\theta(x,y')$

if $y' = (y_1, \dots, y_{n-2}, -y_{n-1}, y_n)$. Since $e^{i\langle y,\zeta\rangle} = e^{iy_{n-1}s - ty_n}$, it follows from (2.3) that $R_{jk}(\zeta) \in \mathbf{R}$. Hence $r(\zeta) \in \mathbf{R}$.

In order to prove (v) we argue by contradiction and assume (using (vi)) that $s_0 e_{n-1} + z_0 e_n \in \mathcal{E}$ for some $s_0 \geq 0$ and some z_0 in the upper half-plane, while $\mathcal{E}_{\mathbf{R}}$ is empty. Set $q_s(z) = r(se_{n-1} + ze_n) - 1$, when $\Im z \geq 0$ and $s \geq s_0$. This is a continuous family of functions which are analytic in the upper half-plane and continuous in the closure of that set. Moreover, there is a positive constant R so that $|q_s(z)| > 1/2$ when $s + |z| > R$. Our assumptions allow us therefore to find a rectangle K in the upper half-plane, so that z_0 is in the interior of K while $q_s(z) \neq 0$ when $s \geq s_0$ and $z \in \partial K$. Set

$$N(s) = (2\pi i)^{-1} \int_K q_s(z)^{-1} \frac{dq_s(z)}{dz}\, dz.$$

Then $N(s)$ is a continuous integer valued function which vanishes for large s. Since $N(s_0) \neq 0$ we obtain a contradiction.

3. Exceptional points and factorization of the scattering matrix.

In this section we shall briefly indicate why the investigation of the exceptional points is important in inverse scattering problems. In order to avoid some technical difficulties we assume in addition to (0.1) that $v(x)$ is an integrable function

The scattering operator is represented by a family $S(\lambda) = I + T(\lambda)$, $\lambda > 0$, of unitary operators on $L^2(S^{n-1})$, where the integral kernel of $T(\lambda)$ (the scattering amplitude) is a continuous function $T(\lambda, \phi, \phi')$ on $\mathbf{R}_+ \times S^{n-1} \times S^{n-1}$ since we have assumed that v is integrable.

We shall say that an operator N on $L^2(S^{n-1})$ is a Volterra operator w.r.t. $\theta \in S^{n-1}$ if its integral kernel $N(\phi, \phi')$ is supported in the set where $\langle \phi - \phi', \theta\rangle \geq 0$ and

$$\sup_\phi \int |N(\phi,\phi')|\, d\phi' < \infty, \quad \sup_{\phi'} \int |N(\phi,\phi')|\, d\phi < \infty.$$

We let $\mathcal{V}_\theta$ be the space of such operators, and we say that the unitary operator S on $L^2(S^{n-1})$ admits a θ-factorization if there exist $N^\pm \in \mathcal{V}_{\pm\theta}$ so that

$$I + N^+ = S(I - N^-). \tag{3.1}$$

Lemma 3.1. *Let $M_\theta(\lambda)$ be the operator on $L^2(S^{n-1})$ which is given by*

$$M_\theta(\lambda)u(\phi) = \lim_{\varepsilon \to +0} r(\lambda\phi + i\varepsilon\theta)u(\phi), \quad u \in L^2(S^{n-1}).$$

There are operators $P_\theta^\pm(\lambda) \in \mathcal{V}_\theta$, which depend continuously on θ and λ, so that

$$I - M_\theta(\lambda) + P_\theta^+(\lambda) = S(\lambda)(I - M_\theta(\lambda) - P_\theta^-(\lambda)), \quad \lambda > 0,\, \theta \in S^{n-1}. \tag{3.2}$$

From this lemma follows the following result.

Theorem 3.2. *Assume that* $\lambda > 0$, $\theta \in S^{n-1}$ *and that there is no* $\zeta \in \mathcal{E}_{\mathbf{R}}$ *with* $\zeta^2 = \lambda^2$ *and* $\theta(\zeta) = \theta$. *Then* $S(\lambda)$ *admits a* θ*-factorization.*

We remark finally that if v is small then (2.1) holds with $K_\theta = 0$ and there are no exceptional points. The equation (3.2) can then be replaced by

$$I + N_\theta^+(\lambda) = S(\lambda)(I - N_\theta^-(\lambda))$$

if we set

$$N_\theta^+(\lambda) = P_\theta^+(\lambda)(I - M_\theta(\lambda))^{-1},\ N_\theta^-(\lambda) = P_\theta^-(\lambda)(I - M_\theta(\lambda))^{-1}.$$

Let

$$W_\pm = \lim_{t \to \pm\infty} e^{itH_v} e^{-itH_0}$$

be the wave operators. Then the operators $N_\theta^\pm = W_\pm^* A_\theta$, which commute with H_0, correspond to the families of operators $N_\theta^\pm(\lambda)$.

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