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NON EMBEDDABLE CR-STRUCTURES (*)

by François TREVES

A CR-structure on a smooth manifold Ω is the datum of a closed (see [5], Ch. 1, Def. 1.1) vector subbundle T' of the complex cotangent bundle $\mathbb{C}T^*\Omega$, such that

$$\mathbb{C}T^*\Omega = T' + \overline{T}' \ . \tag{1}$$

We shall call m the fiber dimension of T'. Note that, by (1), $\dim \Omega \leqslant 2m$. (If $\dim \Omega = 2m$ the structure is a complex one, a case in which we are not interested here). The structure T' is said to be locally integrable or, equivalently, the CR manifold (Ω, T') is said to be locally embeddable if every point of Ω has an open neighborhood over which T' is generated by m closed (or exact) one-forms. A function, or a distribution, f, such that df is a section of T' is said to be a CR function, or distribution. It ought perhaps to be said that CR stands for Cauchy-Riemann.

H. Lewy [3] (1956) has raised the question as to whether a strongly pseudoconvex CR structure, on a (2m-1)-dimensional manifold Ω, is always locally embeddable. Pseudoconvexity is defined by means of the Levi form (see below, (8)). That the answer is no was shown by L. Nirenberg [4] (1972) when $\dim \Omega = 3$, in which case the Levi form is a scalar (and m = 2) .(♦) Here we show that the CR-structures that have non degenerate Levi forms, with one eigenvalue of one sign and all others of the opposite sign, and which are not locally embeddable, are dense (in a sensemade precise below : see Theorem and remarks that follow).

Our view point will be strictly local. We shall hence forth suppose that Ω is an open neighborhood of the origin in an Euclidean space, specifically $\mathbb{R}^{2n+1}$. We shall limit ourselves to the case where

$$n = m - 1 \ . \tag{2}$$

Thus the fiber dimension of $T' \cap \overline{T}'$ is one. We shall begin by assuming that there are m C^∞ functions $Z^1, \dots, Z^m$ in Ω, complex valued, such that $dZ^1, \dots, dZ^m$ span T' at

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(*) The present work is a generalization of some recent joint work, [2], with H. Jacobowitz (Rutgers University).

(♦) For a positive answer to the global embeddability question, when Ω is compact and has dimension $\geqslant 5$, see Boutet de Monvel [1].

each point of Ω. After a contraction of Ω about the origin, possibly a modification of the coordinates in $\mathbb{R}^{2n+1}$, which we denote by $x^1,\ldots,x^m$, $y^1,\ldots,y^n$, and a $\mathbb{C}$-linear substitution on the Z^j's, we may assume that

(3) $$Z^j = x^j + \sqrt{-1}\, y^j, \quad j = 1,\ldots,m-1 \ (= n),$$

(4) $$Z^m = x^m + \sqrt{-1}\, \Phi(x,y),$$

with

(5) $$\Phi \ \underline{\text{real}}, \quad \Phi(0,0) = 0, \qquad d\Phi(0,0) = 0.$$

(see [5], Ch. I, p.20) .

Henceforth we write $z^j = x^j + iy^j$ $(j = 1,\ldots,n)$. But notice that the mapping

(6) $$(x,y) \longmapsto Z(x,y) = (Z^1(x,y),\ldots,Z^m(x,y))$$

defines a <u>diffeomorphism</u> on the (real) hypersurface $Z(\Omega)$ of $\mathbb{C}^m$ defined by the equation

(7) $$y^m = \Phi(x,y'), \quad y' = (y^1,\ldots,y^{m-1}).$$

This justifies that we call (6) a (local) embedding.

Next we introduce the <u>Levi form</u> of the structure, at the origin (without attempting to give here an invariant definition) :

(8) $$Q(\zeta) = \sum_{j,k=1}^{n} \frac{\partial^2\Phi}{\partial z^j \partial \bar{z}^k}(0,0)\zeta^j\bar{\zeta}^k \qquad (\zeta \in \mathbb{C}^n) .$$

Note that

(9) $$\Phi(x',0,y') = \mathrm{Re}\left(\sum_{j,k=1}^{n} b_{jk} z^j z^k\right) + Q(z) + O(|z|^3).$$

It is convenient to introduce the function

$$W = Z^m - \sqrt{-1} \sum_{j,k=1}^{n} b_{jk}\, z^j z^k ,$$

and to use the new coordinate $s = \mathrm{Re}W$ in the place of x^m. Instead of Z^m (see (4)) we shall reason with

(10) $$W = s + i\varphi(z,s),$$

noting that

(11) $$\varphi(z,s) = Q(z) + O(|z|^3 + |s||z| + |s|^2).$$

Our basic hypothesis will be :

(12) The Levi form Q is non degenerate and it has exactly $n-1$ eigenvalues of a given sign, and one of the opposite sign (i.e. it has signature $n-2$).

We shall assume that one eigenvalue of Q is > 0 and $n-1$ are < 0. After a linear substitution on the z^j's we may assume that

(13) $$Q(z) = |z^1|^2 - |z''|^2 ,$$

whre $z'' = (z^2,\dots,z^n)$. By (11) we see that, in a suitable neighborhood of the origin, $U \subset \Omega$,

(14) $$\varphi(z,s) \leqslant 2|z^1|^2 - \frac{1}{2}|z''|^2 + C|s|\,(|z| + |s|),$$

The orthogonal $T'^{\perp}$ of T', for the natural duality between vectors and covectors, is generated over Ω by the following n vector fields :

(15) $$L_j = \frac{\partial}{\partial \bar{z}^j} - i\lambda_j(z,s)\frac{\partial}{\partial s} \quad , \; j = 1,\dots,n ,$$

where the coefficients λ_j are computed by writing that $L_jW = 0$:

(16) $$\lambda_j = \left(1 + i\frac{\partial\varphi}{\partial s}\right)^{-1}\frac{\partial\varphi}{\partial \bar{z}^j} \quad , \quad j = 1,\dots,n .$$

(Incidentally the fact that T' is closed is equivalent to the property that the commutation bracket of any two smooth sections of $T'^{\perp}$ is a section of $T'^{\perp}$).

We may now state our result :

<u>Theorem</u> : <u>Suppose</u> (13) <u>holds</u>. <u>Then there is a function</u> $g \in C^{\infty}(\Omega)$, <u>vanishing to infinite order at the origin</u>, <u>such that the following is true</u> :

(17) <u>There is an open neighborhood</u> U <u>of the origin in</u> Ω <u>such that</u>, <u>for every</u> $j = 1,\ldots,n$,

$$\lambda_j^{\#} = \lambda_j(1 + g/z^1)$$

<u>is a</u> C^{∞} <u>function in</u> U ;

(18) <u>the vector fields</u> $L_j^{\#} = \frac{\partial}{\partial z^j} - i\lambda_j^{\#}\frac{\partial}{\partial s}$ <u>in</u> U $(j = 1,\ldots,n)$ <u>commute pairwise</u> ;

(19) <u>given any open neighborhood</u> $V \subset U$ <u>of the origin</u>, <u>any solution</u> $h \in C^1(V)$ <u>of the equations</u>

(20) $$L_j^{\#}h = 0, \quad j = 1,\ldots,n,$$

<u>has the property that</u> $dh|_0$ <u>is a linear comnination of</u> $dz^1,\ldots,dz^n$.

The meaning of this theorem is, roughly, the following :

Let T' be a CR structure on a manifold Ω of dimension 2n+1. Suppose that $T' \cap \overline{T'}$ is a line bundle (i.e., the structure has " codimension one"). Suppose that, in the neighborhood of a point ω_0 of Ω , the CR structure T' is embedabble, and has a non degenerate Levi form whose signature is equal to n - 2. Then there is another CR structure $T'^{\#}$ in the neighborhood of ω_0, tangent at ω_0 to T' to infinite order, which is <u>not</u> locally embeddable (at ω_0).

<u>Proof of Theorem</u> : Inspired by Nirenberg [4] we select a sequence of compact subsets K_ν $(\nu = 1,2,\ldots)$ in the upper half-plane Im $w > 0$ ($w = s + it$ will denote the variable in $\mathbb{C}^1$) having various properties :

(21) <u>as</u> $\nu \to +\infty$, K_ν <u>converges to</u> $\{0\}$;

(22) <u>the projections of the</u> K_ν <u>into the real axis are pairwise disjoint</u> ;

(23) <u>there is a number</u> $\varepsilon > 0$ <u>such that</u>

$$K_\nu \subset \Gamma^{\varepsilon} = \{s + it\ ; |s| < \varepsilon t\}.$$

We shall furthermore assume that the interior $\overset{\circ}{K}_\nu$ of K_ν is not empty, whatever ν .

We note that, if $s + i\varphi(z,s) \in \Gamma^{\varepsilon}$, we derive from (14) :

$$(\varepsilon^{-1} - C(|z| + |s|))|s| + \frac{1}{2}|z''|^2 \leqslant 2|z^1|^2 ,$$

and therefore, by choosing $\varepsilon > 0$ small enough ,

(24) $$\varepsilon^{-1}|s| + |z''|^2 \leqslant 4|z^1|^2 , \quad (s,z) \in U, W \in \Gamma^{\varepsilon}.$$

According to (11) we have

(25) $$\frac{\partial\varphi}{\partial\bar{z}^j} = \pm z^j + O(|z|^2 + |s|).$$

We note that, by (16), we have :

(26) $$\lambda_j/z^1 = [\pm z^j + O(|z|^2 + |s|)]/z^1 .$$

We select, for each ν, a function $f_\nu \in C^\infty(\mathbb{R}^2)$ having the following properties :

(27) $f_\nu \geqslant 0$ <u>everywhere</u>, $\operatorname{supp} f_\nu \subset K_\nu$, $f_\nu(w_\nu) > 0$ <u>for some</u> $w_\nu \in K_\nu$;

(28) $$f = \sum_{\nu=1}^{\infty} f_\nu \in C^\infty(\mathbb{R}^2) \; ;$$

(29) $$\lambda_j g/z^1 \in C^\infty(U),$$

where

$$g(f\circ W)/[1 + (f\circ W)\,(\log W_s)/z^1] .$$

Let us show that (29) can be achieved (in particular by taking U small enough). Recalling that $W = s + i\varphi(z,s)$, we see that $\log(1 + i\varphi_s)$ is well defined provided U is small; furthermore $\log(1 + i\varphi_s) = O(|z| + |s|)$, hence is $O(|z'|)$ on $\operatorname{supp}(f\circ W)$, by (23) and (24). Since f is flat at the origin, both $(f\circ W)(\log W_s)/z^1$ and $\lambda_j(f\circ W)/z_1$ (cf. (26)) are C^∞ in U, and flat at the origin, whence easily (29).

By differentiating $L_j W = 0$ with respect to s and dividing by W_s we get

(30) $$L_j(\log W_s) = i\lambda_{js} , \; j = 1,\dots,n .$$

A straightforward computation yields

(31) $$L_j g + i\lambda_{js} g^2/z^1 = \lambda_j\, h\ ,\ j = 1,\ldots,n,$$

where h is a certain function of (z,s). We have used the fact that $L_j(f \circ W) = L_j\overline{W}(\frac{\partial f}{\partial w} \circ W)$, and $L_j\overline{W} = L_j(W + \overline{W}) = 2L_j s = -2i\lambda_j$:

(32) $$\lambda_j = \frac{i}{2} L_j\overline{W}\ ,\ j = 1,\ldots,n\ .$$

Note that $L_k\lambda_j = L_j\,\lambda_k$ (hence $[L_j, L_k] = 0$). We have

$$[L_j^{\#}, L_k^{\#}] = [L_j - i\lambda_j \frac{g}{z^1}\frac{\partial}{\partial s}\ ,\ L_k - i\lambda_k \frac{g}{z^1}\frac{\partial}{\partial s}\ ] = -\,iq\frac{\partial}{\partial s}\ ,$$

where

$$\begin{aligned} z^1 q &= L_j(\lambda_k g) - L_k(\lambda_j g) - i\lambda_j \frac{g}{z^1}\frac{\partial}{\partial s}(\lambda_k g) \\ &\quad + i\,\lambda_k \frac{g}{z^1}\frac{\partial}{\partial s}(\lambda_j g) \\ &= \lambda_k(L_j g + i\,\frac{g^2}{z^1}\lambda_{js}) - \lambda_j(L_k g + i\,\frac{g^2}{z^1}\lambda_{ks}) \\ &= 0 \quad \text{according to (31).} \end{aligned}$$

This proves (18).

Finally suppose that $h \in C^1(V)$ is a solution of (20). In particular it is a solution of $L_1^{\#}h = 0$ on the plane $z^2 = \ldots = z^n = 0$. We shall prove below that this implies $h_s(0,0) = 0$. Because of the special form of the equations (20) (see (18)) this implies $\partial_{\bar{z}}h(0,0) = 0$, whence (19).

The proof is reduced to the case where n = 1. We content ourselves with sketching the argument,which is essentially the same as that given, with full details, in [2]. Let us write x, y, z = x + iy, rather than x^1, y^1, z^1, and $L = \frac{\partial}{\partial z} - i\lambda(z,s)\frac{\partial}{\partial s}$ rather than L_1. We have

$$\varphi(z,s) = |z|^2 + O(|z|^3 + |s||z| + |s|^2).$$

By the implicit function theorem there is a C^∞ function, in a neighborhood of zero, $s \longmapsto z(s)$, with $z(0) = 0$, such that, if we set $\varphi_o(s) = \varphi(z(s),s)$, we have

(33) $$\varphi(z,s) - \varphi_o(s) > c_o|z - z(s)|^2 \qquad (c_o > 0).$$

Furthermore $\varphi_o(0) = 0$. We may therefore assume that the intersection of the cone Γ^ε (see (23)) with a small open disk centered at the origin, in the $w = s + it$ plane, is entirely contained in the region

(34) $$t > \varphi_o(s) .$$

We may and shall assume that all the compact sets K_ν are contained in the open set (34), and we shall denote by $\mathcal{R}_o$ the complement of $\overline{\bigcup_\nu K_\nu}$ in (34), by $\mathcal{R}$ the set of points $(z,s) \in \Omega$ such that $W = s + i\varphi(z,s) \in \mathcal{R}_o$. Notice that we have, in $\mathcal{R}$:

(35) $$Lh = 0.$$

Because of (33), when $w = s + it \in \mathcal{R}_o$, the equation $\varphi(z,s) = t$ defines a smooth closed curve in the z-plane, $\gamma(w)$, winding aroun $z(s)$. We can use the parameter $\theta = \mathrm{Arg}(z - z(s))$ on $\gamma(w)$. This defines a smooth map

(36) $$S^1 \times \mathcal{R}_o \ni (\theta,w) \longmapsto (Z,s) \in \mathcal{R}$$

By virtue of (35) we have $dh = A\, dw + B\, dz$ in $\mathcal{R}$, hence

$$\frac{\partial}{\partial \bar{w}}\left(h\frac{\partial z}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(h\, \frac{\partial z}{\partial \bar{w}}\right) \qquad \text{since} \qquad \frac{\partial W}{\partial \bar{w}} = \frac{\partial W}{\partial \theta} = 0 .$$

This implies that the integral $I(w) = \int_{\gamma(w)} h\, dZ$ is a holomorphic function of w in $\mathcal{R}_o$. Since $\gamma(w)$ contracts to the point $z(s)$ when $t = \varphi_o(s)$, we have $I(w) = 0$ on this curve, therefore everywhere in $\mathcal{R}_o$ (note that there is "enough room" for the zeros to propagate around the sets K_ν , thanks to (22)). We select then a smooth closed curve c_ν in $\mathcal{R}_o$ winding around K_ν , such that no point of any set $K_{\nu'}$, $\nu' \neq \nu$, lies inside or on c_ν . We derive

(37) $$\int_{c_\nu} \int_\gamma h(z,s)dz \wedge dw = 0 .$$

The 2-chain $\Sigma_\nu = \{(z,s);(W,z) \in c_\nu \times \gamma\}$ is a kind of torus whose inside we call $\mathcal{T}_\nu$. Stokes theorem implies

(38) $$\iiint_{\mathcal{T}_\nu} dh \wedge dz \wedge dw = 0 .$$

But $dh = A\,dW + B\,dZ + Lh\,d\bar{Z}$, hence (38) reads

(39) $$\iiint_{\mathcal{C}_\nu} (\lambda/z^1)\, g\, h_s\, d\bar{z} \wedge dz \wedge dw = 0$$

since $Lh = i\lambda g\, h_s/z^1$. Near the origin, on supp g(cf.(24))

$$\lambda/z^1 \sim 1 \;,\; g \sim f \circ W \;.$$

If $h_s(0,0)$ were $\neq 0$, in (39) the argument of the integrand would have a well-defined limit as $\nu \to +\infty$: (39) could not hold true for ν large enough. □

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