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TOPOLOGIES, CONTINUITY AND BISIMULATIONS*

J.M. DAVOREN[1]

Abstract. The notion of a *bisimulation relation* is of basic importance in many areas of computation theory and logic. Of late, it has come to take a particular significance in work on the formal analysis and verification of *hybrid control systems*, where system properties are expressible by formulas of the modal μ-calculus or weaker temporal logics. Our purpose here is to give an analysis of the concept of bisimulation, starting with the observation that the zig-zag conditions are suggestive of some form of continuity. We give a topological characterization of bisimularity for preorders, and then use the topology as a route to examining the *algebraic semantics* for the μ-calculus, developed in recent work of Kwiatkowska *et al.*, and its relation to the standard set-theoretic semantics. In our setting, μ-calculus sentences evaluate as clopen sets of an *Alexandroff topology*, rather than as clopens of a (compact, Hausdorff) Stone topology, as arises in the Stone space representation of Boolean algebras (with operators). The paper concludes by applying the topological characterization to obtain the decidability of μ-calculus properties for a class of first-order definable hybrid dynamical systems, slightly extending and considerably simplifying the proof of a recent result of Lafferriere *et al.*

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INTRODUCTION

The notion of a *bisimulation relation* is of basic importance in many areas of computation theory and logic. In the propositional modal μ-calculus, if states x and y of *labeled transition system* (LTS) models $\mathfrak{M}$ and $\mathfrak{N}$ are bisimilar, then in their respective models, x and y satisfy all the same sentences of the language of $\mathbf{L}_\mu$. The corresponding notions of bisimulation-invariance for other formalisms are also well-studied: for example, finitary and infinitary polymodal or temporal logics, and fragments of first-order, infinitary, and monadic second-order logics; [10] is a comprehensive study.

This paper is motivated by the use of bisimulations in recent work on the formal analysis and verification of *hybrid control systems*; see [1,8,9,13,14] and references therein. In that work, the computational model is a structure called a *hybrid automaton*, which is an enrichment of a (*real-valued*) *timed automaton*. Temporal logic or μ-calculus specifications for such systems are interpreted with respect to LTS models $\mathfrak{M}$ over states spaces $X \subseteq Q \times \mathbb{R}^n$, where Q is a finite set of control modes, and the transition relations are of two kinds: *continuous evolution* for some duration of time according to the differential equations modeling a given control mode, and *reset relations* modeling the effects of *discrete jumps* between control modes, which may be controlled or autonomous. The propositional constants denote sets of initial states, guard conditions on the jump transitions, target or desired invariant regions of the state space, and other significant regions of the state space. The systems of interest are those in which each the components of the associated LTS model $\mathfrak{M}$ – the state space, the transition relations and the sets denoted by propositional constants – are *first-order definable* in some structure $\overline{\mathbb{R}} = (\mathbb{R}; <, +, -, \cdot, 0, 1, ...)$ over the reals (or a multi-sorted first-order structure formed from a finite Q and some $\overline{\mathbb{R}}$). For definiteness, take $\overline{\mathbb{R}}$ to be an ordered field, so by the Tarski-Seidenberg elimination of quantifiers, the first-order definable predicates coincide with the semi-algebraic sets defined by Boolean combinations of polynomial inequalities. More restrictedly, take $\overline{\mathbb{R}}$ to be the reals with only order, addition and integer constants, which defines rational polyhedra in $\mathbb{R}^n$; timed automata and so-called *linear* hybrid automata fall in this class [1,7,8]. More generally, and moving beyond decidability for first-order theories $\mathrm{Th}(\overline{\mathbb{R}})$, take $\overline{\mathbb{R}}$ to be an *order-minimal* or *o-minimal structure*; for example, $\mathbb{R}$ as an ordered field together with the exponential function, or finitely many bounded analytic functions [13,18].

To date, the main focus in formal methods for hybrid systems has been on safety/invariance properties of the form "All trajectories of $\mathcal{H}$ starting in a given set I of initial states remain in the set P at all times", or dually, "The complement of P is not $\mathcal{H}$-reachable from I". Various system-specific temporal logics have been developed for the high-level specification of properties, but for the purposes of automatic verification, specification formulas are translated into the μ-calculus since it serves as the common language of model-checking systems. In [6], we bypass translations from temporal logics and show how to directly use the modal μ-calculus to quite simply and clearly express a rich array of properties of hybrid

systems. The modal μ-calculus is particularly suitable as a logic for hybrid systems because its semantics over abstract transition system models, considered as generalized dynamical systems, are *uniform* across the continuous/discrete divide.

In the practice of automatic verification, *symbolic model checking* tools for hybrid and real-time systems such as HyTech and Kronos [1, 7, 8] are programs which take as input a representation of a hybrid system as an LTS model $\mathfrak{M}$, concretely given by explicit first-order definitions in the given language $\mathcal{L}(\overline{\mathbb{R}})$, together with a μ-calculus specification sentence φ, and attempt to compute the value of the denotation set $\|\varphi\|^{\mathfrak{M}}$ as a first-order formula in the language $\mathcal{L}(\overline{\mathbb{R}})$. For finitary modal sentences, there is the well-known and straightforward modal translation built from the first-order definitions of the components of $\mathfrak{M}$. But for infinitary fixed-point sentences, to have a guarantee that the denotation $\|\mu Z.\varphi\|^{\mathfrak{M}}$ is a *finite* union of approximations, it suffices to ensure that the LTS model $\mathfrak{M}$ has a *bisimulation equivalence* $\sim$ of *finite* index. If such is the case, the quotient transition system $\mathfrak{M}_{\sim}$ is a fully-discrete, finite truth-preserving simulacrum of the original system. The proof of the existence of a finite bisimulation quotient is the common core of the many recent results on the decidability of reachability properties – and more generally, μ-calculus expressible properties – for a variety of first-order syntactic classes of hybrid and real-time systems (see [9, 13, 14], and references therein).

In this paper, we re-examine the concept of bisimulation from the viewpoint of general topology. Our point of departure is the observation that the zig-zag conditions cry out to be analyzed as some variant on the theme of continuity. In identifying such topological content, we observe a nice symmetry in subject and object: a *preorder* (reflexive and transitive relation) $\preccurlyeq$ on the state space X is a bisimulation of an LTS model $\mathfrak{M}$, that is, *it respects* the structural components of $\mathfrak{M}$, exactly when the component transition relations and constant sets *respect it*, in the form of its *Alexandroff topology* $\mathcal{T}_{\preccurlyeq}$ on X. In Section 3, we use this topological characterization as a route to examining *algebraic semantics* for the μ-calculus, developed in recent work of Kwiatkowska *et al.* [2,4], and its relation to the standard set-theoretic semantics [12,17,19]. In Section 4, we use our characterization of bisimulations to clarify and slightly extend a recent result of Lafferriere *et al.* [13,14] on the existence of finite bisimulations for a class of hybrid systems definable in an o-minimal structure $\overline{\mathbb{R}}$, and its application to the decidability of μ-calculus sentences. The final Section 5 is a brief discussion of related and future work.

1. Preliminaries

1.1. Basic notation and prerequisites

Our notation and terminology is fairly standard; we review some of it here.

$\mathbb{R}$ and $\mathbb{N}$ denote, respectively, the sets of real and of natural numbers, and $\mathbb{R}^+ \triangleq \{ x \in \mathbb{R} \mid x \geq 0\}$ denotes the non-negative reals, which has the structure of an ordered additive semigroup.

For any set X, $\mathcal{P}(X)$ denotes the family of all subsets of X. We use the term *Boolean algebra of sets* to refer to a family $\mathcal{A} \subseteq \mathcal{P}(X)$ that is a Boolean algebra under the finitary set-theoretic operations of union, intersection and complement. The unit or top element of such an $\mathcal{A}$ is the whole space X, and the zero or bottom element is $\varnothing$. Such an $\mathcal{A}$ is *complete* as a lattice and Boolean algebra when it is closed under arbitrary unions.

The notation $f : X \to Y$ means f is a (single-valued) function with domain the set X and range contained in the set Y.

When $I \geq 1$ is an integer, we may abuse notation by writing $i \in I$, and identifying I with the index set $\{1, ..., I\}$.

We assume the reader is familiar with elementary concepts of general topology, including the order, subspace and product topologies; connectedness of sets; and continuity for (single-valued) functions. The handbook article [16] is a good source for a review.

We also assume a basic familiarity with (classical) first-order logic: first-order formulas and languages; (model-theoretic) structures for first-order languages; and satisfaction and truth. Some prior exposure to modal or temporal logics would be useful, but all necessary concepts and definitions will be developed in the text.

1.2. Relations/set-valued maps

Following Aubin and Frankowska in [3], the notation $r : X \rightsquigarrow Y$ will be used to mean $r : X \to \mathcal{P}(Y)$ is a *set-valued map*, with values $r(x) \subseteq Y$ for $x \in X$, or equivalently, $r \subseteq X \times Y$ is a *relation*, the *graph* of a set-valued map. The set of all points $x \in X$ such that $r(x) \neq \varnothing$ is called the *domain* of the relation r. The expressions:

$$x \xrightarrow{r} y, \qquad x\, r\, y, \qquad (x, y) \in r \quad \text{and} \quad y \in r(x)$$

are to be read as synonymous. The *converse* (or *inverse*) relation $\breve{r} : Y \rightsquigarrow X$ is given simply by $(x, y) \in \breve{r}$ iff $(y, x) \in r$. The *composition* of relations $r : X \rightsquigarrow Y$ and $s : Y \rightsquigarrow Z$ will be written $r \circ s : X \rightsquigarrow Z$ (or simply rs) in sequential (word) order, as is usual in automata theory (the reverse order of functional composition; *cf.* [3]). The *relational sum* $r \cup s : X \rightsquigarrow Y$ of relations $r : X \rightsquigarrow Y$ and $s : X \rightsquigarrow Y$ is just the set-theoretic union.

A relation $r : X \rightsquigarrow Y$ determines two *pre-image operators* (predicate transformers): the *lower* or *existential* pre-image $\sigma(r) : \mathcal{P}(Y) \to \mathcal{P}(X)$ given by:

$$\sigma(r)(B) \triangleq \{x \in X \mid (\exists y \in Y)[\, x \xrightarrow{r} y \ \wedge \ y \in B]\} = \{x \in X \mid r(x) \cap B \neq \emptyset\}$$

for $B \subseteq Y$, while the *upper* or *universal* pre-image operator $\tau(r) : \mathcal{P}(Y) \to \mathcal{P}(X)$ is the dual under set-theoretic complement:

$$\tau(r)(B) \stackrel{\circ}{=} X - \sigma(r)(Y - B) = \{x \in X \mid r(x) \subseteq B\}.$$

In words, $x \in \sigma(r)(B)$ iff *some* r-successor of x lies in B, while $x \in \tau(r)(B)$ iff *all* r-successors of x lie in B. The pre-image operators give the standard relational Kripke semantics for labeled modal operators $\langle a \rangle$ and $[a]$ for relations $a : X \rightsquigarrow X$. In analogy with the *inverse-image* operator of a single-valued function, the pre-image operators are also used to develop purely topological notions of *continuity* for relations/set-valued maps; we return to this in Section 2.2.

In [3], the $\exists$-pre-image is known simply as the *inverse-image*, written $r^{-1}(B)$, and the $\forall$-pre-image is called the *core* operator, written $r^{+1}(B)$. In [2], following earlier work of Sambin and Vaccaro, the $\forall$-pre-image is written r^*, and abstract algebraic operators of that type are written τ_a; our notation is an adaption of the latter. The existential operator $\sigma(r)$ distributes over arbitrary unions and sends the empty set to itself; in the framework of Jónsson and Tarski's foundational work on Boolean algebras with operators [11], $\sigma(r)$ is known as *completely additive* and *normal* with respect to the zero elements of $P(Y)$ and $P(X)$, $\tau(r)$ is *completely multiplicative* (over intersections) and normal with respect to the unit elements of Boolean algebras. What is known as a *normal* diamond operator in the modal logic tradition corresponds to a finitely additive and zero-normal operator in [11].

The *direct-image* or *post-image* operator mapping a set $A \subseteq X$ to its image in Y under r is just $\sigma(\breve{r}) : \mathcal{P}(X) \to \mathcal{P}(Y)$; that is, $r(A) = \sigma(\breve{r})(A)$, so $r(A)$ is the set of all points $y \in Y$ which are *r-reachable* from A, or have some r-predecessor in A. In [11], $\sigma(r)$ and $\sigma(\breve{r})$ are known as *conjugate* operators on $\mathcal{P}(X)$: $A \cap \sigma(r)(B) = \varnothing$ iff $\sigma(\breve{r})(A) \cap B = \varnothing$.

1.3. Transition system models and the modal μ-calculus

Call a pair (Φ, Σ) consisting of a set Φ of propositional constants and a set Σ of transition (action) labels a *modal signature*, and let PVar be a fixed set of propositional variables. The set of formulas $\mathcal{F}_\mu(\Phi, \Sigma)$ in a signature (Φ, Σ) of the propositional modal μ-calculus is generated by the grammar:

$$\varphi ::= p \mid Z \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \langle a \rangle \varphi \mid \mu Z.\varphi$$

for $p \in \Phi$, $Z \in \mathrm{PVar}$, and $a \in \Sigma$, with the proviso that in $\mu Z.\varphi$, the variable Z occur *positively*, *i.e.* each occurrence of Z in φ is within the scope of an even number of negations. Let $\mathcal{S}_\mu(\Phi, \Sigma)$ denote the set of all *sentences* of $\mathcal{F}_\mu(\Phi, \Sigma)$; *i.e.* formulas without any free variables. Also let $\mathcal{F}(\Phi, \Sigma)$ and $\mathcal{S}(\Phi, \Sigma)$ denote, respectively, the set of all (finitary) *modal* formulas and sentences in the signature (Φ, Σ); *i.e.* without any fixed-point quantifiers. Introduce in the usual way the defined logical constants **tt** (true) and **ff** (false), other propositional connectives (we use $\to$ for implication and $\equiv$ for equivalence), and dual modalities $[a]$ and greatest fixed-point quantifier: $[a]\varphi \stackrel{\circ}{=} \neg\langle a \rangle \neg\varphi$ and $\nu Z.\varphi \stackrel{\circ}{=} \neg\mu Z.\neg\varphi[Z := \neg Z]$.

For formulas $\varphi, \psi \in \mathcal{F}_\mu(\Phi, \Sigma)$, let $\varphi[Z := \psi]$ denote the result substituting ψ for all free occurrences of Z. By renaming bound variables in φ if necessary, we can assume such substitutions do not result in the unintended capture of free variables.

Definition 1.1. A *labeled transition system* (*LTS*), or *generalized Kripke model*, of signature (Φ, Σ) is a structure:

$$\mathfrak{M} = \left(X, \{a^{\mathfrak{M}}\}_{a \in \Sigma}, \left\{ \|p\|^{\mathfrak{M}} \right\}_{p \in \Phi} \right)$$

where $X \neq \emptyset$ is the state space (set of worlds, configurations) of arbitrary cardinality; for each transition label $a \in \Sigma$, $a^{\mathfrak{M}} : X \rightsquigarrow X$ is a relation on X; and for each atomic proposition (observation or event label) $p \in \Phi$, $\|p\|^{\mathfrak{M}} \subseteq X$ is a fixed subset of X.

For a given LTS model $\mathfrak{M}$, we write $\mathcal{F}_\mu(\mathfrak{M})$ ($\mathcal{S}_\mu(\mathfrak{M})$) and $\mathcal{F}(\mathfrak{M})$ ($\mathcal{S}(\mathfrak{M})$) to mean, respectively, the set of all μ-calculus formulas (sentences) and the set of all finitary modal formulas (sentences) in the modal signature of $\mathfrak{M}$.

In the standard set-theoretic semantics for the μ-calculus [12, 17, 19] over LTS models $\mathfrak{M}$, propositional variables range over the full power-set algebra $\mathcal{P}(X)$ of the state space. In the more general *algebraic semantics* of Kwiatkowska *et al.* in [2,4], formulas are interpreted with respect to *modal frames* $(\mathfrak{M}, \mathcal{A})$, where $\mathcal{A} \subseteq \mathcal{P}(X)$ is a *modal algebra* for $\mathfrak{M}$: a Boolean algebra of sets which contains each of the constant sets $\|p\|^{\mathfrak{M}}$ and is closed under each of the pre-image operators $\sigma(a^{\mathfrak{M}})$. We give the standard set-theoretic semantics here, and return to the algebraic semantics, and the relationship between the two, in Section 3.

Definition 1.2. Given an LTS model $\mathfrak{M} = (X, \{a^{\mathfrak{M}}\}_{a \in \Sigma}, \{\|p\|^{\mathfrak{M}}\}_{p \in \Phi})$ of modal signature (Φ, Σ), a (propositional, or second-order) *variable assignment* in $\mathfrak{M}$ is any map $\xi : \mathrm{PVar} \to \mathcal{P}(X)$. Each such assignment ξ uniquely extends to a *denotation* map $\|\cdot\|^{\mathfrak{M}}_\xi : \mathcal{F}_\mu(\Phi, \Sigma) \to \mathcal{P}(X)$ inductively defined as follows:

$$\begin{aligned}
\|p\|^{\mathfrak{M}}_\xi &\triangleq \|p\|^{\mathfrak{M}} && \text{for } p \in \Phi \\
\|Z\|^{\mathfrak{M}}_\xi &\triangleq \xi(Z) && \text{for } Z \in \mathrm{PVar} \\
\|\neg\varphi\|^{\mathfrak{M}}_\xi &\triangleq X - \|\varphi\|^{\mathfrak{M}}_\xi \\
\|\varphi_1 \vee \varphi_2\|^{\mathfrak{M}}_\xi &\triangleq \|\varphi_1\|^{\mathfrak{M}}_\xi \cup \|\varphi_2\|^{\mathfrak{M}}_\xi \\
\|\langle a \rangle \varphi\|^{\mathfrak{M}}_\xi &\triangleq \sigma(a^{\mathfrak{M}}) \left(\|\varphi\|^{\mathfrak{M}}_\xi \right) && \text{for } a \in \Sigma \\
\|\mu Z.\varphi\|^{\mathfrak{M}}_\xi &\triangleq \bigcap \left\{ A \in \mathcal{P}(X) \mid \|\varphi\|^{\mathfrak{M}}_{\xi(A/Z)} \subseteq A \right\}
\end{aligned}$$

where for $A \in \mathcal{P}(X)$, the variant assignment $\xi(A/Z) : \mathrm{PVar} \to \mathcal{P}(X)$ is given by: $\xi(A/Z)(W) = \xi(W)$ if $W \neq Z$, and $\xi(A/Z)(W) = A$ if $W = Z$.

For formulas $\varphi \in \mathcal{F}_\mu(\Phi, \Sigma)$ and assignments $\xi : \mathrm{PVar} \to \mathcal{P}(X)$ in $\mathfrak{M}$, we say:

- φ is *satisfied* at state x in $(\mathfrak{M}, \xi)$, written $\mathfrak{M}, \xi, x \vDash \varphi$, iff $x \in \|\varphi\|^{\mathfrak{M}}_{\xi}$;
- φ is *true* in $(\mathfrak{M}, \xi)$, written $\mathfrak{M}, \xi \vDash \varphi$, iff $\|\varphi\|^{\mathfrak{M}}_{\xi} = X$; *i.e.* φ is satisfied at all states x in $(\mathfrak{M}, \xi)$; and
- φ is *true* in $\mathfrak{M}$, written $\mathfrak{M} \vDash \varphi$, iff φ is true in $(\mathfrak{M}, \xi)$ for *all* assignments ξ in $\mathfrak{M}$.

Note that for $\varphi, \psi \in \mathcal{F}_{\mu}(\Phi, \Sigma)$, we have: $\mathfrak{M}, \xi \vDash \varphi \to \psi$ iff $\|\varphi\|^{\mathfrak{M}}_{\xi} \subseteq \|\psi\|^{\mathfrak{M}}_{\xi}$, and for equivalences, $\mathfrak{M}, \xi \vDash \varphi \equiv \psi$ iff $\|\varphi\|^{\mathfrak{M}}_{\xi} = \|\psi\|^{\mathfrak{M}}_{\xi}$. In temporal and modal logics, satisfaction relations $x \in \|\varphi\|^{\mathfrak{M}}_{\xi}$ are usually written $x \vDash^{\mathfrak{M}}_{\xi} \varphi$, or in the forcing notation, $x \Vdash^{\mathfrak{M}}_{\xi} \varphi$.

For sentences $\varphi \in \mathcal{S}_{\mu}(\Phi, \Sigma)$, the denotation $\|\varphi\|^{\mathfrak{M}}_{\xi}$ is independent of variable assignments ξ, so written $\|\varphi\|^{\mathfrak{M}}$. Thus $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}, \xi \vDash \varphi$ for *any* assignment ξ.

The syntactic restriction on formulas $\mu Z.\varphi$ serves to ensure that the operator $\varphi^{\mathfrak{M}}_{\xi,Z} : \mathcal{P}(X) \to \mathcal{P}(X)$ given by $\left(\varphi^{\mathfrak{M}}_{\xi,Z}\right)(A) \triangleq \|\varphi\|^{\mathfrak{M}}_{\xi(A/Z)}$ is $\subseteq$-monotone. In the definition above, $\|\mu Z.\varphi\|^{\mathfrak{M}}_{\xi}$ is defined to be the least *pre*-fixed-point of $\varphi^{\mathfrak{M}}_{\xi,Z}$. By the Tarski-Knaster fixed-point theorem for monotone maps on complete lattices, least *pre*-fixed-points are the same as least fixed-points; thus the inclusion can be replaced with equality. The completeness of $\mathcal{P}(X)$ as a lattice ensures (by the Hitchcock-Park fixed-point theorem) that the set $\|\mu Z.\varphi\|^{\mathfrak{M}}_{\xi}$ may also be characterized as a transfinite union of an $\subseteq$-chain of approximation sets $\|\mu Z.\varphi\|^{\mathfrak{M}}_{\xi,\alpha}$ for ordinals α (of cardinality less than or equal to that of X), beginning with the empty set, applying the $\varphi^{\mathfrak{M}}_{\xi,Z}$ operator at successor ordinals and taking unions at limits. The finite approximation sets are denotations of formulas: for $n < \omega$, $\|\mu Z.\varphi\|^{\mathfrak{M}}_{\xi,n} = \|\varphi^n\|^{\mathfrak{M}}_{\xi}$, where $\varphi^0 \triangleq \mathbf{ff}$ and $\varphi^{n+1} \triangleq \varphi[Z := \varphi^n]$. When the semantic operator $\varphi^{\mathfrak{M}}_{\xi,Z}$ distrubutes over unions of countable $\subseteq$-chains of sets (or more generally, distributes over unions of $\subseteq$-directed families of sets, *i.e.* $\cup$-continuous w.r.t. the Scott topology on $\mathcal{P}(X)$), the ordinal of convergence for $\|\mu Z.\varphi\|^{\mathfrak{M}}_{\xi}$ is at most ω.

1.4. Hybrid systems and their transition systems

A basic hybrid system is essentially a finite collection of dynamical systems together with reset relations between them. The definitions given here are the standard ones from the literature (see, for example [1, 8, 9, 13]), recast from the viewpoint of general topology.

Definition 1.3. A (basic, evolution time-deterministic) *hybrid system* is a structure

$$\mathcal{H} = \left(Q, G, \{X_q,\ \phi_q,\ Init_q,\ Inv_q\}_{q \in Q}, \{r_{q,q'},\ Grd_{q,q'}\}_{(q,q') \in G}\right)$$

where

- Q is a finite set of *discrete states* or *control modes*;

- $G \subseteq Q \times Q$ is the *control graph* of *discrete transitions*;
- for each $q \in Q$,
  - $X_q \subseteq \mathbb{R}^n$ is the state space for mode q;
  - $\phi_q : X_q \times \mathbb{R}^+ \to X_q$ is a continuous semi-flow on X_q (*e.g.* from a system of Lipschitz differential equations or vector field on X_q);
  - $Inv_q \subseteq X_q$ is the set of *invariant* states for mode q, or the *domain of permitted evolution* within mode q;
  - $Init_q \subseteq Inv_q$ is the set of *initial* states for mode q (possibly empty);
- for each discrete transition $(q, q') \in G$,
  - $Grd_{q,q'} \subseteq X_q$ is the *guard set* for the jump from q to q';
  - $r_{q,q'} : X_q \rightsquigarrow X_{q'}$ is a *reset* relation, modeling the effect on the real-valued coordinates of jumping from q to q'.

The *hybrid state space* of the system $\mathcal{H}$ is the set $X_{\mathcal{H}} \triangleq \cup_{q \in Q} \{q\} \times X_q$.

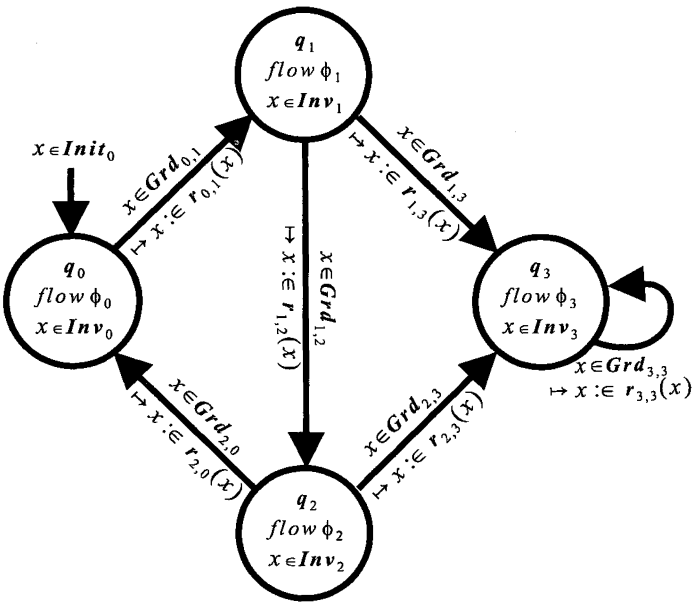

FIGURE 1. Basic hybrid automaton.

For simplicity, assume a fixed number n of real-valued coordinates; *i.e.* $X_q \subseteq \mathbb{R}^n$ for each $q \in Q$. The spaces X_q are taken as equipped with the standard topology as a subspace of $\mathbb{R}^n$, inherited from the order topology on $\mathbb{R}$. By definition, the semi-flows $\phi_q : X_q \times \mathbb{R}^+ \to X_q$ are continuous functions satisfying $\phi_q(x, 0) = x$ and $\phi_q(x, t + s) = \phi_q(\phi_q(x, t), s)$ for all $x \in X_q$ and $t, s \in \mathbb{R}^+$.

Definition 1.4. [1,8,13] Given a hybrid system $\mathcal{H}$, an LTS model $\mathfrak{M}_{\mathcal{H}}$ determined by $\mathcal{H}$ has the following components:

- the state space $X = X_{\mathcal{H}}$;

- for each discrete state $q \in Q$, the (time-abstract) *constrained evolution* relation $e_q : X_q \rightsquigarrow X_q$ defined by:

$$x \xrightarrow{e_q} x' \quad \triangleq \quad (\exists t \in \mathbb{R}^+)[\ x' = \phi_q(x,t)\ \wedge\ (\forall s \in [0,t])\ \phi_q(x,s) \in Inv_q\ ]$$

- for each discrete transition $(q,q') \in G$, the *controlled jump* relation $c_{q,q'} : X_q \rightsquigarrow X_{q'}$ defined by:

$$x \xrightarrow{c_{q,q'}} x' \quad \triangleq \quad x \in Grd_{q,q'}\ \wedge\ x \xrightarrow{r_{q,q'}} x'$$

- for each $q \in Q$, a finite collection of constant sets $A_q \subseteq X_q$, including X_q, $Init_q$, Inv_q, and $Grd_{q,q'}$ for $(q,q') \in G$.

We adopt the notational convention of identifying, when convenient, sets $A_q \subseteq X_q$ and $\{q\} \times A_q \subseteq X$; the relations $e_q : X_q \rightsquigarrow X_q$ and $c_{q,q'} : X_q \rightsquigarrow X_{q'}$ can be "lifted" to relations $X \rightsquigarrow X$ in the obvious (and unique) way. It is immediate that the domain of e_q is Inv_q, and $c_{q,q'}$ is $r_{q,q'}$ restricted to the domain $Grd_{q,q'}$. The transition alphabet for $\mathfrak{M}_\mathcal{H}$ is $\Sigma_\mathcal{H} = \{\mathbf{e}_q\}_{q\in Q} \cup \{\mathbf{c}_{q,q'}\}_{(q,q')\in G}$, and the alphabet $\Phi_\mathcal{H}$ of propositional constants include names for each of the sets A_q.

We return to properties of the constrained evolution relations, and their near-relatives, the orbit relation of a semi-flow, in Section 4.1 below. For now, observe that each of the relations e_q are reflexive on their domains Inv_q, and are also transitive.

Definition 1.5. A *trajectory* of a hybrid system $\mathcal{H}$ is a finite or infinite sequence $\chi = \langle \Delta_i, q_i, \gamma_i \rangle_{i\in I}$ such that for each $i \in I$:

- the duration $\Delta_i \in \mathbb{R}^+ \cup \{\infty\}$, with $\Delta_i = \infty$ only if I is finite and $\max(I) = i$;
- the curve $\gamma_i : [0, \Delta_i] \to X_{q_i}$ is such that $(q_i, \gamma_i(0)) \xrightarrow{e_{q_i}} (q_i, \gamma_i(t))$ for all $t \in [0, \Delta_i]$; and
- $(q_i, \gamma_i(\Delta_i)) \xrightarrow{c_{q_i,q_{i+1}}} (q_{i+1}, \gamma_{i+1}(0))$.

Over $X = X_\mathcal{H}$, the *global $\mathcal{H}$-reachability relation* $h : X \rightsquigarrow X$ is defined by:

$$(q,x) \xrightarrow{h} (q',x') \quad \triangleq \quad (\exists\ \mathcal{H}\text{-trajectory } \chi = \langle \Delta_i, q_i, \gamma_i \rangle_{i\in I} \text{ with } I = \{0,1,...,n\}) : \\ q = q_0,\ x = \gamma_0(0),\ q' = q_n \text{ and } x' = \gamma_n(\Delta_n).$$

Now let $\mathbf{e}$ and $\mathbf{c}$ denote, respectively, the relational sum of the relations e_q for $q \in Q$, and of the relations $c_{q,q'}$ for $(q,q') \in G$. Then the $\mathcal{H}$-reachability relation satisfies the regular expression: $\mathbf{h} = (\mathbf{ec})^*\mathbf{e} = \mathbf{e}(\mathbf{ce})^*$, captured by the dual fixed-point definable modalities:

$$\langle \mathbf{h} \rangle \varphi \ \triangleq\ \mu Z.\ \langle \mathbf{e} \rangle \varphi \vee \langle \mathbf{e} \rangle \langle \mathbf{c} \rangle Z \quad \text{and} \quad [\mathbf{h}]\varphi \ \triangleq\ \nu Z.\ [\mathbf{e}]\varphi \wedge [\mathbf{e}][\mathbf{c}]Z.$$

Proposition 1.6. *Given a hybrid system $\mathcal{H}$, LTS model $\mathfrak{M} = \mathfrak{M}_\mathcal{H}$, μ-calculus formula $\varphi \in \mathcal{F}_\mu(\Phi_\mathcal{H}, \Sigma_\mathcal{H})$, and variable assignment ξ in $\mathfrak{M}$, we have:*

$$\|\langle \mathbf{h} \rangle \varphi\|^\mathfrak{M}_\xi = \sigma(h)\left(\|\varphi\|^\mathfrak{M}_\xi\right) \quad \textit{and} \quad \|[\mathbf{h}]\varphi\|^\mathfrak{M}_\xi = \tau(h)\left(\|\varphi\|^\mathfrak{M}_\xi\right).$$

In words, $[\mathbf{h}]\,\varphi$ denotes the largest subset of $[\mathbf{e}]\,\varphi$ that is invariant under both evolution and controlled jump relations. The *safety* sentence

$$\mathbf{Init} \rightarrow [\mathbf{h}]\,\varphi$$

is *true* in the model $\mathfrak{M} = \mathfrak{M}_{\mathcal{H}}$ exactly when every $\mathcal{H}$-trajectory that starts in *Init* *always* remains within $\|\varphi\|^{\mathfrak{M}}$. As an example of a liveness property, the sentence

$$\varphi \rightarrow [\mathbf{h}]\langle\mathbf{e}\rangle\langle\mathbf{c}\rangle\,\mathbf{tt}$$

is true in $\mathfrak{M}$ exactly when every maximal $\mathcal{H}$-trajectory from a state in $\|\varphi\|^{\mathfrak{M}}$ makes *infinitely many* discrete jumps. This is because $\langle\mathbf{c}\rangle\,\mathbf{tt}$ denotes the domain of c, which is the union of the guard sets $Grd_{q,q'}$.

1.5. Bisimulation relations

Definition 1.7. Given two LTS models $\mathfrak{M}$ and $\mathfrak{N}$ of common modal signature (Φ, Σ), with state spaces X and Y respectively, a relation $\preccurlyeq\colon X \rightsquigarrow Y$ is called a *bisimulation* or *zig-zag* between $\mathfrak{M}$ and $\mathfrak{N}$ iff for $x, x' \in X$, $y, y' \in Y$ and each $a \in \Sigma$ and $p \in \Phi$,

$$\textbf{\textit{Zig}}_a\textbf{:}\quad x \preccurlyeq y \text{ and } x \xrightarrow{a^{\mathfrak{M}}} x' \;\Rightarrow\; (\exists y')[\; y \xrightarrow{a^{\mathfrak{N}}} y' \text{ and } x' \preccurlyeq y'\,]$$

$$\textbf{\textit{Zag}}_a\textbf{:}\quad x \preccurlyeq y \text{ and } y \xrightarrow{a^{\mathfrak{N}}} y' \;\Rightarrow\; (\exists x')[\; x \xrightarrow{a^{\mathfrak{M}}} x' \text{ and } x' \preccurlyeq y'\,]$$

$$\textbf{\textit{Up}}_p\textbf{:}\quad x \preccurlyeq y \text{ and } x \in \|p\|^{\mathfrak{M}} \;\Rightarrow\; y \in \|p\|^{\mathfrak{N}}$$

$$\textbf{\textit{Down}}_p\textbf{:}\quad x \preccurlyeq y \text{ and } y \in \|p\|^{\mathfrak{N}} \;\Rightarrow\; x \in \|p\|^{\mathfrak{M}}.$$

By symmetry, the converse $\succcurlyeq\colon Y \rightsquigarrow X$ will also be a bisimulation between $\mathfrak{N}$ and $\mathfrak{M}$. The relational composition of two bisimulations is also a bisimulation.

The fundamental bisimulation-invariance property for sentences of the μ-calculus is the following.

Proposition 1.8. ([17] Sect. 5.3). *If $\preccurlyeq$ is a bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$, then for all $x \in X$ and $y \in Y$, and all* sentences $\varphi \in \mathcal{S}_\mu(\Phi, \Sigma)$,

$$x \preccurlyeq y \quad\Rightarrow\quad \left[\; x \in \|\varphi\|^{\mathfrak{M}} \Leftrightarrow y \in \|\varphi\|^{\mathfrak{N}} \;\right].$$

Proof. The conditions $\mathbf{Up}_p$ and $\mathbf{Down}_p$ give the base case of the induction, for atomic $p \in \Phi$, and the $\mathbf{Zig}_a$ and $\mathbf{Zag}_a$ conditions give the induction step for the $\langle a\rangle$ modalities. For μ-sentences $\mu Z.\varphi$, one proves $x \in \|\varphi\|^{\mathfrak{M}}_\alpha$ iff $y \in \|\varphi\|^{\mathfrak{N}}_\alpha$ by transfinite induction on ordinals α. □

For states $x \in X$, define

$$\mathcal{S}^{\mathfrak{M}}_\mu(x) \;\triangleq\; \left\{\; \varphi \in \mathcal{S}_\mu(\Phi, \Sigma) \;\mid\; x \in \|\varphi\|^{\mathfrak{M}} \;\right\}$$

to be the set of all μ-sentences satisfied by x in $\mathfrak{M}$. The relation $\approx_{\mathcal{S}_\mu}^{\mathfrak{M},\mathfrak{N}} : X \rightsquigarrow Y$ given by:

$$x \approx_{\mathcal{S}_\mu}^{\mathfrak{M},\mathfrak{N}} y \quad \stackrel{\circ}{\Leftrightarrow} \quad \mathcal{S}_\mu^{\mathfrak{M}}(x) = \mathcal{S}_\mu^{\mathfrak{N}}(y)$$

is that of *logical equivalence* or indistinguishability under μ-calculus sentences; replacing $\mathcal{S}_\mu$ with $\mathcal{S}$ gives logical equivalence under modal sentences. Then the bisimulation invariance property in Proposition 1.8 is the implication:

$$x \preccurlyeq y \quad \Rightarrow \quad x \approx_{\mathcal{S}_\mu}^{\mathfrak{M},\mathfrak{N}} y.$$

When $\mathfrak{M} = \mathfrak{N}$ and $\preccurlyeq = \sim$ is also an equivalence relation on X, $\sim$ is called a *bisimulation equivalence* on $\mathfrak{M}$. In this case, the (single-valued) quotient map $q : \mathfrak{M} \to \mathfrak{M}_\sim$ is a bisimulation between $\mathfrak{M}$ and the *quotient* LTS model $\mathfrak{M}_\sim$; the quotient is defined as:

$$\mathfrak{M}_\sim \stackrel{\circ}{=} \left(X_\sim, \left\{ a^{\mathfrak{M}_\sim} \right\}_{a \in \Sigma}, \left\{ \|p\|^{\mathfrak{M}_\sim} \right\}_{p \in \Phi} \right)$$

where for equivalence classes $C, C' \in X_\sim$, we have $C \xrightarrow{a^{\mathfrak{M}_\sim}} C'$ iff $x \xrightarrow{a^{\mathfrak{M}}} x'$ for some $x \in C$ and $x' \in C'$, iff $C \cap \sigma(\sim)(C') \neq \varnothing$; and for the propositional constants, $\|p\|^{\mathfrak{M}_\sim} \stackrel{\circ}{=} \{C \in X_\sim \mid C \cap \|p\|^{\mathfrak{M}} \neq \varnothing\} = \sigma(\sim)(\|p\|^{\mathfrak{M}})$. The bisimulation conditions ensure that the quotient $\mathfrak{M}_\sim$ is well-defined. When $\sim$ is a bisimulation equivalence on $\mathfrak{M}$, it follows by Proposition 1.8 that for each sentence $\varphi \in \mathcal{S}_\mu(\Phi, \Sigma)$, the denotation set $\|\varphi\|^{\mathfrak{M}}$ is a union of $\sim$ equivalence classes. In particular, if $\sim$ is a bisimulation equivalence of *finite* index I, then for each fixed-point sentence $\mu Z.\varphi \in \mathcal{S}_\mu(\Phi, \Sigma)$, the denotation $\|\mu Z.\varphi\|^{\mathfrak{M}}$ is a finite union of approximations $\|\varphi^n\|^{\mathfrak{M}}$ over $0 \leq n \leq I$, where $\varphi^0 \stackrel{\circ}{=} \mathbf{ff}$ and $\varphi^{n+1} \stackrel{\circ}{=} \varphi[Z := \varphi^n]$ for $n < \omega$. It follows that for each μ-calculus sentence $\mu Z.\varphi \in \mathcal{S}_\mu(\Phi, \Sigma)$, there is a finitary modal sentence $\psi \in \mathcal{S}(\Phi, \Sigma)$ such that $\mathfrak{M} \vDash \mu Z.\varphi \equiv \psi$.

2. Bisimulations and continuity

2.1. Bisimulation preorders

In our analysis of bisimulation relations, we narrow the focus and consider relations $\preccurlyeq : X \rightsquigarrow X$ on single LTS model $\mathfrak{M}$. A bisimulation $\preccurlyeq$ is a *structuring* on the state space X in a manner which *preserves* the component transition relations $a^{\mathfrak{M}} : X \rightsquigarrow X$ and constant sets $\|p\|^{\mathfrak{M}}$. In order to manifest this notion of preservation as a continuity property, we seek to recast the relational zig-zag clauses as conditions on the preservation of families of sets of states.

Definition 2.1. Given a relation $r : X \rightsquigarrow X$, we call a set $A \subseteq X$:

- *up-r-closed* iff $\sigma(\breve{r})(A) \subseteq A$ iff $A \subseteq \tau(r)(A)$;
- *down-r-closed* iff $\sigma(r)(A) \subseteq A$ iff $A \subseteq \tau(\breve{r})(A)$.

Let $Up(r), Dn(r) \subseteq \mathcal{P}(X)$ denote, respectively, the families of all up-r-closed and down-r-closed subsets of X.

In temporal logic or in the topological dynamics of set-valued functions, up-r-closed sets $A \subseteq X$ are also called *positive-* or *future-invariant* under r. When $r = \preccurlyeq$ is a preorder or partial order, it is usually written $\uparrow A = A$. For $r = a^{\mathfrak{M}}$ a transition relation of $\mathfrak{M}$, a set $\|\varphi\|^{\mathfrak{M}}$ is respectively, up-$a^{\mathfrak{M}}$-invariant or down-$a^{\mathfrak{M}}$-invariant, exactly when $\mathfrak{M} \models \varphi \to [a]\varphi$ or $\mathfrak{M} \models \langle a\rangle\varphi \to \varphi$. For arbitrary relations $r : X \rightsquigarrow X$, each of the families $Up(r)$ and $Dn(r)$ are closed under *both* arbitrary unions and arbitrary intersections, since the pre-image operators $\sigma(r)$ and $\tau(r)$ are completely additive and completely multiplicative respectively, and we can exploit the duality between r and $\breve{r}$. Moreover, the two families are duals under complement: $A \in Up(r)$ iff $-A \in Dn(r)$. Thus the family of sets $UpDn(r) \triangleq Up(r) \cap Dn(r)$ is a complete Boolean algebra.

We now further narrow the focus to *preorders* (reflexive and transitive relations) $\preccurlyeq$: $X \rightsquigarrow X$. In this case: $A \in UpDn(\preccurlyeq)$ iff $\sigma(\succcurlyeq)\sigma(\preccurlyeq)(A) = A$ iff $\sigma(\preccurlyeq)(A) = A = \tau(\preccurlyeq)(A)$ iff A is a (disjoint) union of $\preccurlyeq$-*clusters*; that is, sets $C \subseteq X$ such that for all $x, y \in C$, $x \preccurlyeq y$ (all pairs of points in C are mutually $\preccurlyeq$-accessible).

Proposition 2.2. *Given an LTS* $\mathfrak{M} = (X, \{a^{\mathfrak{M}}\}_{a\in\Sigma}, \{\|p\|^{\mathfrak{M}}\}_{p\in\Phi})$, *and a preorder* $\preccurlyeq$ *on* X, *we have for each* $a \in \Sigma$ *and* $p \in \Phi$, *and all* $A \in \mathcal{P}(X)$,

$$\begin{array}{rll}
\preccurlyeq \textit{ satisfies } \mathbf{Zig}_a & \text{iff} & A \in Up(\preccurlyeq) \;\Rightarrow\; \sigma(a^{\mathfrak{M}})(A) \in Up(\preccurlyeq) \\
\preccurlyeq \textit{ satisfies } \mathbf{Zag}_a & \text{iff} & A \in Dn(\preccurlyeq) \;\Rightarrow\; \sigma(a^{\mathfrak{M}})(A) \in Dn(\preccurlyeq) \\
\preccurlyeq \textit{ satisfies } \mathbf{Up}_p & \text{iff} & \|p\|^{\mathfrak{M}} \in Up(\preccurlyeq) \\
\preccurlyeq \textit{ satisfies } \mathbf{Down}_p & \text{iff} & \|p\|^{\mathfrak{M}} \in Dn(\preccurlyeq).
\end{array}$$

Proof. The condition $\mathbf{Zig}_a$ for $\preccurlyeq$ is equivalent to the relational inclusion:

$$\succcurlyeq \circ a^{\mathfrak{M}} \subseteq a^{\mathfrak{M}} \circ \succcurlyeq$$

and this is in turn equivalent to the set-inclusion:

$$\sigma(\succcurlyeq)\sigma(a^{\mathfrak{M}})(A) \subseteq \sigma(a^{\mathfrak{M}})\sigma(\succcurlyeq)(A)$$

for all $A \in \mathcal{P}(X)$. Then using the reflexivity of $\preccurlyeq$, so $A \in Up(\preccurlyeq)$ iff $A = \sigma(\succcurlyeq)(A)$, the stated equivalence follows. For the $\mathbf{Zag}_a$ condition, replace $\succcurlyeq$ by $\preccurlyeq$. The equivalence for $\mathbf{Up}_p$ and $\mathbf{Down}_p$ is immediate from Definition 1.7. □

2.2. Semi-continuity of relations

For relations/set-valued maps, the purely topological notion of continuity was introduced by Kuratowski and Bouligand in the 1930's, and generalizes that for single-valued functions.

Definition 2.3. Given a topological space $(X, \mathcal{T})$, let $\mathcal{O}(\mathcal{T}) = \mathcal{T}$ and $\mathcal{C}(\mathcal{T})$ denote, respectively, the open and closed sets of $\mathcal{T}$.

A relation $r : (X, \mathcal{T}) \rightsquigarrow (Y, \mathcal{S})$ is called:

$$\begin{array}{rlll} \textit{lower semi-continuous (l.s.c.)} & \textit{iff} & U \in \mathcal{O}(\mathcal{S}) \;\Rightarrow\; \sigma(r)(U) \in \mathcal{O}(\mathcal{T}) \\ \textit{upper semi-continuous (u.s.c.)} & \textit{iff} & U \in \mathcal{O}(\mathcal{S}) \;\Rightarrow\; \tau(r)(U) \in \mathcal{O}(\mathcal{T}) \\ & \textit{iff} & C \in \mathcal{C}(\mathcal{S}) \;\Rightarrow\; \sigma(r)(C) \in \mathcal{C}(\mathcal{T}) \\ \textit{continuous} & \textit{iff} & \text{both l.s.c. and u.s.c.} \end{array}$$

Let $Clop(\mathcal{T}) = \mathcal{O}(\mathcal{T}) \cap \mathcal{C}(\mathcal{T})$ denote the Boolean algebra (under the finitary set-theoretic operations) of *clopen* subsets of $(X, \mathcal{T})$. The two semi-continuity properties together imply that for every $A \in Clop(\mathcal{S})$, we have $\sigma(r)(A) \in Clop(\mathcal{T})$. In particular, the domain $\mathrm{dom}(r) \triangleq \sigma(r)(Y) \in Clop(\mathcal{T})$, since $Y \in Clop(\mathcal{S})$.

A related notion of continuity for relations is examined in [2] Section 9.1, where in the context of Stone duality, the interest is in spaces $(X, \mathcal{A})$, where $\mathcal{A}$ is a Boolean algebra of sets that serves as a clopen basis for a topology $\mathcal{T}_{\mathcal{A}}$ on X; the open sets in $\mathcal{T}_{\mathcal{A}}$ are arbitrary unions of clopens, and dually, the closed sets are arbitrary intersections. A Boolean algebra of sets $\mathcal{A} \subseteq \mathcal{P}(X)$ is both *perfect* (every ultrafilter of $\mathcal{A}$ is determined by a point $x \in X$) and *reduced* (every pair of distinct points in X can be separated by sets $A, -A \in \mathcal{A}$) exactly when the topology $\mathcal{T}_{\mathcal{A}}$ is a Stone space (compact, Hausdorff and totally disconnected). In [2], a relation $r : (X, \mathcal{A}) \rightsquigarrow (Y, \mathcal{B})$ is said to be continuous if for all $B \in \mathcal{B}$, $\tau(r)(B) \in \mathcal{A}$.

2.3. Preorders and Alexandroff topologies

Given a preorder $\preccurlyeq$ on X, the *Alexandroff topology* $\mathcal{T}_{\preccurlyeq}$ on X determined by $\preccurlyeq$ is simply $\mathcal{T}_{\preccurlyeq} = \mathcal{O}(\mathcal{T}_{\preccurlyeq}) = Up(\preccurlyeq)$ and $\mathcal{C}(\mathcal{T}_{\preccurlyeq}) = Dn(\preccurlyeq)$. Thus $\mathcal{T}_{\preccurlyeq}$ is closed under arbitrary intersections as well as unions, and for all $A \subseteq X$,

$$int_{\mathcal{T}_{\preccurlyeq}}(A) = \tau(\preccurlyeq)(A) \qquad \text{and} \qquad cl_{\mathcal{T}_{\preccurlyeq}}(A) = \sigma(\preccurlyeq)(A).$$

In particular, $Clop(\mathcal{T}_{\preccurlyeq}) = UpDn(\preccurlyeq)$ is a complete Boolean algebra. The topology $\mathcal{T}_{\preccurlyeq}$ has as a basis the collection of all sets $B_{\preccurlyeq}(x) \triangleq \sigma(\succcurlyeq)(\{x\}) = \{y \in X \mid x \preccurlyeq y\}$, and $B_{\preccurlyeq}(x)$ is the intersection of all open sets in $\mathcal{T}_{\preccurlyeq}$ containing x.

More generally, a topology $\mathcal{T}$ on X is called *Alexandroff* if it has the property that for every point $x \in X$, there is a *smallest open set* containing x. In particular, every *finite* topology on a (arbitrary) set X is Alexandroff. For a preorder $\preccurlyeq$ on X, the topology $\mathcal{T}_{\preccurlyeq}$ is of course Alexandroff. Going the other way, any topology $\mathcal{T}$ on X determines a relation $\preccurlyeq_{\mathcal{T}}$ on X, called the *specialization preorder* of $\mathcal{T}$, given by:

$$x \preccurlyeq_{\mathcal{T}} y \quad \text{iff} \quad (\forall U \in \mathcal{T})[\, x \in U \Rightarrow y \in U \,].$$

Note that $\preccurlyeq_{\mathcal{T}}$ is a *partial order* exactly when $\mathcal{T}$ is T_0, and is trivial (the identity relation) when $\mathcal{T}$ is T_1. Alexandroff topologies are those that can be completely recovered from their specialization preorder: for any preorder $\preccurlyeq$ on X, $\preccurlyeq_{\mathcal{T}_{\preccurlyeq}} = \preccurlyeq$,

and if $\mathcal{T}$ is Alexandroff, then $\mathcal{T}_{\preccurlyeq_\mathcal{T}} = \mathcal{T}$. The Alexandroff topology on a preordered space can also be seen as a crude cousin of the Scott topology $\mathcal{T}_\sqsubseteq$ on a dcpo $(X, \sqsubseteq)$, which satisfies $\preccurlyeq_{\mathcal{T}_\sqsubseteq} = \sqsubseteq$; see [16], Section 2.4.

In the modal logic tradition, preorders give the relational Kripke semantics for **S4** modalities, with $\sigma(\preccurlyeq)$ interpreting $\Diamond$ and $\tau(\preccurlyeq)$ interpreting $\Box$. From work of McKinsey and Tarski in the 1940's, **S4** also admits a more general topological semantics in addition to the (historically later) relational Kripke semantics using preorders. The axioms for $\Box$ correspond to those of an arbitrary topological interior operator $\text{int}_\mathcal{T}$, and dually, $\Diamond$ corresponds to topological closure. Alexandroff topologies arise when one correlates the two semantics (see [5], where they go by the name D-topology, for "digital"). In earlier work on hybrid systems [15], Alexandroff spaces arising from *finite sub-topologies* of standard topologies on $X \subseteq \mathbb{R}^n$ (by the name "small" or AD-topologies) are used to model the conversion of sensor data into an input signal to a finite control automaton ([15], Sect. 5).

2.4. Topological characterization of bisimulation preorders

It follows immediately from Proposition 2.2 and Definition 2.3 that if $(X, \mathcal{T})$ is an Alexandroff space, then $a^\mathfrak{M} : (X, \mathcal{T}) \rightsquigarrow (X, \mathcal{T})$ is l.s.c. with respect to $\mathcal{T}$ iff $\preccurlyeq_\mathcal{T}$ satisfies $\mathbf{Zig}_a$, and $a^\mathfrak{M}$ is u.s.c. with respect to $\mathcal{T}$ iff $\preccurlyeq_\mathcal{T}$ satisfies $\mathbf{Zag}_a$. The Alexandroff hypothesis is essential for this characterization of lower semi-continuity, but for *arbitrary* topological spaces $(X, \mathcal{T})$, upper semi-continuity implies $\preccurlyeq_\mathcal{T}$ satisfies $\mathbf{Zag}_a$ (in longer words, $a^\mathfrak{M}$ is upper-$\preccurlyeq_\mathcal{T}$-monotonic); see [16], Section 4.4.

We now have our topological characterization of bisimulation preorders.

Proposition 2.4. *Let $\mathfrak{M} = (X, \{a^\mathfrak{M}\}_{a\in\Sigma}, \{\|p\|^\mathfrak{M}\}_{p\in\Phi})$ be an LTS model and let $\mathcal{T}$ be an Alexandroff topology X. Then:*

$$\begin{array}{ll} & \preccurlyeq_\mathcal{T} \text{ is a bisimulation preorder on } \mathfrak{M} \\ \text{iff} & \text{for each } a \in \Sigma, \quad a^\mathfrak{M} : (X, \mathcal{T}) \rightsquigarrow (X, \mathcal{T}) \text{ is continuous, and} \\ & \text{for each } p \in \Phi, \quad \|p\|^\mathfrak{M} \in \mathit{Clop}(\mathcal{T}). \end{array}$$

Moreover, the preorder

$$x \preccurlyeq_{\mathit{Clop}(\mathcal{T})} y \quad \text{iff} \quad (\forall A \in \mathit{Clop}(\mathcal{T}))[\, x \in A \;\Rightarrow\; y \in A \,]$$

includes $\preccurlyeq_\mathcal{T}$ and is symmetric, thus an equivalence relation $\sim_{\mathit{Clop}(\mathcal{T})}$. When $\preccurlyeq_\mathcal{T}$ is a bisimulation preorder on $\mathfrak{M}$, $\sim_{\mathit{Clop}(\mathcal{T})}$ is a bisimulation equivalence.

The last statement also follows from Proposition 2.2 and Definition 2.3, using the fact that $\mathit{Clop}(\mathcal{T}) = \mathit{Up}(\sim_{\mathit{Clop}(\mathcal{T})}) = \mathit{Dn}(\sim_{\mathit{Clop}(\mathcal{T})})$. Note that although $\preccurlyeq_\mathcal{T}$ and $\succcurlyeq_\mathcal{T}$ are both bisimulations if either is such, the topological equivalence (Stone T_0 quotient) $\sim_\mathcal{T} = (\preccurlyeq_\mathcal{T} \cap \succcurlyeq_\mathcal{T})$ can fail to be a bisimulation. If $B_\mathcal{T}(x) = B_{\preccurlyeq_\mathcal{T}}(x)$ and $C_\mathcal{T}(x) = cl_\mathcal{T}(\{x\})$ are, respectively, the smallest open and the smallest closed sets containing a point x, then under $\sim_\mathcal{T}$, the equivalence classes are $E_\mathcal{T}(x) =$

$B_T(x) \cap C_T(x)$. In contrast, the equivalence class $D_{Clop(T)}(x)$, is the smallest clopen or $\preccurlyeq_T$-*cluster* containing both $B_T(x)$ and $C_T(x)$.

More generally, if $\sim$ is any equivalence relation on X, and $\mathcal{T}_\sim$ is the Alexandroff topology of $\sim$, then the basic open sets are just the equivalence classes under $\sim$, and $\mathcal{T}_\sim = Clop(\mathcal{T}_\sim) = Up(\sim) = Dn(\sim)$ is the complete Boolean algebra of all unions of equivalence classes. The *bisimulation equivalence* conditions $\mathbf{UpDn}_p$ and $\mathbf{ZigZag}_a$ and reduce, respectively, to the requirement that $\|p\|^{\mathfrak{M}} \in UpDn(\sim)$, and that $UpDn(\sim)$ be closed under $\sigma(a^{\mathfrak{M}})$.

In the light of our excursion into general topology, we restate the basic truth-preservation property of bisimulations from Proposition 1.8.

Proposition 2.5. *Let* $\mathfrak{M} = (X, \{a^{\mathfrak{M}}\}_{a\in\Sigma}, \{\|p\|^{\mathfrak{M}}\}_{p\in\Phi})$ *be an LTS model and let* $\preccurlyeq$ *be a bisimulation preorder on* $\mathfrak{M}$.

Then for every sentence $\varphi \in \mathcal{S}_\mu(\Phi, \Sigma)$,

$$\sigma(\preccurlyeq)\left(\|\varphi\|^{\mathfrak{M}}\right) \;=\; \|\varphi\|^{\mathfrak{M}} \;=\; \tau(\preccurlyeq)\left(\|\varphi\|^{\mathfrak{M}}\right)$$

hence $\|\varphi\|^{\mathfrak{M}} \in Clop(\mathcal{T}_\preccurlyeq) = UpDn(\preccurlyeq)$.

Proof. The truth-preservation property is: $\sigma(\preccurlyeq)(\|\varphi\|^{\mathfrak{M}}) \subseteq \|\varphi\|^{\mathfrak{M}} \subseteq \tau(\preccurlyeq)(\|\varphi\|^{\mathfrak{M}})$, and the reflexivity of $\preccurlyeq$ gives the rest of the inclusions. □

3. Algebraic approaches to the modal μ-calculus

3.1. Modal algebras and modal frames

For bisimulation preorders on $\mathfrak{M}$, the algebras of sets $Clop(\mathcal{T}_\preccurlyeq)$ are clearly of interest since they contain the denotations in $\mathfrak{M}$ of all μ-calculus sentences. An algebraic approach to the semantics of the μ-calculus is taken up in the recent work of Kwiatkowska *et al.* in [2, 4]. The enterprise in those papers is to extend the framework of Stone duality for Boolean algebras to modal algebras with fixed-points, and in the process, give an algebraic completeness proof for Kozen's axiomatization $\mathbf{L}_\mu$ of the μ-calculus, using a Henkin-style canonical model construction over the space of ultrafilters of the Lindenbaum algebra of the logic $\mathbf{L}_\mu$. Their language for the μ-calculus contains logical constants **ff** and **tt**, but no alphabet Φ of propositional constants. We make the obvious extension.

Definition 3.1. A structure $(\mathcal{A}, \{\sigma_a^{\mathcal{A}}\}_{a\in\Sigma}, \{\|p\|^{\mathcal{A}}\}_{p\in\Phi})$ is called a *modal algebra* of signature (Φ, Σ), with carrier $\mathcal{A}$, iff

(1) $(\mathcal{A}; \vee, \wedge, \neg, 0, 1)$ is a Boolean algebra, with lattice order $\leqslant$;
(2) for each $p \in \Phi$, $\|p\|^{\mathcal{A}} \in \mathcal{A}$;
(3) for each $a \in \Sigma$, $\sigma_a^{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is a finitely additive and normal operator with values in $\mathcal{A}$: $\sigma_a^{\mathcal{A}}(A \vee B) = \sigma_a^{\mathcal{A}}(A) \vee \sigma_a^{\mathcal{A}}(B)$ and $\sigma_a^{\mathcal{A}}(0) = 0$.

For modal formulas φ, the denotation $\|\varphi\|^{\mathcal{A}}_{\xi}$ in $\mathcal{A}$ with respect to variable assignments $\xi : \mathrm{PVar} \to \mathcal{A}$ is defined as usual by induction on formulas, parallel to Definition 1.2.

Such a structure is called a *modal μ-algebra* if for each formula $\mu Z.\varphi \in \mathcal{F}_{\mu}(\Phi, \Sigma)$, the $\leqslant$-monotone operator $A \mapsto \|\varphi\|^{\mathcal{A}}_{\xi(A/Z)}$ has a least pre-fixed-point in $\mathcal{A}$, in which case:

$$\begin{aligned} \|\mu Z.\varphi\|^{\mathcal{A}}_{\xi} &\triangleq \bigwedge\{A \in \mathcal{A} \mid \|\varphi\|^{\mathcal{A}}_{\xi(A/Z)} \leqslant A\} \\ &= \bigwedge\{A \in \mathcal{A} \mid \|\varphi\|^{\mathcal{A}}_{\xi(A/Z)} = A\}. \end{aligned}$$

Definition 3.2. A *modal frame* of signature (Φ, Σ) is a pair $(\mathfrak{M}, \mathcal{A})$ consisting of an LTS model $\mathfrak{M} = (X, \{a^{\mathfrak{M}}\}_{a\in\Sigma}, \{\|p\|^{\mathfrak{M}}\}_{p\in\Phi})$ and a modal algebra $\mathcal{A}$ *for* $\mathfrak{M}$, by which we mean:

(1) $\mathcal{A}$ is a Boolean algebra under the finitary set-theoretic operations;
(2) $\mathcal{A}$ contains each of the sets $\|p\|^{\mathfrak{M}}$ for $p \in \Phi$; and
(3) $\mathcal{A}$ is closed under each of the pre-image operators $\sigma(a^{\mathfrak{M}})$ for $a \in \Sigma$.

A *modal μ-frame* is a modal frame $(\mathfrak{M}, \mathcal{A})$ such that $\mathcal{A}$ is a modal μ-algebra. An LTS model $\mathfrak{M}$ is identified with the modal μ-frame $(\mathfrak{M}, \mathcal{P}(X))$.

So by definition, modal algebras $\mathcal{A}$ for $\mathfrak{M}$ are *subalgebras* of the full power-set algebra $\mathcal{P}(X)$, considered as a Boolean algebra under the finitary set-theoretic operations, and a modal algebra with respect to the operators $\sigma(a^{\mathfrak{M}})$. The definition also entails that for each purely modal sentence $\varphi \in \mathcal{S}(\Phi, \Sigma)$, the denotation set $\|\varphi\|^{\mathfrak{M}} \in \mathcal{A}$. Define

$$\mathcal{S}^{\mathfrak{M}} \triangleq \{\|\varphi\|^{\mathfrak{M}} \mid \varphi \in \mathcal{S}(\Phi, \Sigma)\}$$

to be the family of all denotations in $\mathfrak{M}$ of modal sentences $\mathcal{S}(\Phi, \Sigma)$, and likewise define $\mathcal{S}^{\mathfrak{M}}_{\mu}$ by the denotations of μ-calculus sentences in $\mathfrak{M}$. Then $\mathcal{S}^{\mathfrak{M}}$ and $\mathcal{S}^{\mathfrak{M}}_{\mu}$ are both modal algebras for $\mathfrak{M}$, and $\mathcal{S}^{\mathfrak{M}}_{\mu}$ is a modal μ-algebra: an assignment ξ in $\mathcal{S}^{\mathfrak{M}}_{\mu}$ maps variables V_i to sets $\|\psi_i\|^{\mathfrak{M}}$, so for any formula $\mu Z.\varphi \in \mathcal{F}_{\mu}(\Phi, \Sigma)$, we have $\|\mu Z.\varphi\|^{\mathfrak{M}}_{\xi} = \|\mu Z.\varphi[V_i := \psi_i]\|^{\mathfrak{M}} \in \mathcal{S}^{\mathfrak{M}}_{\mu}$. Thus $\mathcal{S}^{\mathfrak{M}}$ ($\mathcal{S}^{\mathfrak{M}}_{\mu}$) is the *intersection* of all modal algebras (modal μ-algebras) for $\mathfrak{M}$.

In the Stone duality algebraic approach of [2,4], one takes a modal algebra $\mathcal{A}$ and generates a topology $\mathcal{T}_{\mathcal{A}}$ by taking $\mathcal{A}$ as a clopen basis. Here, we take an (Alexandroff) topology $\mathcal{T}$, and consider the algebra $Clop(\mathcal{T})$. In both cases, the algebras of clopens provide the denotations of modal formulas, but we switch the dynamic between the algebra and the topology.

3.2. Semantic agreement and bisimulations

In this section, we examine the relationship between the standard set-theoretic semantics in LTS models [12,17,19], and the algebraic semantics over modal frames or μ-frames. For purely *modal* formulas $\varphi \in \mathcal{F}(\Phi, \Sigma)$, the semantics in $(\mathfrak{M}, \mathcal{P}(X))$

and in any modal frame $(\mathfrak{M}, \mathcal{A})$ are in agreement: $\|\varphi\|^{\mathfrak{M}}_{\xi} = \|\varphi\|^{\mathcal{A}}_{\xi}$ for all variable assignments $\xi : \mathrm{PVar} \to \mathcal{A}$. But in general, they part company on μ-formulas, since the smallest set in $\mathcal{A}$ such that some condition holds will in general be larger than the smallest of all subsets of X such that the same condition holds. This motivates the following definition.

Definition 3.3. Given an LTS model $\mathfrak{M}$ and a modal μ-algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ for $\mathfrak{M}$, we say the frame $(\mathfrak{M}, \mathcal{A})$ is *in semantic agreement* with the underlying model $\mathfrak{M}$ if for all formulas $\varphi \in \mathcal{F}_{\mu}(\Phi, \Sigma)$ and all assignments ξ in $\mathcal{A}$, we have: $\|\varphi\|^{\mathcal{A}}_{\xi} = \|\varphi\|^{\mathfrak{M}}_{\xi}$.

In other words, such algebras $\mathcal{A}$ yield the "true" denotation of formulas, as determined by the standard set-theoretic semantics in $\mathfrak{M}$. In establishing semantic agreement, the task is to show that for assignments ξ in $\mathcal{A}$, each set $\|\mu Z.\varphi\|^{\mathfrak{M}}_{\xi}$ is in $\mathcal{A}$; it then follows readily that $\|\mu Z.\varphi\|^{\mathfrak{M}}_{\xi}$ is the least pre-fixed-point of $A \mapsto \|\varphi\|^{\mathcal{A}}_{\xi(A/Z)} = \|\varphi\|^{\mathfrak{M}}_{\xi(A/Z)}$, with induction on the complexity of formulas. The available means to prove $\|\mu Z.\varphi\|^{\mathfrak{M}}_{\xi} \in \mathcal{A}$ is by transfinite induction that each of the α-approximations $\|\mu Z.\varphi\|^{\mathfrak{M}}_{\xi,\alpha} \in \mathcal{A}$. If $\mathcal{A}$ is a complete, then closure under unions at limit ordinals is immediate; the task then reduces to proving closure under the $\varphi^{\mathfrak{M}}_{\xi,Z}$ operator.

In particular, the algebra of standard denotations of μ-calculus sentences $\mathcal{S}^{\mathfrak{M}}_{\mu}$ is always in semantic agreement with $\mathfrak{M}$, thus it is the smallest modal μ-algebra for $\mathfrak{M}$ in semantic agreement with $\mathfrak{M}$.

Our analysis of bisimulation preorders leads to a simple condition for semantic agreement.

Proposition 3.4. *If $\preccurlyeq$ is a bisimulation preorder on an LTS model $\mathfrak{M}$, then $(\mathfrak{M}, Clop(\mathcal{T}_{\preccurlyeq}))$ is in semantic agreement with $\mathfrak{M}$.*

Proof. From Proposition 2.4, $Clop(\mathcal{T}_{\preccurlyeq})$ is a modal algebra for $\mathfrak{M}$, since it contains each $\|p\|^{\mathfrak{M}}$ and is closed under $\sigma(a^{\mathfrak{M}})$. The completeness of $Clop(\mathcal{T}_{\preccurlyeq})$ as a Boolean algebra ensures that it is also a μ-algebra, since the relevant pre-fixed-points exist in $Clop(\mathcal{T}_{\preccurlyeq})$. From Proposition 2.5, for all *sentences* $\varphi \in \mathcal{S}_{\mu}(\Phi, \Sigma)$, we have $\|\varphi\|^{\mathfrak{M}} \in Clop(\mathcal{T}_{\preccurlyeq})$. To prove that $\|\varphi\|^{\mathfrak{M}}_{\xi} \in Clop(\mathcal{T}_{\preccurlyeq})$ for all *formulas* $\varphi \in \mathcal{F}_{\mu}(\Phi, \Sigma)$ and any assignment ξ in $Clop(\mathcal{T}_{\preccurlyeq})$, use transfinite induction as above. □

Corollary 3.5. *If $\mathcal{A}$ is any* complete *modal algebra for an LTS model $\mathfrak{M}$, then $(\mathfrak{M}, \mathcal{A})$ is in semantic agreement with $\mathfrak{M}$.*

Proof. Consider the equivalence relation $\sim_{\mathcal{A}}$ on X defined by:

$$x \sim_{\mathcal{A}} y \quad \text{iff} \quad (\forall A \in \mathcal{A})[\, x \in A \Leftrightarrow y \in A \,].$$

In virtue of the closure conditions on $\mathcal{A}$ as a modal algebra for $\mathfrak{M}$, $\sim_{\mathcal{A}}$ is a bisimulation equivalence on $\mathfrak{M}$, and by the completeness of $\mathcal{A}$ as a Boolean algebra, $\mathcal{A} = Clop(\mathcal{T}_{\sim_{\mathcal{A}}})$. The result then follows from Proposition 3.4. □

4. Finite bisimulations of hybrid systems

4.1. Semi-flows and their orbit relations

In the definition of an LTS model of a hybrid system, the continuous evolution relations e_q are defined in terms of the semi-flows ϕ_q, but in a non-elementary way. We start by investigating a more primitive relation determined by a semi-flow.

Definition 4.1. Given a semi-flow $\phi : X \times \mathbb{R}^+ \to X$ on a topological space X, define the (positive) *orbit relation* $f : X \rightsquigarrow X$ of ϕ by:

$$x \xrightarrow{f} x' \quad \overset{\circ}{\Leftrightarrow} \quad (\exists t \in \mathbb{R}^+) \;\; x' = \phi(x,t).$$

For each point $x \in X$, the set $f(x) \overset{\circ}{=} \{\phi(x,t) \mid t \in \mathbb{R}^+\}$ is the *positive orbit* of x under ϕ.

The pre-image operators of the orbit relation are such that $x \in \sigma(f)(A)$ iff the flow from x reaches A at *some* time $t \in \mathbb{R}^+$, while $x \in \tau(f)(A)$ iff the flow from x remains inside A for *all* time $t \in \mathbb{R}^+$, for $A \in \mathcal{P}(X)$.

Observe that by the semi-group properties of a semi-flow ϕ, the orbit relation f is both *reflexive* and *transitive*. It is also (weakly) *connected* in the sense that

$$x \xrightarrow{f} x_1 \text{ and } x \xrightarrow{f} x_2 \;\Rightarrow\; \left[x_1 \xrightarrow{f} x_2 \text{ or } x_2 \xrightarrow{f} x_1\right].$$

Equivalently, the postive orbit $f(x)$ of any point x is *linearly pre-ordered* by f, *i.e.* linearly ordered modulo cycles. The conjunction of reflexivity, transitivity and connectedness entails that f is (weakly) *n-fold connected*: if $x \xrightarrow{f} x_i$ for $i = 1, ..., n$, then there exists a permutation π on the letters $\{1, ..., n\}$ such that $x \xrightarrow{f} x_{\pi(1)} \xrightarrow{f} x_{\pi(2)} \xrightarrow{f} \cdots \xrightarrow{f} x_{\pi(n)}$.

When a semi-flow $\phi : X \times \mathbb{R}^+ \to X$ is in fact a *flow*, which means each of the functions $\phi_t : X \to X$ for $t \in \mathbb{R}^+$ is invertible, then the relational converse $\breve{f}$ coincides with the orbit relation of the reverse flow $\phi^{-1} : X \times \mathbb{R}^+ \to X$ given by $\phi^{-1}(x,t) \overset{\circ}{=} (\phi_t)^{-1}(x)$.

Implicit in the definition of continuous transitions e_q as evolution constrained within Inv_q is the idea that the domains Inv_q be *convex* with respect to their flows ϕ_q, in the sense that any integral curve of ϕ_q connecting two points in Inv_q should remain within Inv_q at all intermediate points; wandering outside Inv_q and then returning is to be ruled out. The general form of this notion of "in-between-ness" is captured in the following definition.

Definition 4.2. Given a relation $r : X \rightsquigarrow X$, we call a set $A \subseteq X$ *r-convex* if for all $x, y, z \in X$,

$$\text{if} \quad x, y \in A \;\text{ and }\; x \xrightarrow{r} z \xrightarrow{r} y, \quad \text{then} \quad z \in A.$$

Equivalently, $\sigma(\breve{r})(A) \cap \sigma(r)(A) \subseteq A$.

For $f : X \rightsquigarrow X$ the orbit relation of a semi-flow ϕ, a set $A \subseteq X$ is f-convex iff for all $x \in A$ and all $t \in \mathbb{R}^+$, if $\phi(x,t) \in A$ then for all $s \in [0,t]$, $\phi(x,s) \in A$. Moreover, when A is f-convex, then the relation $e : X \rightsquigarrow X$ of evolution constrained within A given by:

$$x \xrightarrow{e} x' \quad \overset{\circ}{\Leftrightarrow} \quad (\exists t \in \mathbb{R}^+)[x' = \phi(x,t) \ \wedge \ (\forall s \in [0,t]) \ \phi(x,s) \in A]$$

(as in the definition of an LTS model of a hybrid system) admits the decomposition $e = f \cap (A \times A)$, hence the pre-image operators satisfy: $\sigma(e)(Z) = A \cap \sigma(f)(Z \cap A)$. In concrete examples of hybrid systems in the literature, the domains of evolution Inv_q are invariably f_q-convex.

The property of f-convexity is identified by Lafferriere *et al.* in [13] under the name property **(P)**, and is of fundamental importance in their construction of a finite bisimulation for classes of hybrid systems, to which we now turn.

4.2. Finite bisimulations of o-minimal flows

Definition 4.3. [18], Chap. 1. Let $\overline{\mathbb{R}} = (\mathbb{R}; <, ...)$ be a (model-theoretic) structure over the reals $\mathbb{R}$ equipped with at least a dense linear order without endpoints, and let $\mathcal{L}(\overline{\mathbb{R}})$ be the first-order language of $\overline{\mathbb{R}}$. The structure $\overline{\mathbb{R}}$ is said to be *o-minimal* if every set $A \subseteq \mathbb{R}$ definable in $\mathcal{L}(\overline{\mathbb{R}})$ is a finite union of (open) intervals and points.

The term *o-minimal structure* is also used to refer to any sequence $(\mathcal{S}_n)_{n \in \mathbb{N}}$ of Boolean algebras $\mathcal{S}_n \subseteq \mathcal{P}(\mathbb{R}^n)$ of $\overline{\mathbb{R}}$-definable sets such that for each $n \in \mathbb{N}$:

(1) the sets $\{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_i = x_j\} \in \mathcal{S}_n$;
(2) if $A \in \mathcal{S}_n$ then $A \times \mathbb{R} \in \mathcal{S}_{n+1}$ and $\mathbb{R} \times A \in \mathcal{S}_{n+1}$;
(3) if $A \in \mathcal{S}_{n+1}$ then $\pi(A) \in \mathcal{S}_n$, where π is the projection onto the first n coordinates;
(O1) $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < x_2\} \in \mathcal{S}_2$;
(O2) the sets in $\mathcal{S}_1$ are exactly the finite unions of open intervals and points.

Van den Dries' monograph [18] is a comprehensive and highly readable study of spaces and maps definable in o-minimal structures, and their "tameness" as manifested in a cell-decomposition property: any definable set $A \subseteq \mathbb{R}^n$ has only finitely many connected components (with respect to the standard topology on $\mathbb{R}^n$ inherited from the order on $\mathbb{R}$). For our purposes, all we need is one consequence of cell-decomposition: a uniform bound result for definable relations $r : X \rightsquigarrow Y$, which in [18] go by the name *definable families* $(r_x)_{x \in X}$.

Lemma 4.4. *Fix an o-minimal structure $\overline{\mathbb{R}}$. Given an $\overline{\mathbb{R}}$-definable space $X \subseteq \mathbb{R}^n$, an $\overline{\mathbb{R}}$-definable relation $r : X \rightsquigarrow X$ and an $\overline{\mathbb{R}}$-definable set $A \subseteq X$, there is a positive integer $N(r, A)$ such that the number of connected components of the set $r(x) \cap A$ is bounded by $N(r, A)$, independent of $x \in X$.*

Proof. Apply [18], Corollary 3.3.6. □

The following result and its proof are the product of a close analysis of the construction of a finite bisimulation in [13]. Our reformulation of the task as the construction of a finite topology yields a conceptual clarification of that work.

Theorem 4.5. [13] *Let $\overline{\mathbb{R}} = (\mathbb{R}; <, +, 0, ...)$ be an o-minimal structure expanding the reals as an ordered Abelian group. Suppose $\mathfrak{M} = (X, f, \{P_k\}_{k\in K})$ is a LTS model that is first-order definable in $\overline{\mathbb{R}}$, with $X \subseteq \mathbb{R}^n$, $f : X \rightsquigarrow X$ the positive orbit relation of an invertible flow $\phi : X \times \mathbb{R}^+ \to X$, and K finite. Then $\mathfrak{M}$ has a bisimulation equivalence of finite index, which is also first-order definable in $\overline{\mathbb{R}}$.*

Proof. By Proposition 2.4, it suffices to produce a finite topology $\mathcal{T}$ on X such that each $P_k \in Clop(\mathcal{T})$ and f is both u.s.c. and l.s.c. with respect to $\mathcal{T}$.

Stage 1: Let $\{A_j\}_{j\in J}$ be a list of the (non-empty) *atoms* of the finite Boolean algebra generated by the sets P_k. Each A_j is a finite Boolean combination of literals over $\{P_k\}_{k\in K}$, and hence $\overline{\mathbb{R}}$-definable. Without loss of generality, we may assume for convenience that the modal signature of $\mathfrak{M}$ includes for each j a propositional constant $\mathbf{A}_j$ denoting A_j.

Stage 2: By Lemma 4.4, for each $j \in J$, there is an integer N_j such that for all $x \in X$, the number of connected components of the set $f(x) \cap A_j$ is bounded by N_j.

Now consider the modal operator defined by

$$((\mathbf{f}))Z \;\triangleq\; Z \wedge [\mathbf{f}](\neg Z \to [\mathbf{f}]\neg Z) \;\equiv\; Z \wedge [\mathbf{f}](\langle\mathbf{f}\rangle Z \to Z).$$

Thus $((\mathbf{f}))Z$ denotes the set of points in Z from which if the flow ever leaves Z, it never returns. Now for each $j \in J$, consider the further partition of A_j into sets A_j^n recursively defined by the modal formulas:

$$\begin{aligned} \mathbf{A}_j^0 &\triangleq \mathbf{A}_j \wedge [\mathbf{f}]\mathbf{A}_j \\ \mathbf{A}_j^{n+1} &\triangleq ((\mathbf{f}))\left(\mathbf{A}_j \wedge \neg \textstyle\bigvee_{k=0}^n \mathbf{A}_j^k\right). \end{aligned}$$

Thus A_j^0 is the "f-sink" of A_j: that part of A_j from which the flow never leaves (possibly empty), and A_j^{n+1} is the result of applying $((\mathbf{f}))$ to the n-th remainder $A_j - \bigcup_{k=0}^n A_j^k$. Hence each of the sets A_j^n is $\overline{\mathbb{R}}$-definable. The finiteness of this subpartition process is established by the lemma:

Lemma 4.6. *For each $j \in J$, each $n \geq 2$, and each $x \in A_j^n$, the set $f(x) \cap A_j$ has at least n connected components. Hence $A_j^n = \varnothing$ for all $n > N_j$.*

The proof of Lemma 4.6 depends on four claims:

Claim 4.7. For each $j \in J$, and each $n \geq 0$, the set A_j^n is f-convex.

Claim 4.8. For each $j \in J$, each $n \geq 2$ and for each $x \in A_j^n$, there exists $y_{n-1}, ..., y_1, x_{n-1}, ..., x_1 \in X$ such that

$$x = x_n \xrightarrow{f} y_{n-1} \xrightarrow{f} x_{n-1} \xrightarrow{f} y_{n-2} \xrightarrow{f} \cdots \xrightarrow{f} y_1 \xrightarrow{f} x_1$$

and for $1 \leq k < n$, $x_k \in f(x) \cap A_j^k$ and $y_k \in f(x) - A_j$.

Claim 4.9. For any set $A \subseteq X$, and $x \in X$,

$$\text{if } A \text{ is } f\text{-convex, then the set } f(x) \cap A \text{ is connected.}$$

Claim 4.10. ([13], Lem. 5.3) For $\overline{\mathbb{R}}$-definable sets $A, C \subseteq X$, and $x \in X$,

$$\text{if } C \text{ is a connected component of } f(x) \cap A, \text{ then } C \text{ is } f\text{-convex.}$$

Proof of Lemma 4.6. Fix $n \geq 2$ and $x \in A_j^n$. By Claims 4.7 and 4.9, for each k, $0 \leq k \leq n$, the set $f_q(x) \cap A_j^k$ is (path) connected, and thus contained in a connected component of $f(x) \cap A_j$. Now fix k, $1 \leq k \leq n$, and let C^k be the connected component of $f(x) \cap A_j$ that contains $f(x) \cap A_j^k$. By Claim 4.10, the set $C^k = f(x) \cap C^k$ is f-convex. Now suppose, for a contradiction, that $C^k \cap A_j^m \neq \varnothing$ for some $m \neq k$, $1 \leq m \leq n$; w.l.o.g., assume $k < m$. Then by Claim 4.8, starting from $x_m \in C^k \cap A_j^m \subseteq f(x) \cap A_j^m$, the relation leaves A_j (and hence C^k) at least once before returning to $f(x) \cap A_j^k \subseteq C^k$, contradicting the f-convexity of C^k. Thus $C^k \cap A_j^m = \varnothing$ for all $m \neq k$, $1 \leq m \leq n$, hence $C^k = f(x) \cap A_j^k$ is a connected component of $f(x) \cap A_j$. Hence $f(x) \cap A_j$ has at least n connected components. □

Claims 4.7 and 4.8 can be proved using only the *transitivity* of f, together with the definition of the partition sequence A_j^k. Claim 4.9 is immediate from the definition of a semi-flow, its positive orbit relation, and f-convexity. Claim 4.10 is a reformulation of Lemma 5.3 of [13]; the proof given there makes essential use of the assumption that the semi-flow is invertible, together with o-minimal definability.

Let $\{S_i\}_{i \in I}$ be a list of all the non-empty sets A_j^k, for $j \in J$ and $k \leq N_j$, and again for convenience, assume the modal signature of $\mathfrak{M}$ includes a propositional constant $\mathbf{S}_i$ denoting S_i, for each i. Now $\{S_i\}_{i \in I}$ forms an f-convex partition of the state space X. This means that any curve of the flow ϕ will pass through a partition set S_i *at most once*, since by f-convexity, if the flow passes through S_i and leaves, it never returns to S_i, and if the flow enters S_i from some other S_j, then it has never passed through S_i before.

Stage 3: We now build a finite topology $\mathcal{T}$ on X such that each $S_i \in Clop(\mathcal{T})$ and f is both u.s.c. and l.s.c. with respect to $\mathcal{T}$.

Let $\mathrm{Sg} \triangleq \{+, -\}$, and for modal formulas $\varphi \in \mathcal{F}(\mathfrak{M})$, define $+\varphi \triangleq \varphi$ and $-\varphi \triangleq \neg\varphi$. Consider the collection of modal sentences $\mathcal{G}$ generated by the grammar:

$$\psi \ ::= \ \mathbf{S}_i \ \mid \mathbf{S}_i \ \wedge \ \eta \langle \mathbf{f} \rangle \psi$$

where $i \in I$ and $\eta \in \mathrm{Sg}$. Thus each sentence $\psi \in \mathcal{G}$ is uniquely characterized by an alternating sequence $\alpha = (i, \eta_1, j_1, ..., \eta_n, j_n) \in I \times (\mathrm{Sg} \times I)^n$, for some $n \geq 0$,

where we define

$$\psi(\alpha) \doteq \mathbf{S}_i \wedge \eta_1\langle\mathbf{f}\rangle(\mathbf{S}_{j_1} \wedge \eta_2\langle\mathbf{f}\rangle(\mathbf{S}_{j_2} \wedge \eta_3\langle\mathbf{f}\rangle(\ldots \wedge \eta_{n-1}\langle\mathbf{f}\rangle(\mathbf{S}_{j_{n-1}} \wedge \eta_n\langle\mathbf{f}\rangle\mathbf{S}_{j_n})\ldots))).$$

Define the *degree* of such an alternating sequence α to be n, which is just the *modal degree* (depth of nesting of modal operators) of the sentence $\psi(\alpha)$. Let $\mathcal{G}(I)$ denote the finite subcollection of sentences $\psi(\alpha) \in \mathcal{G}$ such that $\alpha \in \bigcup_{0\leq n\leq I} I \times (\mathrm{Sg} \times I)^n$, and for each $i \in I$, let $\mathcal{G}(I,i)$ denote the subcollection of all $\psi(\alpha) \in \mathcal{G}(I)$ such that α begins with i. So for $\psi \in \mathcal{G}(I)$, the sentence $\psi \to \mathbf{S}_i$ is true in $\mathfrak{M}$ iff $\psi \in \mathcal{G}(I,i)$.

We now take $\mathcal{T}$ to be the finite topology generated from the sets $\|\psi\|^{\mathfrak{M}}$ for $\psi \in \mathcal{G}(I)$ by closing under finite unions and intersections.

Lemma 4.11. *$\mathcal{T}$ is closed under complement, hence $\mathcal{T} = Clop(\mathcal{T})$ and for each $i \in I$, $S_i \in Clop(\mathcal{T})$.*

Proof. The sets in $\mathcal{T}$ are modally defined by disjunctions and conjunctions of sentences in $\mathcal{G}(I)$, so it suffices to show that $\|\neg\psi\|^{\mathfrak{M}} \in \mathcal{T}$ for each $\psi \in \mathcal{G}(I)$. Proceed by induction on the modal degree of ψ, using the equivalence in $\mathfrak{M}$:

$$\neg(\,\mathbf{S}_i \wedge \eta\langle\mathbf{f}\rangle\psi(\alpha)\,) \equiv \left(\bigvee_{j\neq i,\, j\in I} \mathbf{S}_j\right) \vee (\,\mathbf{S}_i \wedge \neg\eta\langle\mathbf{f}\rangle\psi(\alpha))\,.$$

□

Lemma 4.12. *The flow relation f is both u.s.c. and l.s.c. with respect to $\mathcal{T}$.*

Proof. By Lemma 4.11, it suffices to show that the Boolean algebra $\mathcal{T} = Clop(\mathcal{T})$ is closed under $\sigma(f)$, so f is l.s.c.; the u.s.c. property will then follow by Boolean duality. For each of the generating formulas $\psi \in \mathcal{G}(I)$, it is immediate that $\|\langle\mathbf{f}\rangle\psi\|^{\mathfrak{M}} \in \mathcal{T}$, since in $\mathfrak{M}$,

$$\langle\mathbf{f}\rangle\psi(\alpha) \equiv \bigvee_{i\in I}(\,\mathbf{S}_i \wedge \langle\mathbf{f}\rangle\psi(\alpha)\,) \equiv \bigvee_{i\in I}\psi((i,+)^\frown\alpha).$$

And since $\langle\mathbf{f}\rangle$ distributes over disjunctions, we have $\left\|\langle\mathbf{f}\rangle\left(\bigvee_{j\in J}\varphi_j\right)\right\|^{\mathfrak{M}} \in \mathcal{T}$ whenever $\|\varphi_j\|^{\mathfrak{M}} \in \mathcal{T}$ for each $j \in J$.

To conclude the proof, observe that every *atom* of the algebra $\mathcal{T} = Clop(\mathcal{T})$ is modally defined by a conjunction $\bigwedge_{k\in K}\psi_k$, where each conjunct $\psi_k \in \mathcal{G}(I,i)$ for some one $i \in I$. Since each set in $\mathcal{T}$ is modally representable as a finite disjunction of atoms, the required closure under $\langle\mathbf{f}\rangle$ follows from two further claims.

Claim 4.13. For each $i \in I$ and $\psi \in \mathcal{G}(I,i)$, the following sentence is true in $\mathfrak{M}$:

$$\psi \to [\mathbf{f}](\,\mathbf{S}_i \to \psi)\,.$$

Claim 4.13 is proved by induction on modal degree, using the connectedness and transitivity of f together with the f-convexity of the sets S_i. The sentence asserts that whenever $x \in \|\psi\|^{\mathfrak{M}}$, then every flow successor of x that is in S_i is in fact in $\|\psi\|^{\mathfrak{M}}$.

Claim 4.14. For each $i \in I$ and each finite family $\psi_k \in \mathcal{G}(I,i)$ for $k \in K$, the following equivalence is true in $\mathfrak{M}$:

$$\langle \mathbf{f} \rangle \left(\bigwedge_{k \in K} \psi_k \right) \equiv \bigwedge_{k \in K} \langle \mathbf{f} \rangle \psi_k.$$

For Claim 4.14, the left-to-right implication is always true. For the converse, fix $x \in \left\| \bigwedge_{k \in K} \langle \mathbf{f} \rangle \psi_k \right\|^{\mathfrak{M}}$, where each $\psi_k \in \mathcal{G}(I,i)$. So for each $k \in K$, there is an $x_k \in \|\psi_k\|^{\mathfrak{M}}$ such that $x \xrightarrow{f} x_k$. Thus each $x_k \in S_i$. Then since f is K-fold connected, there exists a permutation π on the letters $\{1, \ldots, K\}$ such that $x \xrightarrow{f} x_{\pi(1)} \xrightarrow{f} x_{\pi(2)} \xrightarrow{f} \cdots \xrightarrow{f} x_{\pi(K)}$. Set $x^* = x_{\pi(K)}$. Then $x^* \in S_i$ and $x \xrightarrow{f} x_k \xrightarrow{f} x^*$ for each $k \in K$. By Claim 4.13, since $x^* \in S_i$ is an f-successor of each of the points $x_k \in \|\psi_k\|^{\mathfrak{M}}$, we must have $x^* \in \|\psi_k\|^{\mathfrak{M}}$ for each $k \in K$. Hence $x \in \left\| \langle \mathbf{f} \rangle \left(\bigwedge_{k \in K} \psi_k \right) \right\|^{\mathfrak{M}}$, as required.

This concludes the proof of Theorem 4.5. □

4.3. Finite topologies for hybrid systems

Theorem 4.5 yields a finite topology for an o-minimal LTS model equipped with a single flow relation. In applying this to an LTS model of a hybrid system, we can separately produce for each discrete control mode $q \in Q$, a finite topology $\mathcal{T}_q$ on the space $X_q \subseteq \mathbb{R}^n$ such that the flow relation $f_q : X_q \rightsquigarrow X_q$ is continuous w.r.t. $\mathcal{T}_q$, and for any finite number of constant sets $A_q \subseteq X_q$, we can ensure $A_q \in Clop(\mathcal{T}_q) = \mathcal{T}_q$. When the domains of evolution Inv_q are f_q-convex, the equation $\sigma(e_q)(Z) = Inv_q \cap \sigma(f_q)(Z \cap Inv_q)$ entails that e_q will also be continuous w.r.t. $\mathcal{T}_q$.

The difficulty comes in dealing with the reset relations $r_{q,q'}$. The required compatibility property between the topologies $\mathcal{T}_q$ and $\mathcal{T}_{q'}$ is the continuity of the relation $r_{q,q'} : (X_q, \mathcal{T}_q) \rightsquigarrow (X_{q'}, \mathcal{T}_{q'})$. However, in the absence of special assumptions on the reset relations, we have no reason to believe that the appropriate continuity properties would hold.

The solution in [13] is to make the radical restriction to reset relations which are *set-valued constant*, which means $r = A \times B$, so $r(x) = B$ for all $x \in A = \mathrm{dom}(r)$. The existential pre-image of a constant set-valued $r : X \rightsquigarrow Y$ satisfies $\sigma(r)(Z) = A$ if $Z \cap B \neq \varnothing$ and $\sigma(r)(Z) = \varnothing$ otherwise. Hence for any topologies $\mathcal{T}$ on X and $\mathcal{S}$ on Y, $r : (X, \mathcal{T}) \rightsquigarrow (Y, \mathcal{S})$ is continuous exactly when the domain $A \in Clop(\mathcal{T})$.

We consider a slightly more general class of relations: we say $r : X \rightsquigarrow Y$ is *piecewise set-valued constant* when r is the relational sum of a finite family of set-valued constant relations $r_k = A_k \times B_k$, so $r = \bigcup_{k \in K} (A_k \times B_k)$. Then for

any topologies $\mathcal{T}$ on X and $\mathcal{S}$ on Y, $r : (X, \mathcal{T}) \rightsquigarrow (Y, \mathcal{S})$ is continuous exactly when $A_k \in Clop(\mathcal{T})$ for each $k \in K$. Note that the sets A_k may overlap, and if $x \in A_{k_1} \cap A_{k_2}$ then $r(x) = B_{k_1} \cup B_{k_2}$.

Theorem 4.15. *Let $\overline{\mathbb{R}} = (\mathbb{R}; <, +, 0, ...)$ be an o-minimal structure expanding the reals as an ordered Abelian group. Suppose $\mathcal{H} = (Q, G, \{X_q, \phi_q, Init_q, Inv_q\}_{q \in Q}, \{r_{q,q'}, Grd_{q,q'}\}_{(q,q') \in G})$ is a hybrid system each of whose components are first-order definable in $\overline{\mathbb{R}}$, and where each of the reset relations $r_{q,q'}$ are piecewise set-valued constant, and let $\mathfrak{M}_{\mathcal{H}}$ be an LTS model for $\mathcal{H}$ which includes among its constant sets each of the pieces that form the domains of the reset relations. Then $\mathfrak{M}_{\mathcal{H}}$ has a finite bisimulation equivalence of finite index.*

When the structure $\overline{\mathbb{R}}$ is such that the first-order theory $\mathrm{Th}(\overline{\mathbb{R}})$ is decidable, then under the hypotheses of Theorem 4.15, for any μ-calculus sentence $\varphi \in \mathcal{S}(\mathfrak{M}_{\mathcal{H}})$, we can effectively decide whether $\mathfrak{M}_{\mathcal{H}} \models \varphi$. To derive μ-calculus decidability from Theorem 4.15 for more general o-minimal structures, we need to examine the decidability of the relevant *modal fragments* of the first-order language $\mathcal{L}(\overline{\mathbb{R}})$, and identify decidable modal algebras of first-order formulas. The work of Lafferriere *et al.* in [14] on the decidability of hybrid systems whose flows are defined using the exponential function (arising from linear vector fields $\dot{x} = Ax$ where the matrix A is nilpotent, diagonalizable or has purely imaginary eigenvalues) can be recast in this light.

5. Discussion and conclusion

Our investigation of topological content in the notion of a bisimulation relation has shed some new light on the nature of the structure-preservation conditions, and in application to hybrid systems, our characterization as a continuity property is put to useful service.

For the μ-calculus, topics of further research include a deeper examination of the Henkin-style canonical model construction in [2,4] and its semantic agreement with the standard set-theoretic semantics, in order to properly relate the algebraic completeness result in that work with Walukiewicz's completeness result in [19].

Regarding the construction of finite bisimulations for hybrid systems, several lines of inquiry present themselves. In the proof of Theorem 4.5, it would be more satisfying to produce a finite topology $\mathcal{T}$ for which $\mathcal{T} \neq Clop(\mathcal{T})$; one approach is to consider only formulas encoding *positive* accessibility information between f-convex partition blocks. There is a clear need to identify more general reset relations for which the required continuity properties can be established. And moving beyond time-determinism in the continuous dynamics requires a deeper study of set-valued and parametrized semi-flows and their orbit relations.

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