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<http://www.numdam.org/item?id=CTGDC_2010__51_2_83_0>

BIPULLBACKS AND CALCULUS OF FRACTIONS
by Enrico M. VITALE

Dedicated to Francis Borceux on the occasion of his 60th birthday

RÉSUMÉ. Nous démontrons que la classe des équivalences faibles entre groupoïdes internes dans une catégorie régulière protomodulaire est une "congruence à biproduits fibrés" et, par conséquent, elle admet un calcul à droite des fractions. Comme application, nous montrons que les foncteurs monoïdaux entre groupoïdes internes dans les groupes et les homomorphismes entre 2-algèbres de Lie strictes sont les fractions des foncteurs internes par rapport aux équivalences faibles.

RÉSUMÉ. We prove that the class of weak equivalences between internal groupoids in a regular protomodular category is a bipullback congruence and, therefore, has a right calculus of fractions. As an application, we show that monoidal functors between internal groupoids in groups and homomorphisms of strict Lie 2-algebras are fractions of internal functors with respect to weak equivalences.

1. Introduction

It is well known that any monoidal category is monoidally equivalent to a strict one. This is not true for strong monoidal functors: not every strong monoidal functor is naturally isomorphic to a strict one (i.e., to a functor F such that the structural isomorphisms $FA \otimes FB \to F(A \otimes B)$ and $I \to FI$ are identities). An important example of this fact is given by Schreier theory of group extensions. In fact, let A and B be groups and write $D(A)$ for A seen as a discrete internal groupoid in the category Grp of groups, and $OUT(B)$ for the internal groupoid in Grp corresponding to the crossed module $B \to Aut(B)$ of inner automorphisms.

Then internal (= strict) functors from $D(A)$ to $OUT(B)$ correspond to split extensions of A through B, whereas monoidal functors from $D(A)$ to $OUT(B)$ correspond to arbitrary extensions of A by B.

The previous example leads to the following question: what is the precise relation between the 2-category of internal groupoids and internal functors in *Grp* and the 2-category of internal groupoids in *Grp* and monoidal functors? The same question can be asked working internally to the category *Lie* of Lie K-algebras (for K a fixed field), replacing monoidal functors by homomorphisms of strict Lie 2-algebras (precise definitions are in Section 7).

A possible answer to the previous questions is suggested by the fact that if $F: \mathbb{C} \to \mathbb{D}$ is an internal functor in *Grp* which is a weak equivalence (i.e., full, faithful and essentially surjective on objects) then the quasi-inverse functor $F^{-1}: \mathbb{D} \to \mathbb{C}$ is no longer an internal functor, but it is still a monoidal one. More precisely, we prove that:

1. The 2-category of internal groupoids in *Grp* and monoidal functors is the 2-category of fractions of the 2-category of internal groupoids and internal functors in *Grp* with respect to weak equivalences.

2. The 2-category of internal groupoids in *Lie* and homomorphisms is the 2-category of fractions of the 2-category of internal groupoids and internal functors in *Lie* with respect to weak equivalences.

The paper is organized as follows:

- In Section 2 we recall some basic facts on bicategories of fractions established by D. Pronk in [16]. We then revisit the right calculus of fractions for classes of 1-cells using bipullbacks.

- In Section 3 we show that, for a category $\mathcal{C}$ with finite limits, the 2-category $Grpd(\mathcal{C})$ of internal groupoids and internal functors has bipullbacks. More precisely, we show that the standard homotopy pullback in $Grpd(\mathcal{C})$ also satisfies the universal property of a bipullback.

- Using bipullbacks, we show in Section 4 that if $\mathcal{C}$ is regular, then the class of weak equivalences in $Grpd(\mathcal{C})$ has a right calculus of fractions.

- In Section 5 we refine the previous result showing that if $\mathcal{C}$ is regular and protomodular, then weak equivalences satisfy the "2 $\Rightarrow$ 3" property and therefore they are a bipullback congruence, a notion inspired by Bénabou's approach to categories of fractions (see [4]).

- In the last two sections we choose as base category $\mathcal{C}$ the category of groups (Section 6) and the category of Lie K-algebras (Section 7) and we prove the results announced above.

Since *Grp* and *Lie* are Mal'cev categories, internal categories coincide with internal groupoids (see [11]). This is the reason why we restrict our attention to internal groupoids.

Let me finish with some comments. The result established in Section 6 is not at all a surprise. In fact, if we work with isomorphism classes of internal functors, then Proposition 6.4 becomes a result on categories of fractions (not on 2-categories of fractions) quite easy to prove directly and also easy to deduce using the Quillen model structures studied in [13] and in [15]. So, in my opinion, what is interesting is not the result *per se* but the fact that the 2-categorical nature of its proof requires the use of bipullbacks, whereas other kinds of 2-dimensional limits (like homotopy pullbacks) are not convenient in this context (see the Introduction in [4] for some comments on bilimits). Concerning the analogous result for Lie algebras stated in Section 7, I think it is interesting for a completely different reason. The notion of monoidal functor is a well-established one, whereas the notion of homomorphism of Lie 2-algebras is much more recent, so Proposition 7.4 could help to understand the 2-dimensional theory of Lie algebras.

Notation: the composite of $f \colon A \to B$ and $g \colon B \to C$ is written $f \cdot g$ or fg.

Terminology: bicategory means bicategory with invertible 2-cells.

2. Bicategories of fractions

2.1 Categories of fractions have been introduced by P. Gabriel and M. Zisman in [14] (see also Ch. 5 in [5]). If $\mathcal{C}$ is a category and Σ a class of arrows in $\mathcal{C}$, the category of fractions of $\mathcal{C}$ with respect to Σ is a functor

$$P_\Sigma \colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$$

universal among all functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{A}$ such that $\mathcal{F}(s)$ is an isomorphism for all $s \in \Sigma$. This can be restated saying that for every category $\mathcal{A}$

$$P_\Sigma \cdot - \;:\; Funct(\mathcal{C}[\Sigma^{-1}], \mathcal{A}) \to Funct_\Sigma(\mathcal{C}, \mathcal{A})$$

is an equivalence of categories, where $Funct_\Sigma(\mathcal{C}, \mathcal{A})$ is the category of functors making the elements of Σ invertible. If the class Σ has a right calculus of fractions, then $\mathcal{C}[\Sigma^{-1}]$ has a quite simple description:

Proposition 2.2 *(Gabriel-Zisman) Assume that Σ satisfies the following conditions:*

CF1. Σ contains all identities;

CF2. Σ is closed under composition;

CF3. For every pair $f\colon A \to B \leftarrow C\colon g$ with $g \in \Sigma$ there exist $g'\colon P \to A$ and $f'\colon P \to C$ such that $g' \cdot f = f' \cdot g$ and $g' \in \Sigma$;

CF4. If a pair of parallel arrows is coequalized by an element of Σ, then it is also equalized by an element of Σ.

Then the objects of $\mathcal{C}[\Sigma^{-1}]$ are those of $\mathcal{C}$ and an arrow from A to B is $\mathcal{C}[\Sigma^{-1}]$ is a class of spans

$$A \xleftarrow{\;s\;} I \xrightarrow{\;f\;} B$$

with $s \in \Sigma$. Two spans (s, I, f) and (s', I', f') are equivalent if there exist arrows x, x' in $\mathcal{C}$ such that $x \cdot s = x' \cdot s' \in \Sigma$ and $x \cdot f = x' \cdot f'$.

The analogous problem for bicategories has been solved by D. Pronk in [16]. For an introduction to bicategories see [3] or Ch. 7 in [5] where 2-categories are also discussed.

Definition 2.3 *(Pronk)* Let $\mathcal{B}$ be a bicategory and Σ a class of 1-cells in $\mathcal{B}$. The bicategory of fractions of $\mathcal{B}$ with respect to Σ is a homomorphism of bicategories

$$P_\Sigma\colon \mathcal{B} \to \mathcal{B}[\Sigma^{-1}]$$

universal among all homomorphisms $\mathcal{F}\colon \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$. This can be restated saying that for every bicategory $\mathcal{A}$

$$P_\Sigma \cdot - \; : \; Hom(\mathcal{B}[\Sigma^{-1}], \mathcal{A}) \to Hom_\Sigma(\mathcal{B}, \mathcal{A})$$

is a biequivalence of bicategories, where $Hom_\Sigma(\mathcal{B}, \mathcal{A})$ is the bicategory of those homomorphisms $\mathcal{F}$ such that $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$.

Definition 2.4 *(Pronk)* Let $\mathcal{B}$ be a bicategory and Σ a class of 1-cells in $\mathcal{B}$. The class Σ has a right calculus of fractions if the following conditions hold:

BF1. Σ contains all equivalences;

BF2. Σ is closed under composition;

BF3. For every pair $F\colon \mathbb{A} \to \mathbb{B} \leftarrow \mathbb{C}\colon G$ with $G \in \Sigma$ there exist $G'\colon \mathbb{P} \to \mathbb{A}$, $F'\colon \mathbb{P} \to \mathbb{C}$ and $\varphi\colon G' \cdot F \Rightarrow F' \cdot G$ with $G' \in \Sigma$;

BF4. For every $\alpha\colon F \cdot W \Rightarrow G \cdot W$ with $W \in \Sigma$ there exist $V \in \Sigma$ and $\beta\colon V \cdot F \Rightarrow V \cdot G$ such that $V \cdot \alpha = \beta \cdot W$, and for any other $V' \in \Sigma$ and $\beta'\colon V' \cdot F \Rightarrow V' \cdot G$ such that $V' \cdot \alpha = \beta' \cdot W$ there exist U, U' and $\varepsilon\colon U \cdot V \Rightarrow U' \cdot V'$ such that $U \cdot V \in \Sigma$ and

$$
\begin{array}{ccc}
U \cdot V \cdot F & \xrightarrow{\;U\cdot\beta\;} & U \cdot V \cdot G \\
{\scriptstyle \varepsilon\cdot F}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon\cdot G} \\
U' \cdot V' \cdot F & \xrightarrow[\;U'\cdot\beta'\;]{} & U' \cdot V' \cdot G
\end{array}
$$

commutes;

BF5. If $\alpha\colon F \Rightarrow G$ is a 2-cell, then $F \in \Sigma$ if and only if $G \in \Sigma$.

If the class Σ has a right calculus of fractions, the bicategory $\mathcal{B}[\Sigma^{-1}]$ can be described in a way similar to that recalled in Proposition 2.2. Here we do not give full details because what we will use in Sections 6 and 7 is the following useful result:

Proposition 2.5 *(Pronk) Let $\mathcal{B}$ be a bicategory and Σ a class of 1-cells in $\mathcal{B}$ which has a right calculus of fractions. Consider a bifunctor $\mathcal{F}\colon \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F}(S)$ is an equivalence for all $S \in \Sigma$ and let $\widehat{\mathcal{F}}\colon \mathcal{B}[\Sigma^{-1}] \to \mathcal{A}$ be its extension. Then $\widehat{\mathcal{F}}$ is a biequivalence provided that $\mathcal{F}$ satisfies the following conditions:*

EF1. $\mathcal{F}$ is surjective up to equivalence on objects;

EF2. $\mathcal{F}$ is full and faithful on 2-cells;

EF3. For every 1-cell F in $\mathcal{A}$ there exist 1-cells G and W in $\mathcal{B}$ with W in Σ and a 2-cell $\mathcal{F}(G) \Rightarrow \mathcal{F}(W) \cdot F$.

(In [16] it is stated that conditions EF1-EF3 are also necessary for $\widehat{\mathcal{F}}$ being a biequivalence. This is not true, as proved by M. Dupont in [12].)

2.6 Recall that a diagram

$$
\begin{array}{ccc}
\mathbb{P} & \xrightarrow{\ F'\ } & \mathbb{C} \\
{\scriptstyle G'}\downarrow & \overset{\varphi}{\diagup} & \downarrow{\scriptstyle G} \\
\mathbb{A} & \xrightarrow[\ F\ ]{} & \mathbb{B}
\end{array}
$$

in a bicategory $\mathcal{B}$ is a bipullback of F and G if for any other diagram

$$
\begin{array}{ccc}
\mathbb{X} & \xrightarrow{\ K\ } & \mathbb{C} \\
{\scriptstyle H}\downarrow & \overset{\psi}{\diagup} & \downarrow{\scriptstyle G} \\
\mathbb{A} & \xrightarrow[\ F\ ]{} & \mathbb{B}
\end{array}
$$

there exists a fill-in, that is a triple $(L\colon \mathbb{X} \to \mathbb{P},\ \alpha\colon L \cdot G' \Rightarrow H,\ \beta\colon L \cdot F' \Rightarrow K)$ such that

$$
\begin{array}{ccc}
L \cdot G' \cdot F & \xrightarrow{\ L \cdot \varphi\ } & L \cdot F' \cdot G \\
{\scriptstyle \alpha \cdot F}\downarrow & & \downarrow{\scriptstyle \beta \cdot G} \\
H \cdot F & \xrightarrow[\ \psi\ ]{} & K \cdot G
\end{array}
$$

commutes, and for any other fill-in (L', α', β') there exists a unique $\lambda \colon L' \Rightarrow L$ such that

$$L' \cdot G' \xrightarrow{\lambda \cdot G'} L \cdot G' \qquad L' \cdot F' \xrightarrow{\lambda \cdot F'} L \cdot F'$$

with α', α to H and β', β to K

commute.

Remark 2.7 1. Bipullbacks are determined uniquely up to equivalence.

2. A 1-cell $W \colon \mathbb{B} \to \mathbb{A}$ is called full and faithful if for every $\mathbb{X}$ the hom-functor

$$\mathcal{B}(\mathbb{X}, W) \colon \mathcal{B}(\mathbb{X}, \mathbb{B}) \to \mathcal{B}(\mathbb{X}, \mathbb{A})$$

is full and faithful in the usual sense. Consider now the following diagrams, the first one being a bipullback,

$$
\begin{array}{ccc}
\mathbb{K} \xrightarrow{W_2} \mathbb{B} & & \mathbb{B} \xrightarrow{id} \mathbb{B} \\
W_1 \downarrow \quad {\scriptstyle w} \quad \downarrow W & & id \downarrow \quad {\scriptstyle W} \quad \downarrow W \\
\mathbb{B} \xrightarrow{W} \mathbb{A} & & \mathbb{B} \xrightarrow{W} \mathbb{A}
\end{array}
$$

Let $(D_W \colon \mathbb{B} \to \mathbb{K}, \delta_1 \colon D_W \cdot W_1 \Rightarrow id, \delta_2 \colon D_W \cdot W_2 \Rightarrow id)$ be the fill-in of the second diagram through the first one. Then W is full and faithful iff the second diagram is a bipullback iff the diagonal D_W is an equivalence.

Proposition 2.8 *Let $\mathcal{B}$ be a bicategory with bipullbacks and Σ a class of 1-cells in $\mathcal{B}$. Assume that Σ satisfies the following conditions:*

BP1. Σ contains all equivalences;

BP2. Σ is closed under composition;

BP3. Σ is stable under bipullbacks;

BP4. If W is in Σ, then the diagonal D_W is in Σ;

BP5. If $\alpha \colon F \Rightarrow G$ is a 2-cell, then $F \in \Sigma$ if and only if $G \in \Sigma$.

Then Σ has a right calculus of fractions.

Proof. Clearly BP3 implies BF3. We have to show that BF4 holds. Consider the following diagrams, the first one being a bipullback,

$$
\begin{array}{ccc}
\mathbb{K} \xrightarrow{W_2} \mathbb{B} & \mathbb{B} \xrightarrow{id} \mathbb{B} & \mathbb{C} \xrightarrow{G} \mathbb{B} \\
W_1 \downarrow \;\; {}^{w} \nearrow \;\; \downarrow W & id \downarrow \;\; {}^{W} \nearrow \;\; \downarrow W & F \downarrow \;\; {}^{\alpha} \nearrow \;\; \downarrow W \\
\mathbb{B} \xrightarrow{W} \mathbb{A} & \mathbb{B} \xrightarrow{W} \mathbb{A} & \mathbb{B} \xrightarrow{W} \mathbb{A}
\end{array}
$$

Let $(D_W \colon \mathbb{B} \to \mathbb{K}, \delta_1, \delta_2)$ be the fill-in of the second diagram through the first one, and $(H \colon \mathbb{C} \to \mathbb{K}, \alpha_1, \alpha_2)$ the fill-in of the third diagram through the first one. Consider also the bipullback

$$
\begin{array}{ccc}
\mathbb{D} & \xrightarrow{V} & \mathbb{C} \\
L \downarrow & {}^{\varphi} \nearrow & \downarrow H \\
\mathbb{B} & \xrightarrow{D_W} & \mathbb{K}
\end{array}
$$

and define $\beta \colon V \cdot F \Rightarrow V \cdot G$ as follows

$$
VF \xrightarrow{V\alpha_1^{-1}} VHW_1 \xrightarrow{\varphi^{-1}W_1} LD_W W_1 \xrightarrow{L\delta_1} L \xrightarrow{L\delta_2^{-1}} LD_W W_2 \xrightarrow{\varphi W_2} VHW_2 \xrightarrow{V\alpha_2} VG
$$

Observe that since $W \in \Sigma$, then $D_W \in \Sigma$ by BP4, and then $V \in \Sigma$ by BP3. Moreover, the condition $V \cdot \alpha = \beta \cdot W$ follows from the fill-in condition on $(D_W, \delta_1, \delta_2)$ and (H, α_1, α_2).

Let $\beta' \colon V' \cdot F \Rightarrow V' \cdot G$ be such that $V' \in \Sigma$ and $V' \cdot \alpha = \beta' \cdot W$. We obtain two fill-in of

$$
\begin{array}{ccc}
\mathbb{D}' & \xrightarrow{V' \cdot F} & \mathbb{B} \\
V' \cdot F \downarrow & {}^{V'FW} \nearrow & \downarrow W \\
\mathbb{B} & \xrightarrow{W} & \mathbb{A}
\end{array}
$$

through the bipullback $(\mathbb{K}, W_1, W_2, w)$: the first one is

$$
(\, \mathbb{D}' \xrightarrow{V'} \mathbb{C} \xrightarrow{F} \mathbb{B} \xrightarrow{D_W} \mathbb{K} \,,\, V' \cdot F \cdot \delta_1, V' \cdot F \cdot \delta_2 \,)
$$

and the second one is

$$
(\, \mathbb{D}' \xrightarrow{V'} \mathbb{C} \xrightarrow{H} \mathbb{K} \,,\, V' \cdot \alpha_1, \, V'HW_2 \xrightarrow{V'\alpha_2} V'G \xrightarrow{(\beta')^{-1}} V'F \,)
$$

By the universal property of $(\mathbb{K}, W_1, W_1, w)$, there exists a unique $\beta^* \colon V' \cdot F \cdot D_W \Rightarrow V' \cdot H$ such that

$$V' \cdot F \cdot D_W \cdot W_1 \xrightarrow{\beta^* \cdot W_1} V' \cdot H \cdot W_1$$
$$\searrow{\scriptstyle V' \cdot F \cdot \delta_1} \qquad \swarrow{\scriptstyle V' \cdot \alpha_1}$$
$$V' \cdot F$$

and

$$\begin{array}{ccc} V' \cdot F \cdot D_W \cdot W_2 & \xrightarrow{\beta^* \cdot W_2} & V' \cdot H \cdot W_2 \\ {\scriptstyle V' \cdot F \cdot \delta_2}\downarrow & & \downarrow{\scriptstyle V' \cdot \alpha_2} \\ V' \cdot F & \xrightarrow[\beta']{} & V' \cdot G \end{array}$$

commute. Let $(U \colon \mathbb{D}' \to \mathbb{D}, \eta \colon U \cdot L \Rightarrow V' \cdot F, \varepsilon \colon U \cdot V \Rightarrow V')$ be the fill-in of

$$\begin{array}{ccc} \mathbb{D}' & \xrightarrow{V'} & \mathbb{C} \\ {\scriptstyle V' \cdot F}\downarrow & \nearrow{\scriptstyle \beta^*} & \downarrow{\scriptstyle H} \\ \mathbb{B} & \xrightarrow[D_W]{} & \mathbb{K} \end{array}$$

through the bipullback $(\mathbb{D}, L, V, \varphi)$. If we choose $U' = id$, we have $\varepsilon \colon U \cdot V \Rightarrow U' \cdot V'$. Since $V' \in \Sigma$, then also $U' \cdot V'$ and $U \cdot V$ are in Σ because of BP1, BP2 and BP5. It remains to check the compatibility of ε, β and β' as in BF4, but this is just a diagram chasing. $\blacksquare$

3. Bipullbacks in $Grpd(\mathcal{C})$

The aim of this section is to prove the following result:

Proposition 3.1 *Let $\mathcal{C}$ be a category with finite limits, and let $Grpd(\mathcal{C})$ be the 2-category of internal groupoids, internal functors and internal natural transformations in $\mathcal{C}$. The 2-category $Grpd(\mathcal{C})$ has bipullbacks.*

3.2 Let us fix notation (details can be found in Ch. 7 of [5] or in Appendix 3 of [7]):

- An internal groupoid $\mathbb{C}$ is represented by

$$C_1 \times_{c,d} C_1 \xrightarrow{\; m \;} C_1 \underset{c}{\overset{d}{\underset{\longleftarrow}{\rightrightarrows}}} C_0 \qquad C_1 \xrightarrow{\; i \;} C_1$$

where the following diagram is a pullback

$$
\begin{array}{ccc}
C_1 \times_{c,d} C_1 & \xrightarrow{\;\pi_2\;} & C_1 \\
{\scriptstyle \pi_1} \downarrow & & \downarrow {\scriptstyle d} \\
C_1 & \xrightarrow{\;c\;} & C_0
\end{array}
$$

- An internal functor $F \colon \mathbb{C} \to \mathbb{D}$ is represented by

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\;F_1\;} & D_1 \\
{\scriptstyle d} \downdownarrows {\scriptstyle c} & & {\scriptstyle d} \downdownarrows {\scriptstyle c} \\
C_0 & \xrightarrow{\;F_0\;} & D_0
\end{array}
$$

- An internal natural transformation $\alpha \colon F \Rightarrow G \colon \mathbb{C} \to \mathbb{D}$ is represented by

$$
\begin{array}{ccc}
C_1 & \overset{F_1}{\underset{G_1}{\rightrightarrows}} & D_1 \\
{\scriptstyle d} \downdownarrows {\scriptstyle c} \quad {\scriptstyle \alpha} \nearrow & & {\scriptstyle d} \downdownarrows {\scriptstyle c} \\
C_0 & \overset{F_0}{\underset{G_0}{\rightrightarrows}} & D_0
\end{array}
$$

3.3 It is helpful to start recalling that in $Grpd(Set)$ bipullbacks are comma-squares. With the notations of 2.6:

- an object in $\mathbb{P}$ is a triple $(a_0 \in A_0, b_1 \colon F_0(a_0) \to G_0(c_0), c_0 \in C_0)$,

- an arrow from (a_0, b_1, c_0) to (a_0', b_1', c_0') is a pair of arrows $(a_1 \colon a_0 \to a_0', c_1 \colon c_0 \to c_0')$ such that $F_1(a_1) \cdot b_1' = b_1 \cdot G_1(c_1)$,

- $G' \colon \mathbb{P} \to \mathbb{A}$ and $F' \colon \mathbb{P} \to \mathbb{C}$ are the obvious projections, and $\varphi(a_0, b_1, c_0) = b_1$,

- $L_0(x_0) = (H_0(x_0), \psi(x_0), K_0(x_0))$, $L_1(x_1) = (H_1(x_1), K_1(x_1))$, $\alpha = id$ and $\beta = id$,

- $\lambda(x_0) = (\alpha'(x_0), \beta'(x_0))$.

3.4 The description of bipullbacks in $Grpd(Set)$ recalled in 3.3 indicates that the first step to obtain bipullbacks in $Grpd(\mathcal{C})$ is to construct from an internal groupoid $\mathbb{B}$ a new internal groupoid $\vec{\mathbb{B}}$ whose objects are arrows in $\mathbb{B}$ and whose arrows are commutative squares in $\mathbb{B}$. The construction of $\vec{\mathbb{B}}$ is quite standard:

$$\vec{\mathbb{B}} = \left(\vec{B}_1 \times_{\vec{c},\vec{d}} \vec{B}_1 \xrightarrow{\vec{m}} \vec{B}_1 \underset{\vec{c}}{\overset{\vec{d}}{\underset{\longleftarrow}{\rightrightarrows}}} B_1 \qquad \vec{B}_1 \xrightarrow{\vec{i}} \vec{B}_1 \right)$$

- $\vec{B}_1$ is defined by the following pullback

$$
\begin{array}{ccc}
\vec{B}_1 & \xrightarrow{m_2} & B_1 \times_{c,d} B_1 \\
{\scriptstyle m_1}\downarrow & & \downarrow{\scriptstyle m} \\
B_1 \times_{c,d} B_1 & \xrightarrow{m} & B_1
\end{array}
$$

- $\vec{d} = m_1 \cdot \pi_1$ and $\vec{c} = m_2 \cdot \pi_2$,

- $\vec{e}$ is the unique factorization through $\vec{B}_1$ of the following commutative diagram

$$
\begin{array}{ccccc}
B_1 & \xrightarrow{<d,1>} & B_0 \times B_1 & \xrightarrow{e \times 1} & B_1 \times_{c,d} B_1 \\
{\scriptstyle <1,c>}\downarrow & & & & \downarrow{\scriptstyle m} \\
B_1 \times B_0 & \xrightarrow{1 \times e} & B_1 \times_{c,d} B_1 & \xrightarrow{m} & B_1
\end{array}
$$

- we leave to the reader the task of describing $\vec{m}$ and $\vec{i}$.

3.5 The internal groupoid $\vec{\mathbb{B}}$ is equipped with two internal functors $\delta, \gamma \colon \vec{\mathbb{B}} \to \mathbb{B}$ specified by

$$
\begin{array}{ccc}
\vec{B}_1 \xrightarrow{\delta_1 = m_2 \cdot \pi_1} B_1 & \qquad & \vec{B}_1 \xrightarrow{\gamma_1 = m_1 \cdot \pi_2} B_1 \\
{\scriptstyle \vec{d}}\downdownarrows{\scriptstyle \vec{c}} \quad\quad d\downdownarrows c & & {\scriptstyle \vec{d}}\downdownarrows{\scriptstyle \vec{c}} \quad\quad d\downdownarrows c \\
\vec{B}_1 \xrightarrow{\delta_0 = d} B_0 & & \vec{B}_1 \xrightarrow{\gamma_0 = c} B_0
\end{array}
$$

and it tourns out that to give an internal natural transformation $\alpha\colon F \Rightarrow G\colon \mathbb{A} \to \mathbb{B}$ is the same as giving an internal functor $\alpha\colon \mathbb{A} \to \vec{\mathbb{B}}$ such that $\alpha \cdot \delta = F$ and $\alpha \cdot \gamma = G$. Indeed, the internal functor α is specified by

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\alpha_1} & \vec{B}_1 \\
d \Big\| c & & \vec{d} \,\Big\| \vec{c} \\
A_0 & \xrightarrow{\alpha} & B_1
\end{array}
$$

where α_1 is the unique factorization through $\vec{B}_1$ of the following commutative diagram

$$
\begin{array}{ccccc}
A_1 & \xrightarrow{\langle 1,c \rangle} & A_1 \times A_0 & \xrightarrow{F_1 \times \alpha} & B_1 \times_{c,d} B_1 \\
{\scriptstyle \langle d,1 \rangle}\Big\downarrow & & & & \Big\downarrow {\scriptstyle m} \\
A_0 \times A_1 & \xrightarrow{\alpha \times G_1} & B_1 \times_{c,d} B_1 & \xrightarrow{\quad m \quad} & B_1
\end{array}
$$

3.6 We are ready to prove Proposition 3.1. We use the notations of 2.6.

Proof. Given $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{C} \to \mathbb{B}$ in $Grpd(\mathcal{C})$, a bipullback

$$
\begin{array}{ccc}
\mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\
{\scriptstyle G'}\Big\downarrow & {\scriptstyle \varphi}\ \nearrow\!\!\!\!\!/ & \Big\downarrow {\scriptstyle G} \\
\mathbb{A} & \xrightarrow{F} & \mathbb{B}
\end{array}
$$

is given by the following limit in $Grpd(\mathcal{C})$ (recall that $Grpd(\mathcal{C})$ has limits computed componentwise in $\mathcal{C}$)

$$
\begin{array}{ccccc}
 & & \mathbb{P} & & \\
 & \swarrow^{G'} & \Big\downarrow^{\varphi} & \searrow^{F'} & \\
\mathbb{A} & & \vec{\mathbb{B}} & & \mathbb{C} \\
 & \searrow_{F}\ \swarrow_{\delta} & & \searrow_{\gamma}\ \swarrow_{G} & \\
 & \mathbb{B} & & \mathbb{B} &
\end{array}
$$

Indeed, any diagram

$$
\begin{array}{ccc}
\mathbb{X} & \xrightarrow{K} & \mathbb{C} \\
H \downarrow & \psi \nearrow & \downarrow G \\
\mathbb{A} & \xrightarrow{F} & \mathbb{B}
\end{array}
$$

produces a commutative diagram

$$
\begin{array}{ccccc}
 & & \mathbb{X} & & \\
 & \overset{H}{\swarrow} & \downarrow \psi & \overset{K}{\searrow} & \\
\mathbb{A} & & \vec{\mathbb{B}} & & \mathbb{C} \\
 & \overset{F}{\searrow} \; \overset{\delta}{\swarrow} & & \overset{\gamma}{\searrow} \; \overset{G}{\swarrow} & \\
 & \mathbb{B} & & \mathbb{B} &
\end{array}
$$

so that following the universal property of $\mathbb{P}$ as a limit there exists a unique $L \colon \mathbb{X} \to \mathbb{P}$ such that $L \cdot G' = H$, $L \cdot F' = K$ and $L \cdot \varphi = \psi$. (In other words, $(\mathbb{P}, G', F', \varphi)$ is the standard homotopy pullback of F and G.)

Clearly, $(L, \alpha = id, \beta = id)$ is a fill-in of $(\mathbb{X}, H, K, \psi)$ through $(\mathbb{P}, G', F', \varphi)$. Let (L', α', β') be another fill-in of $(\mathbb{X}, H, K, \psi)$ through $(\mathbb{P}, G', F', \varphi)$. We have to show that there exists a unique $\lambda \colon L' \Rightarrow L$ such that $\lambda \cdot G' = \alpha'$ and $\lambda \cdot F' = \beta'$. Define:

- τ_1 to be the unique factorization through $B_1 \times_{c,d} B_1$ of the following diagram

$$
\begin{array}{ccccc}
X_0 & \xrightarrow{\beta'} & C_1 & \xrightarrow{G_1} & B_1 \\
L'_0 \downarrow & & & & \downarrow d \\
P_0 & \xrightarrow{\varphi} & B_1 & \xrightarrow{c} & B_0
\end{array}
$$

- τ_2 to be the unique factorization through $B_1 \times_{c,d} B_1$ of the following diagram

$$
\begin{array}{ccccc}
X_0 & \xrightarrow{\hspace{2em}\psi\hspace{2em}} & & & B_1 \\
\alpha' \downarrow & & & & \downarrow d \\
A_1 & \xrightarrow{F_1} & B_1 & \xrightarrow{c} & B_0
\end{array}
$$

- τ to be the unique factorization through $\vec{B}_1$ of the following diagram

$$
\begin{array}{ccc}
X_0 & \xrightarrow{\ \tau_2\ } & B_1 \times_{c,d} B_1 \\
{\scriptstyle \tau_1}\big\downarrow & & \big\downarrow{\scriptstyle m} \\
B_1 \times_{c,d} B_1 & \xrightarrow[\ m\ ]{} & B_1
\end{array}
$$

Finally, λ is the unique factorization through P_1 of the following diagram

$$
\begin{array}{ccccc}
 & & X_0 & & \\
 & {\scriptstyle \alpha'}\swarrow & \big\downarrow{\scriptstyle \tau} & \searrow{\scriptstyle \beta'} & \\
A_1 & & \vec{B}_1 & & C_1 \\
 & {\scriptstyle m_2\cdot\pi_1}\searrow\ \swarrow & & \searrow\ \swarrow{\scriptstyle m_1\cdot\pi_2} & \\
{\scriptstyle F_1}\quad & B_1 & & B_1 & \quad{\scriptstyle G_1}
\end{array}
$$

Clearly, $\lambda \cdot G' = \alpha'$ and $\lambda \cdot F' = \beta'$. To check that $\lambda \cdot d = L_0'$ and $\lambda \cdot c = L_0$, the naturality of λ, and its uniqueness is a diagram chasing using that $\{G_1', \varphi_1, F_1'\}$, $\{m_1, m_2\}$ and $\{\pi_1, \pi_2\}$ are jointly monomorphic. ∎

4. Weak equivalences in $Grpd(\mathcal{C})$

Definition 4.1 *(Bunge-Paré)* Let $F \colon \mathbb{C} \to \mathbb{B}$ be in $Grpd(\mathcal{C})$.

1. F is essentially surjective on objects if

$$
C_0 \times_{F_0,d} D_1 \xrightarrow{\ t_2\ } D_1 \xrightarrow{\ c\ } D_0
$$

   is a regular epimorphism, where t_2 is given by the following pullback

$$
\begin{array}{ccc}
C_0 \times_{F_0,d} D_1 & \xrightarrow{\ t_2\ } & D_1 \\
{\scriptstyle t_1}\big\downarrow & & \big\downarrow{\scriptstyle d} \\
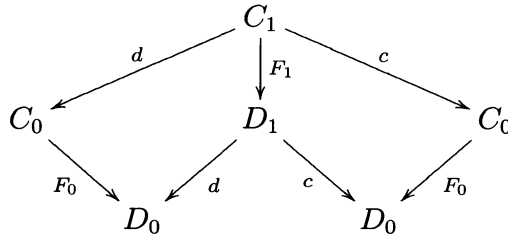C_0 & \xrightarrow[\ F_0\ ]{} & D_0
\end{array}
$$

2. F is a weak equivalence if it is full and faithful (see 2.7) and essentially surjective on objects.

The previous definition is due to M. Bunge and R. Paré (see [10]). In [13] a more general notion of weak equivalence involving a Grothendieck topology on $\mathcal{C}$ has been considered. Since in Sections 6 and 7 the base category $\mathcal{C}$ is regular, I adopt for the moment the definition of Bunge and Paré. More on this point is contained in 5.10.

Next lemma is well-known and we only sketch the proof.

Lemma 4.2 *Let $F\colon \mathbb{C} \to \mathbb{D}$ be in $Grpd(\mathcal{C})$.*

1. *F is full and faithful if and only if the following is a limit diagram*

2. *F is an equivalence if and only if it is full and faithful and*

$$C_0 \times_{F_0,d} D_1 \xrightarrow{t_2} D_1 \xrightarrow{c} D_0$$

   is a split epimorphism.

Proof. 1. If the diagram is a limit diagram and $\alpha\colon G\cdot F \Rightarrow H\cdot F\colon \mathbb{X} \to \mathbb{D}$ is an internal natural transformation, then $\alpha \cdot d = G_0 \cdot F_0$ and $\alpha \cdot c = H_0 \cdot F_0$. By the universal property of C_1 we get a unique $\beta\colon X_0 \to C_1$ such that $\beta \cdot d = G_0$, $\beta \cdot d = H_0$ and $\beta \cdot F_1 = \alpha$. So we have $\beta\colon G \Rightarrow H$ such that $\beta \cdot F = \alpha$. (The naturality of β follows from that of α.) Conversely, any commutative diagram
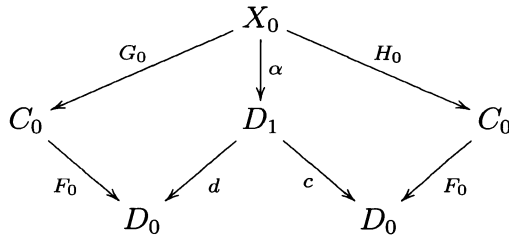
gives rise to internal functors $G, H \colon \mathbb{X} \to \mathbb{C}$ with discrete domain

$$
\begin{array}{ccc}
X_0 & \underset{H_0 \cdot e}{\overset{G_0 \cdot e}{\rightrightarrows}} & C_1 \\
1 \Big\downarrow\Big\downarrow 1 & & d \Big\downarrow\Big\downarrow c \\
X_0 & \underset{H_0}{\overset{G_0}{\rightrightarrows}} & C_0
\end{array}
$$

and to an internal natural transformation $\alpha \colon G \cdot F \Rightarrow H \cdot F$. To give an internal natural transformation $\beta \colon G \Rightarrow H$ such that $\beta \cdot F = \alpha$ means precisely to give a factorization $\beta \colon X_0 \to C_1$ of (G_0, α, H_0) through (d, F_1, c).

2. Let F be an equivalence and consider an internal natural transformation $\beta \colon G \cdot F \Rightarrow Id_{\mathcal{D}}$. Since $\beta \cdot d = G_0 \cdot F_0$, there exists a unique $j \colon D_0 \to C_0 \times_{F_0, d} D_1$ such that $j \cdot t_1 = G_0$ and $j \cdot t_2 = \beta$. Therefore $j \cdot t_2 \cdot c = \beta \cdot c = id$.

Conversely, if $j \colon D_0 \to C_0 \times_{F_0, d} D_1$ such that $j \cdot t_2 \cdot c = id$, we can construct a quasi-inverse internal functor $G \colon \mathbb{D} \to \mathbb{C}$ as follows: first define G_0 by

$$
G_0 = j \cdot t_1 \colon D_0 \to C_0 \times_{F_0, d} D_1 \to C_0
$$

Then, define $j_1 \colon D_1 \to D_1$ by

$$
j_1 = \;< d \cdot j \cdot t_2, 1, c \cdot j \cdot t_2 \cdot i > \cdot (m \times 1) \cdot m \colon D_1 \to D_1 \times_{c, d} D_1 \times_{c, d} D_1 \to D_1
$$

Finally, since F is full and faithful, by the first part of the lemma we get a unique arrow $G_1 \colon D_1 \to C_1$ such that $G_1 \cdot d = d \cdot G_0$, $G_1 \cdot F_1 = j_1$ and $G_1 \cdot c = c \cdot G_0$. $\blacksquare$

Corollary 4.3 *Every equivalence in Grpd($\mathcal{C}$) is a weak equivalence. The converse is true provided that in $\mathcal{C}$ the axiom of choice holds (i.e., regular epimorphisms split).*

4.4 Regular categories have been introduced by M. Barr in [2] (see also Ch. 2 in [6]). In a regular category regular epimorphisms behave well: they are closed under composition and finite products, stable under pullbacks, and if a composite arrow $f \cdot g$ is a regular epimorphism, then g is a regular epimorphism. It follows that if $F \colon \mathbb{C} \to \mathbb{D}$ is in Grpd($\mathcal{C}$) with $\mathcal{C}$ regular and if F_0 is a regular epimorphism, then F is essentially surjective on objects.

Proposition 4.5 *Let C be a regular category and let Σ be the class of weak equivalences in $Grpd(C)$. Then Σ has a right calculus of fractions.*

Proof. Since by Proposition 3.1 $Grpd(C)$ has bipullbacks, to prove that Σ has a right calculus of fractions we check conditions BP1–BP5 in Proposition 2.8.

BP1 is given by Corollary 4.3, BP4 follows from 2.7 and BP5 is an exercise for the reader.

BP2: full and faithful internal functors are closed under composition because so they are in $Grpd(Set)$. Assume now that $F \colon \mathbb{A} \to \mathbb{B}$ and $G \colon \mathbb{B} \to \mathbb{C}$ are essentially surjective. Consider the following pullbacks

$$
\begin{array}{ccc}
A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 \\
{\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d} \\
A_0 & \xrightarrow{F_0} & B_0
\end{array}
\qquad
\begin{array}{ccc}
B_0 \times_{G_0,d} C_1 & \xrightarrow{t_2} & C_1 \\
{\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle d} \\
B_0 & \xrightarrow{G_0} & C_0
\end{array}
\qquad
\begin{array}{ccc}
A_0 \times_{F_0 G_0,d} C_1 & \xrightarrow{\tau_2} & C_1 \\
{\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle d} \\
A_0 & \xrightarrow{F_0 \cdot G_0} & C_0
\end{array}
$$

The essential surjectivity of $F \cdot G$ comes from the commutativity of the following diagram

$$
\begin{array}{ccc}
A_0 \times_{F_0,d} B_1 \times_{G_1 c,d} C_1 & \xrightarrow{t_2 \times 1} B_1 \times_{G_1 c,d} C_1 \xrightarrow{c \times 1} & B_0 \times_{G_0,d} C_1 \\
{\scriptstyle 1 \times G_1 \times 1}\downarrow & & \downarrow{\scriptstyle t_2} \\
A_0 \times_{F_0 G_0,d} C_1 \times_{c,d} C_1 & & C_1 \\
{\scriptstyle 1 \times m}\downarrow & & \downarrow{\scriptstyle c} \\
A_0 \times_{F_0 G_0,d} C_1 & \xrightarrow{\tau_2} C_1 \xrightarrow{c} & C_0
\end{array}
$$

BP3: full and faithful internal functors are stable under bipullbacks because so they are in $Grpd(Set)$ (use 3.3) and $Grpd(C)(\mathbb{X}, -) \colon Grpd(C) \to Grpd(Set)$ preserves bipullbacks. Consider now a bipullback

$$
\begin{array}{ccc}
\mathbb{P} & \xrightarrow{F'} & \mathbb{C} \\
{\scriptstyle G'}\downarrow & {\scriptstyle \varphi}\nearrow\!\!\!\!\!/ & \downarrow{\scriptstyle G} \\
\mathbb{A} & \xrightarrow{F} & \mathbb{B}
\end{array}
$$

and assume that F is essentially surjective. Following the description of $\mathbb{P}$ given at the beginning of 3.6, we have a limit diagram in $\mathcal{C}$
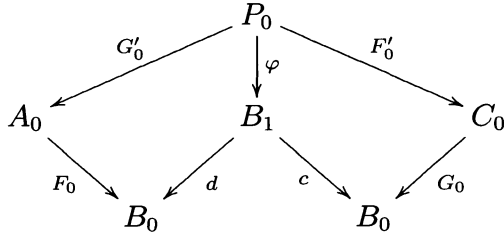
But such a limit can be obtained performing two pullbacks as follows
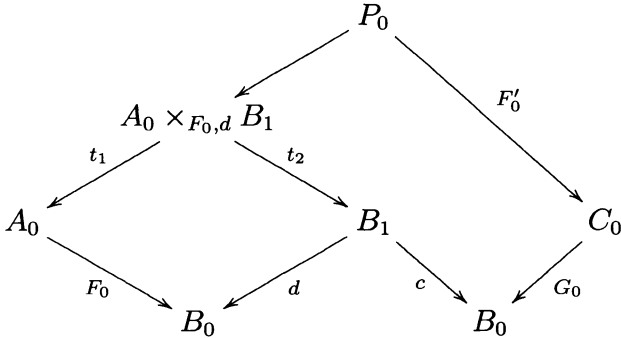
Since by assumption $t_2 \cdot c \colon A_0 \times_{F_0,d} B_1 \to B_1 \to B_0$ is a regular epimorphism, F_0' also is a regular epimorphism and then F' is essentially surjective (see 4.4). ∎

5. Bipullback congruences

Next definition is the direct bicategorical generalization of the notion of pullback congruence introduced by J. Bénabou in [4].

Definition 5.1 Let $\mathcal{B}$ be a bicategory with bipullbacks and Σ a class of 1-cells in $\mathcal{B}$. The class Σ is a bipullback congruence if the following conditions hold:

BC1. Σ contains all equivalences;

BC2. Σ satisfies the "$2 \Rightarrow 3$" property: let $F\colon \mathbb{C} \to \mathbb{D}$ and $G\colon \mathbb{D} \to \mathbb{E}$ be 1-cells in $\mathcal{B}$; if two of F, G and $F \cdot G$ are in Σ, then the third one is in Σ;

BC3. Σ is stable under bipullbacks;

BC4. If $\alpha\colon F \Rightarrow G$ is a 2-cell, then $F \in \Sigma$ if and only if $G \in \Sigma$.

Proposition 5.2 *Let $\mathcal{B}$ be a bicategory with bipullbacks. Any bipullback congruence has a right calculus of fractions.*

Proof. It is enough to prove that a bipullback congruence Σ satisfies condition BP3 in Proposition 2.8. Let $W\colon \mathbb{B} \to \mathbb{A}$ be in Σ and let $(D_W\colon \mathbb{B} \to \mathbb{K}, \delta_1\colon D_W \cdot W_1 \Rightarrow id, \delta_2\colon D_W \cdot W_2 \Rightarrow id)$ be the diagonal fill-in as in 2.7. By BC1, $id \in \Sigma$, and then by BC4 $D_W \cdot W_1 \in \Sigma$. Since by BC3 $W_1 \in \Sigma$, we conclude by BC2 that $D_W \in \Sigma$. $\blacksquare$

5.3 Protomodular categories have been introduced by D. Bourn in [8] (see also [7]). Since we are concerned only with regular categories, we can consider the next lemma, proved in [9], as a definition of protomodular category. This lemma makes also evident the analogy between bipullback congruences and regular protomodular categories: in a regular protomodular category pullbacks satisfies the "$2 \Rightarrow 3$" property. This analogy will be made precise in Proposition 5.5.

Lemma 5.4 *(Bourn-Gran) Let $\mathcal{C}$ be a regular category. The following conditions are equivalent:*

1. $\mathcal{C}$ is protomodular;

2. In any commutative diagram

where b is a regular epimorphism, if the left hand square and the outer rectangle are pullbacks, then the right hand square is a pullback.

Proposition 5.5 *Let $\mathcal{C}$ be a regual protomodular category. The class of weak equivalences in $Grpd(\mathcal{C})$ is a bipullback congruence.*

Let $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$ be in $Grpd(\mathcal{C})$. In order to prove Proposition 5.5 we need two lemmas on the shape of certains limits. The proof is routine.

Lemma 5.6 *Consider the pullbacks*

$$
\begin{array}{ccc}
A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 \\
t_1 \downarrow & & \downarrow d \\
A_0 & \xrightarrow{F_0} & B_0
\end{array}
\qquad
\begin{array}{ccc}
B_1 \times_{c,F_0} A_0 & \xrightarrow{s_2} & A_0 \\
s_1 \downarrow & & \downarrow F_0 \\
B_1 & \xrightarrow{c} & B_0
\end{array}
$$

and the commutative diagrams

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\quad c \quad} & A_0 \\
{\scriptstyle <d,F_1>}\downarrow & (1) & \downarrow F_0 \\
A_0 \times_{F_0,d} B_1 & \xrightarrow[t_2]{} B_1 \xrightarrow[c]{} & B_0
\end{array}
\qquad
\begin{array}{ccc}
A_1 & \xrightarrow{\quad d \quad} & A_0 \\
{\scriptstyle <F_1,c>}\downarrow & (2) & \downarrow F_0 \\
B_1 \times_{c,F_0} A_0 & \xrightarrow[s_1]{} B_1 \xrightarrow[d]{} & B_0
\end{array}
$$

Then $F\colon \mathbb{A} \to \mathbb{B}$ is full and faithful iff (1) is a pullback iff (2) is a pullback.

Lemma 5.7 *Consider the pullbacks*

$$
\begin{array}{ccc}
A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 \\
t_1 \downarrow & & \downarrow d \\
A_0 & \xrightarrow{F_0} & B_0
\end{array}
\quad
\begin{array}{ccc}
C_1 \times_{c,G_0} B_0 & \xrightarrow{s_2} & B_0 \\
s_1 \downarrow & & \downarrow G_0 \\
C_1 & \xrightarrow{c} & C_0
\end{array}
\quad
\begin{array}{ccc}
A_0 \times_{F_0G_0,d} C_1 & \xrightarrow{\tau_2} & C_1 \\
\tau_1 \downarrow & & \downarrow d \\
A_0 & \xrightarrow{F_0 \cdot G_0} & C_0
\end{array}
$$

and the commutative diagrams

$$
\begin{array}{ccc}
A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} B_1 \xrightarrow{<G_1,c>} & C_1 \times_{c,G_0} B_0 \\
& & \downarrow s_1 \\
t_1 \downarrow & (3) & C_1 \\
& & \downarrow d \\
A_0 & \xrightarrow{F_0} B_0 \xrightarrow{G_0} & C_0
\end{array}
\qquad
\begin{array}{ccc}
A_0 \times_{F_0,d} B_1 & \xrightarrow{t_2} & B_1 & \xrightarrow{c} & B_0 \\
{\scriptstyle 1\times G_1}\downarrow & (4) & & & \downarrow G_0 \\
A_0 \times_{F_0G_0,d} C_1 & \xrightarrow{\tau_2} & C_1 & \xrightarrow{c} & C_0
\end{array}
$$

Then (3) is a pullback iff (4) is a pullback.

5.8 We are ready to prove Proposition 5.5.

Proof. Let Σ be the class of weak equivalences in $Grpd(\mathcal{C})$. We have to show that condition BC2 holds, since the other conditions have been checked in the proof of Proposition 4.5. More precisely, given $F\colon \mathbb{A} \to \mathbb{B}$ and $G\colon \mathbb{B} \to \mathbb{C}$ in $Grpd(\mathcal{C})$ such that $F \cdot G \in \Sigma$, we have to prove that $F \in \Sigma$ iff $G \in \Sigma$. There are two not obvious steps. (The protomodularity of $\mathcal{C}$ is needed only for the first step.)

1. If $F \cdot G$ is full and faithful and F is a weak equivalence, then G is full and faithful. Consider the following commutative diagram

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\quad c \quad} & A_0 \\
{\scriptstyle <d,F_1>}\downarrow & & \downarrow{\scriptstyle F_0} \\
A_0 \times_{F_0,d} B_1 \xrightarrow{\ t_2\ } B_1 & \xrightarrow{\ c\ } & B_0 \\
{\scriptstyle 1\times G_1}\downarrow & & \downarrow{\scriptstyle G_0} \\
A_0 \times_{F_0 G_0,d} C_1 \xrightarrow{\ \tau_2\ } C_1 & \xrightarrow{\ c\ } & C_0
\end{array}
$$

Since F is full and faithful, by Lemma 5.6 the top square is a pullback. Since $F \cdot G$ is full and faithful, by Lemma 5.6 the outer rectangle is a pullback. Since F is essentially surjective, the second row is a regular epimorphism. Following Lemma 5.4 the bottom square is a pullback. Therefore, by Lemma 5.7, the outer rectangle of the following commutative diagram is a pullback

$$
\begin{array}{ccc}
A_0 \times_{F_0,d} B_1 \xrightarrow{\ t_2\ } B_1 & \xrightarrow{<G_1,c>} & C_1 \times_{c,G_0} B_0 \\
 & & \downarrow{\scriptstyle s_1} \\
{\scriptstyle t_1}\downarrow \qquad {\scriptstyle d}\downarrow & & C_1 \\
 & & \downarrow{\scriptstyle d} \\
A_0 \xrightarrow{\ F_0\ } B_0 & \xrightarrow{\ G_0\ } & C_0
\end{array}
$$

Since the left hand square is a pullback by definition and the second column is a split epimorphism, by Lemma 5.4 the right hand square is a pullback. By Lemma 5.6 again we conclude that G is full and faithful.

2. If $F \cdot G$ is essentially surjective and G is full and faithful, then F is

essentially surjective. Consider the following pullback (notations as in Lemma 5.7)

$$
\begin{array}{ccc}
Q & \xrightarrow{\ \lambda_2\ } & B_0 \\
\left\downarrow{\scriptstyle \lambda_1}\right. & & \left\downarrow{\scriptstyle G_0}\right. \\
A_0 \times_{F_0 G_0, d} C_1 \xrightarrow[\tau_2]{} & C_1 \xrightarrow[c]{} & C_0
\end{array}
$$

By assumption $\tau_2 \cdot c$ is a regular epimorphism, so that λ_2 also is a regular epimorphism. Since G is full and faithful, there exists $\lambda \colon Q \to B_1$ such that $\lambda \cdot d = \lambda_1 \cdot \tau_1 \cdot F_0$, $\lambda \cdot G_1 = \lambda_1 \cdot \tau_2$ and $\lambda \cdot c = \lambda_2$. From the first equation on λ, we deduce the existence of $\mu \colon Q \to A_0 \times_{F_0, d} B_1$ such that $\mu \cdot t_1 = \lambda_1 \cdot \tau_1$ and $\mu \cdot t_2 = \lambda$. Finally, $\mu \cdot t_2 \cdot c = \lambda \cdot c = \lambda_2$, so that $t_2 \cdot c$ is a regular epimorphism. (Note that we need only the existence of λ, not its uniqueness. In other words we only use the "fullness" of G, and not its "faithfulness".) ∎

5.9 Observe that, contrarily to Lemma 5.4, Proposition 5.5 is not a characterization of regular protomodular categories. Indeed, if $\mathcal{C}$ is *Set* (more generally, if in $\mathcal{C}$ the axiom of choice holds) then weak equivalences in *Grpd*$(\mathcal{C})$ are the same that equivalences (see Corollary 4.3), and the class of equivalences obviously is a bipullback congruence.

5.10 G. Janelidze pointed out to me that condition 2 in Lemma 5.4 holds in any protomodular (not necessarily regular) category $\mathcal{C}$ provided that the arrow b is a pullback stable strong epimorphism. This fact has an interesting consequence. Indeed, Proposition 4.5 holds when $\mathcal{C}$ is any finitely complete category and Σ is the class of "weak $\mathcal{E}$-equivalences", where:

- $\mathcal{E}$ is any class of arrows that behaves well (in the sense explained in 4.4) and contains the split epimorphisms,

- an internal functor F is a weak $\mathcal{E}$-equivalence if it is full and faithful and essentially $\mathcal{E}$-surjective (that is, the arrow $t_2 \cdot c \colon C_0 \times_{F_0, d} D_1 \to D_1 \to D_0$ of Definition 4.1 is in $\mathcal{E}$).

Therefore, Proposition 5.5 holds for weak $\mathcal{E}$-equivalences in any protomodular category $\mathcal{C}$ provided that $\mathcal{E}$ behaves well, contains the split

epimorphisms and is contained in the class of pullback stable strong epimorphisms. Examples are:

i. the class of pullback stable regular epimorphisms,

ii. the class of pullback stable regular epimorphisms that are effective descent morphisms.

6. Monoidal functors

All along this section we fix $\mathcal{C} = Grp$, the category of groups, which is a regular and protomodular category. I use additive notation for groups.

6.1 The aim of this section is to prove that the 2-category MON described hereunder is the bicategory of fractions of $Grpd(\mathcal{C})$ with respect to weak equivalences.

1. Objects of MON are internal groupoids in Grp. Note that since the forgetful functor $Grp \to Set$ preserves finite limits, any object of MON is also a groupoid in the usual sense.

2. 1-cells $F \colon \mathbb{A} \to \mathbb{B}$ in MON are monoidal functors, that is, pairs (F, F_2) where F is a (not necessarily internal) functor and

$$F_2 = \{F_2^{a,b} \colon Fa + Fb \to F(a+b)\}_{a,b \in A_0}$$

is a natural family of arrows in $\mathbb{B}$ satisfying the cocycle condition

$$
\begin{array}{ccc}
Fa + Fb + Fc & \xrightarrow{\;1+F_2^{b,c}\;} & Fa + F(b+c) \\
{\scriptstyle F_2^{a,b}+1}\downarrow & & \downarrow{\scriptstyle F_2^{a,b+c}} \\
F(a+b) + Fc & \xrightarrow[\;F_2^{a+b,c}\;]{} & F(a+b+c)
\end{array}
$$

(and suitable $F_0 \colon 0 \to F0$ is uniquely determined by F and F_2).

3. 2-cells $\lambda \colon F \Rightarrow G$ in MON are monoidal natural transformations, that is, natural transformations such that the following diagram

commutes

$$
\begin{array}{ccc}
Fa + Fb & \xrightarrow{\ F_2^{a,b}\ } & F(a+b) \\
{\scriptstyle \lambda_a + \lambda_b} \downarrow & & \downarrow {\scriptstyle \lambda_{a+b}} \\
Ga + Gb & \xrightarrow[\ G_2^{a,b}\ ]{} & G(a+b)
\end{array}
$$

Remark 6.2 1. The 2-category $Grpd(\mathcal{C})$ embeds into the 2-category MON: internal functors $F\colon \mathbb{A} \to \mathbb{B}$ are precisely those monoidal functors for which all the $F_2^{a,b}$ are identites. Indeed, in this case the naturality of F_2 corresponds to the fact that $F_1\colon A_1 \to B_1$ is a group homomorphism, and the cocycle condition is verified because $e\colon B_0 \to B_1$ is a group homomorphism.

2. The embedding $\mathcal{F}\colon Grpd(\mathcal{C}) \to MON$ is full and faithful on 2-cells. Indeed, if $F_2^{a,b} = id = G_2^{a,b}$, then the fact that λ is monoidal corresponds to the fact that $\lambda\colon A_0 \to B_1$ is a group homomorphism.

3. The embedding $\mathcal{F}\colon Grpd(\mathcal{C}) \to MON$ preserves weak equivalences. In fact, the forgetful functor $Grp \to Set$ preserves and reflects finite limits and regular epimorphisms (this is because Grp is an algebraic category, see Ch. 3 in [6]), so that weak equivalences in $Grpd(\mathcal{C})$ and in MON are 1-cells which are full, faithful and essentially surjective in the usual sense.

4. In MON weak equivalences coincide with equivalences. Indeed, if $F\colon \mathbb{A} \to \mathbb{B}$ is a weak equivalence, any quasi-inverse $G\colon \mathbb{B} \to \mathbb{A}$ can be equipped with a monoidal structure as follows: choose, for each $x \in B_0$, an arrow $\beta_x\colon F(Gx) \to x$ so to have a natural transformation $\beta\colon G \cdot F \Rightarrow Id$. Then define

$$
G_2^{x,y}\colon Gx + Gy \to G(x+y)
$$

to be the unique arrow making the following diagram commutative

$$
\begin{array}{ccc}
F(Gx+Gy) & \xrightarrow{\ F(G_2^{x,y})\ } & F(G(x+y)) \\
{\scriptstyle F_2^{Gx,Gy}} \uparrow & & \downarrow {\scriptstyle \beta_{x+y}} \\
F(Gx)+F(Gy) & \xrightarrow[\ \beta_x+\beta_y\ ]{} & x+y
\end{array}
$$

It is straightforward to check naturality and cocycle condition for G_2 and that β is monoidal. Moreover, we get a monoidal natural transformation $\alpha \colon F \cdot G \Rightarrow Id$ via the equation $F(\alpha_a) = \beta_{Fa}$.

5. The above construction of G_2 makes clear that even if F is a weak equivalence in $Grpd(\mathcal{C})$ in general G is in MON but not in $Grpd(\mathcal{C})$.

Lemma 6.3 *The 2-category MON has bipullbacks. Moreover, given 1-cells $F \colon \mathbb{A} \to \mathbb{B}$ and $G \colon \mathbb{C} \to \mathbb{B}$, it is possible to choose a bipullback of F and G*

$$
\begin{array}{ccc}
\mathbb{P} & \overset{F'}{\longrightarrow} & \mathbb{C} \\
{\scriptstyle G'}\downarrow & {\scriptstyle \varphi} & \downarrow{\scriptstyle G} \\
\mathbb{A} & \underset{F}{\longrightarrow} & \mathbb{B}
\end{array}
$$

in such a way that F' and G' are internal functors in Grp.

Proof. The construction of the pullback $\mathbb{P}$ is as in 3.3. The interesting point is that, even if F and G are monoidal (not necessarily internal) functors, $\mathbb{P}$ is an internal groupoid in Grp and not just a monoidal category. Indeed, if
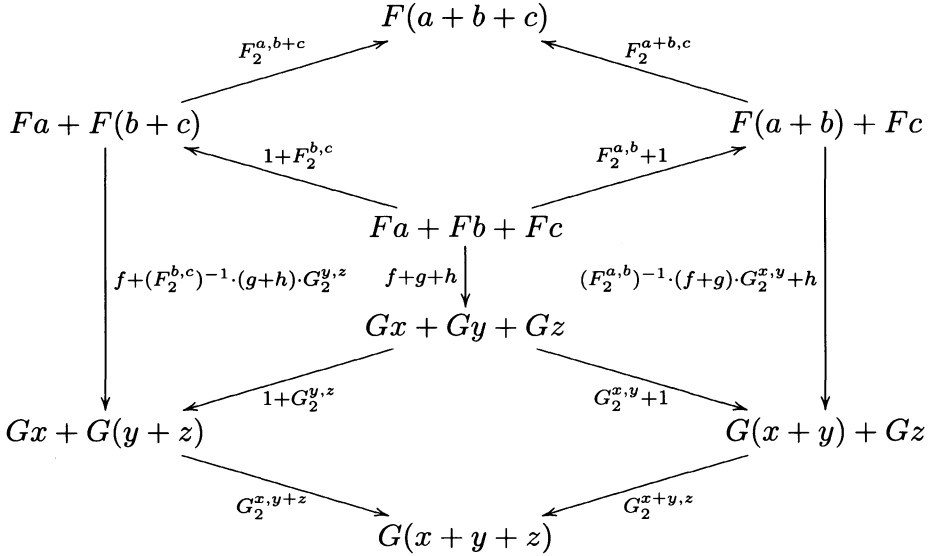
$$(a, f \colon Fa \to Gx, x) \quad \text{and} \quad (b, g \colon Fb \to Gy, y)$$

are objects in $\mathbb{P}$, their tensor product $(a, f \colon Fa \to Gx, x) + (b, g \colon Fb \to Gy, y)$ is given by

$$\left(a + b, \; F(a+b) \overset{(F_2^{a,b})^{-1}}{\longrightarrow} Fa + Fb \overset{f+g}{\longrightarrow} Gx + Gy \overset{G_2^{x,y}}{\longrightarrow} G(x+y), \, x+y\right)$$

If $(c, h \colon Fc \to Gz, z)$ is a third object in $\mathbb{P}$, to check that the above tensor product is strictly associative easily reduces to the commutativity of the
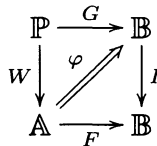
following diagram

$$F(a+b+c)$$

$F_2^{a,b+c}$ $\qquad\qquad$ $F_2^{a+b,c}$

$$Fa + F(b+c) \qquad\qquad\qquad\qquad F(a+b) + Fc$$

$1+F_2^{b,c}$ $\qquad\qquad$ $F_2^{a,b}+1$

$$Fa + Fb + Fc$$

$f+(F_2^{b,c})^{-1}\cdot(g+h)\cdot G_2^{y,z}$ $\qquad$ $f+g+h$ $\qquad$ $(F_2^{a,b})^{-1}\cdot(f+g)\cdot G_2^{x,y}+h$

$$Gx + Gy + Gz$$

$1+G_2^{y,z}$ $\qquad\qquad$ $G_2^{x,y}+1$

$$Gx + G(y+z) \qquad\qquad\qquad\qquad G(x+y) + Gz$$

$G_2^{x,y+z}$ $\qquad\qquad$ $G_2^{x+y,z}$

$$G(x+y+z)$$

that is, to the cocycle condition on F_2 and G_2.

The fact that F' and G' are internal functors is obvious. ∎

Proposition 6.4 *The embedding* $\mathcal{F}\colon Grpd(\mathcal{C}) \to MON$ *is the bicategory of fractions of* $Grpd(\mathcal{C})$ *with respect to the class of weak equivalences.*

Proof. Let Σ be the class of weak equivalences in $Grpd(\mathcal{C})$. From Proposition 4.5 we know that Σ has a right calculus of fractions. Moreover, by 6.2.3 and 6.2.4, $\mathcal{F}(W)$ is an equivalence for every $W \in \Sigma$. It remains to check conditions EF1–EF3 in Proposition 2.5: EF1 is obvious and EF2 is precisely 6.2.2. As far as EF3 is concerned, consider a 1-cell $F\colon \mathbb{A} \to \mathbb{B}$ in MON and perform the bipullback of F along the identity 1-cell I as in Lemma 6.3

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{G} & \mathbb{B} \\ W\downarrow & \overset{\varphi}{\nearrow} & \downarrow I \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array}$$

so that both W and G are internal functors. Since equivalences are stable under bipullbacks, W is an equivalence in MON and therefore it

is a weak equivalence in $Grpd(\mathcal{C})$. Finally, $\varphi \colon \mathcal{F}(W) \cdot F \Rightarrow \mathcal{F}(G)$ is the 2-cell needed in EF3. Following Proposition 2.5, $\mathcal{F} \colon Grpd(\mathcal{C}) \to MON$ is the bicategory of fractions with respect to Σ. ∎

Remark 6.5 Observe that we cannot expect to describe a class larger than the class of monoidal functors as fractions of internal functors with respect to weak equivalences. Indeed, the existence of a 2-cell $\mathcal{F}(W) \cdot F \Rightarrow \mathcal{F}(G)$ as in condition EF3 implies that F is monoidal.

7. Homomorphisms of strict Lie 2-algebras

In this section the base category $\mathcal{C}$ is the category Lie of Lie algebras over a fixed field K, which is a regular and protomodular category. The situation is completely analogous to the situation described in Section 6 for groups. The reason is that the forgetful functors $Lie \to Vect$ (where $Vect$ is the category of vector spaces over K) and $Vect \to Set$ preserve and reflect finite limits and regular epimorphisms (because Lie and $Vect$ are algebraic categories) and moreover in $Vect$ the axiom of choice holds (because every vector space is free and therefore regular projective).

7.1 The aim of this section is to prove that the 2-category LIE described hereunder is the bicategory of fractions of $Grpd(\mathcal{C})$ with respect to weak equivalences.

1. Objects of LIE are internal groupoids in Lie, also called strict Lie 2-algebras in [1].

2. 1-cells $F \colon \mathbb{A} \to \mathbb{B}$ in LIE are internal functors in $Vect$ equipped with a family of arrows in $\mathbb{B}$

$$F_2 = \{F_2^{a,b} \colon [Fa, Fb] \to F[a, b]\}_{a,b \in A_0}$$

which is natural, bilinear, antisymmetric, and satisfies the follow-

ing Jacobi condition

$$[Fa, [Fb, Fc]] \xLeftrightarrow{\quad} [[Fa, Fb], Fc] + [Fb, [Fa, Fc]]$$

$$\begin{array}{ccc}
[Fa, [Fb, Fc]] & = & [[Fa, Fb], Fc] + [Fb, [Fa, Fc]] \\
{\scriptstyle [1, F_2^{b,c}]}\Big\downarrow & & \Big\downarrow{\scriptstyle [F_2^{a,b}, 1] + [1, F_2^{a,c}]} \\
[Fa, F[b, c]] & & [F[a, b], Fc] + [Fb, F[a, c]] \\
{\scriptstyle F_2^{a, [b,c]}}\Big\downarrow & & \Big\downarrow{\scriptstyle F_2^{[a,b],c} + F_2^{b,[a,c]}} \\
F[a, [b, c]] & = & F[[a, b], c] + F[b, [a, c]]
\end{array}$$

These 1-cells are simply called homomorphisms in [1], where in fact they are defined for more general semi-strict Lie 2-algebras.

3. 2-cells $\lambda \colon F \Rightarrow G$ in *LIE* are internal natural transformations in *Vect* such that the following diagram commutes

$$\begin{array}{ccc}
[Fa, Fb] & \xrightarrow{F_2^{a,b}} & F[a, b] \\
{\scriptstyle [\lambda_a, \lambda_b]}\Big\downarrow & & \Big\downarrow{\scriptstyle \lambda_{[a,b]}} \\
[Ga, Gb] & \xrightarrow[G_2^{a,b}]{} & G[a, b]
\end{array}$$

Remark 7.2   1. The 2-category $Grpd(\mathcal{C})$ embeds into the 2-category *LIE*: internal functors $F \colon \mathbb{A} \to \mathbb{B}$ are precisely those homomorphisms for which all the $F_2^{a,b}$ are identites. The embedding $\mathcal{F} \colon Grpd(\mathcal{C}) \to LIE$ is full and faithful on 2-cells, and preserves weak equivalences.

2. In *LIE* weak equivalences coincide with equivalences. Indeed, let $F \colon \mathbb{A} \to \mathbb{B}$ be a weak equivalence in *LIE*. Then F is also a weak equivalence in the 2-category of internal groupoids and internal functors in *Vect*. Since in *Vect* the axiom of choice holds, F has a quasi-inverse $G \colon \mathbb{B} \to \mathbb{A}$ which is an internal functor in *Vect* (see Corollary 4.3). Now G can be equipped with a structure of homomorphism as follows: consider the internal (in *Vect*) natural transformation $\beta \colon G \cdot F \Rightarrow Id$ and define

$$G_2^{x,y} \colon [Gx, Gy] \to G[x, y]$$

to be the unique arrow making the following diagram commutative

$$
\begin{array}{ccc}
F[Gx, Gy] & \xrightarrow{\ F(G_2^{x,y})\ } & F(G[x,y]) \\[4pt]
{\scriptstyle F_2^{Gx,Gy}}\Big\uparrow & & \Big\downarrow{\scriptstyle \beta_{[x,y]}} \\[4pt]
[F(Gx), F(Gy)] & \xrightarrow[\ [\beta_x,\beta_y]\ ]{} & [x,y]
\end{array}
$$

Lemma 7.3 *The 2-category LIE has bipullbacks. Moreover, given 1-cells $F \colon \mathbb{A} \to \mathbb{B}$ and $G \colon \mathbb{C} \to \mathbb{B}$, it is possible to choose a bipullback of F and G*

$$
\begin{array}{ccc}
\mathbb{P} & \xrightarrow{\ F'\ } & \mathbb{C} \\[4pt]
{\scriptstyle G'}\Big\downarrow & {\scriptstyle \varphi}\ \nearrow & \Big\downarrow{\scriptstyle G} \\[4pt]
\mathbb{A} & \xrightarrow[\ F\ ]{} & \mathbb{B}
\end{array}
$$

in such a way that F' and G' are internal functors in Lie.

Proof. Once again the point is that, even if F and G are homomorphisms, the bipullback $\mathbb{P}$ constructed as in 3.3 is an internal groupoid in *Lie* and not just a semi-strict Lie 2-algebra. Indeed, the Lie operation in $\mathbb{P}$ is defined by

$$
([a,b],\ F[a,b] \xrightarrow{\ (F_2^{a,b})^{-1}\ } [Fa, Fb] \xrightarrow{\ [f,g]\ } [Gx, Gy] \xrightarrow{\ G_2^{x,y}\ } G[x,y]\ ,\ [x,y])
$$

and the Jacobi identity is strictly verified thanks to the Jacobi condition on F_2 and G_2. ∎

Proposition 7.4 *The embedding $\mathcal{F} \colon Grpd(\mathcal{C}) \to LIE$ is the bicategory of fractions of $Grpd(\mathcal{C})$ with respect to the class of weak equivalences.*

Proof. The proof is analogous to that of Proposition 6.4 and we omit details. ∎

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