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FIBRATIONS AND CLASSIFYING SPACES: AN AXIOMATIC APPROACH II

by Peter I. BOOTH

Résumé

L'utilisation du théorème de représentabilité de Brown pour produire une classification des espaces de fibrations n'est pas possible faute d'une difficulté technique de théorie d'ensemble. En particulier, ceci est le cas pour le résultat unifiant les classifications présenté dans l'article antécédent. Nous introduisons ici le concept de fibration universelle par deux nouvelles définitions, et démontrons l'équivalence de celles-ci. Nous profitons de ce résultat pour enfin circonvenir le problème soulevé ci-haut.

1 Introduction

A set theoretical difficulty occurs when one seeks to use the Brown Representability Theorem to produce a theory of classifying spaces for fibrations [M, p.(vi)]. In particular, this applies to the unifying classifying result (theorem 8.1) of [B2]. We will prove the equivalence of two alternative definitions of universal fibration, and use this result to eliminate the aforementioned problem.

We recall the concept of an *$\mathcal{E}$-fibration*, characterized by the property that it satisfies the $\mathcal{E}WCHP$ [B1, p.136] relative to a *category* $(\mathcal{E}, U)$ *of enriched spaces*, or *$\mathcal{E}$-spaces* [B1, p.129], in which the fibres of these fibrations are required to lie. If F is a given $\mathcal{E}$-space, then $\mathcal{F}$ will denote the *category of fibres generated by* F. Thus $\mathcal{F}$ consists of all $\mathcal{E}$-spaces that are $\mathcal{E}$-homotopy equivalent to F, together with all $\mathcal{E}$-homotopy equivalences between these $\mathcal{E}$-spaces. If B is a space, then $\mathcal{F}FHE(B)$ will denote the collection of all *$\mathcal{F}$-fibre homotopy equivalence classes*, or *$\mathcal{F}FHE$ classes*, of $\mathcal{F}$-fibrations over B. The above result of [B2], when applied to $\mathcal{F}$-fibrations, invokes the assumption that $\mathcal{E}$ is *$\mathcal{E}FHE$ set-valued*, i.e. that, for all choices of $\mathcal{F}$ in $\mathcal{E}$ and of a CW-complex B, the collection $\mathcal{F}FHE(B)$ is a set.

However, in the cases of the types of fibrations that we wish to classify, it is not obvious that the corresponding $\mathcal{E}FHE$ set-valued conditions are satisfied. *The objective of this paper is to produce a satisfactory resolution of that difficulty.*

Recalling the concept of a *category of well enriched spaces* [B2, def.2.3], we introduce (definitions 5.1(ii)) a condition - that of $\mathcal{E}$ carrying the structure of a *proper* category of well enriched spaces - whose validity is easily verified for the important examples. We define a subcategory $\kappa\mathcal{E}$ of $\mathcal{E}$, by putting an upper bound κ on the cardinality of the underlying sets of acceptable $\mathcal{E}$-spaces. Assuming that $\mathcal{E}$ is well behaved in the above senses and that κ is greater than or equal to the cardinality of the continuum, we show (proposition 5.2) that $\kappa\mathcal{E}$ carries the structure of a category of well enriched spaces and is $\kappa\mathcal{E}FHE$ set-valued. It follows, using theorem 8.1 of [B2], that universal $\kappa\mathcal{F}$-fibrations exist, for each $\kappa\mathcal{F}$ in $\kappa\mathcal{E}$. We then have to show that these universal $\kappa\mathcal{F}$-fibrations are also universal $\mathcal{F}$-fibrations. Our verification of this hypothesis requires an understanding of the relationship between two alternative definitions of universal fibration, a topic that takes up most of this paper.

In section 2, we review some ideas concerning *fibred mapping spaces.* We use these ideas in section 3, and some techniques from [B2] in section 4, to show that the *grounded universal* and *weakly contractible universal* concepts are often equivalent. In section 5, the set theoretical difficulty is eliminated from the classification result of [B2]. A brief discussion of examples is given in section 6.

The conventions, notation and terminology of [B1] and [B2] will normally be used in this paper; in particular we work in the context of the category $\mathcal{T}$ of *cg (= compactly generated) spaces* [B1, p.128-129]. A space B will be said to be *weak Hausdorff* if the diagonal subset is closed in the cg-ified product space $B \times B$.

We recall some notation and terminology. If $s : D \to B$ and $t : D \to C$ are maps, then the map $D \to B \times C, d \to (s(d), t(d))$ will be denoted by (s, t).

The symbols $\simeq, \simeq^0, \simeq^A, \simeq_{\mathcal{F}}$ and $\simeq_{\mathcal{F}_B}$ will denote homotopies in the *free, pointed, under A, in $\mathcal{F}$* and "*$\mathcal{F}$-map over the space B*" senses, respectively. A pair of sections to a fibration are *vertically homotopic* if there is a homotopy between them consisting of sections to the fibration.

A map that satisfies the *covering homotopy property*, or CHP, is a *Hurewicz fibration.* One that satisfies the *weak covering homotopy property*, or $WCHP$, is a *Dold fibration.*

We assume, from this point on, that $(\mathcal{E}, U)$ is a category of enriched spaces. This assumption appears in a stronger form in theorem 4.5 and section 5.

2 Fibred Mapping Spaces

If $p: X \to B$ is a map and $b \in B$, then $X|b$ will denote the fibre $p^{-1}(b)$ of p over b. If, for each $b \in B$, $X|b$ carries the structure of an $\mathcal{E}$-space, then p will be said to be an *$\mathcal{E}$-overspace* [B1, p.130].

Let $f: D \to B$ be a map, and $p: X \to B$ and $q: Y \to C$ be $\mathcal{E}$-overspaces. We recall the concept of the *induced $\mathcal{E}$-overspace* $p_f: X \sqcap D \to D$, obtained by "*pulling p back over f*" [B1, p.130]. Also there is a fibred mapping space $X \square Y$ with underlying set $\bigcup_{b \in B, c \in C} \mathcal{E}(X|b, Y|c)$ and a map $p\square q: X \square Y \to B \times C$, with $(p\square q)(\xi) = (b, c)$, where $\xi \in \mathcal{E}(X|b, Y|c)$, $b \in B$ and $c \in C$ [B1, p.131-132]. We use the symbols $p\square_1 q: X\square Y \to B$ and $p\square_2 q: X \square Y \to C$ to denote the maps obtained by composing $p\square q$ with the projections $\pi_B: B \times C \to B$ and $\pi_C: B \times C \to C$, respectively. So if $\xi \in \mathcal{E}(X|b, Y|c)$, where $b \in B$ and $c \in C$, then $(p\square_1 q)(\xi) = b$ and $(p\square_2 q)(\xi) = c$.

In this section, we present some basic properties of $p\square_1 q$. Some of these results are stated and proved in section 2 of [B4], others are proved here.

We recall that an *$\mathcal{E}$-pairwise map* $<\gamma, \delta>: p \to q$, i.e. from p to q, consists of a pair of maps $\gamma: X \to Y$ and $\delta: B \to C$ such that $q\gamma = \delta p$, with the further property that $\gamma|(X|b): X|b \to Y|\delta(b)$ is an $\mathcal{E}$-map for all $b \in B$.

Theorem 2.1 : Fibred exponential law. *(= theorem 2.1 of [B4]). Let $p: X \to B$ and $q: Y \to C$ be $\mathcal{E}$-overspaces, and $f: D \to B$ be a map, where B is a weak Hausdorff space. Then there is a bijective correspondence between:*

(i) the set of $\mathcal{E}$-pairwise maps $<\gamma,\delta>$ from p_f to q, and

(ii) the set of maps $\gamma^0: D \to X\square Y$ over B, i.e. with $(p\square_1 q)\gamma^0 = f$. This is determined by $\gamma(x,d) = \gamma^0(d)(x)$ and by $\delta = (p\square_2 q)\gamma^0$, where $p(x) = f(d)$.

Corollary 2.2 *(= corollary 2.2 of [B4]). There is a bijective correspondence between :*

(i) the set of $\mathcal{E}$-pairwise maps $<\gamma,\delta>$ from p to q, and

(ii) the set of sections γ^0 to $p\square_1 q$. This is determined by $\gamma(x) = \gamma^0(b)(x)$ and by $\delta = (p\square_2 q)\gamma^0$, where $p(x) = b$.

Corollary 2.3 *(a) There is a bijective correspondence between:*

(i) $\mathcal{E}$-pairwise homotopies $<\Gamma,\Delta>$ from $p\times 1_I: X\times I \to B\times I$ to q, and

(ii) homotopies $\Gamma^0: B\times I \to X\square Y$ over B, i.e such that $(p\square_1 q)\Gamma^0$ is the projection $B\times I\to B$. This is determined by $\Gamma(x,t) = \Gamma^0(b,t)(x)$ and $\Delta = (p\square_2 q)\Gamma^0$, where $p(x) = b$.

(b) Let $<\gamma,\delta>$ and $<\zeta,\eta>$ be $\mathcal{E}$-pairwise maps from p to q, γ^0 and ζ^0 be sections to $p\square_1 q$, $<\Gamma,\Delta>$ be an $\mathcal{E}$-pairwise homotopy between $<\gamma,\delta>$ and $<\zeta,\eta>$, and Γ^0 be a vertical homotopy between γ^0 and ζ^0. If $<\Gamma,\Delta>$ corresponds to Γ^0 in the sense of (a), then $<\gamma,\delta>$ corresponds γ^0 and $<\zeta,\eta>$ corresponds ζ^0 in the sense of corollary 2.2.

Proof. (a) This follows from theorem 2.1 if we take $D = B\times I, f$ the projection $B\times I\to B$, and identify p_f with $p\times 1_I$.

(b) This is an immediate consequence of (a) and corollary 2.2.

Let F be an $\mathcal{E}$-space and $q: Y\to C$ is an $\mathcal{E}$-overspace. Then there is an injection from the set $\bigcup_{c\in C}\mathcal{E}(F,Y|c)$ to the *set* $\mathcal{T}(F,Y)$, that takes each $\mathcal{E}$-map $F\to Y|c$ to the composite map $F\to Y|c\subset Y$. We define $Prin_F Y$ to be the space with underlying set $\bigcup_{c\in C}\mathcal{E}(F,Y|c)$ and the strong (cg-sense) topology, relative to this injection into the *space* $\mathcal{T}(F,Y)$. Thus the injection is a homeomorphism into, from $Prin_F Y$ to $\mathcal{T}(F,Y)$.

Corollary 2.4 *(= corollary 2.4 of [B4]). If $b \in B$, then the fibre of $p\Box_1 q$ over b is $Prin_{X|b}Y$.*

The following result should not be confused with proposition 2.6 of [B4].

Proposition 2.5 *Let B be a weak Hausdorff space. If $p: X \to B$ and $q: Y \to C$ are $\mathcal{E}$-fibrations, then $p\Box_1 q$ is a Dold fibration.*

Proof. We know from [B1, prop.4.5] that $p\Box q$ is a Dold fibration. Now $p\Box_1 q = \pi_B(p\Box q)$ and so the result follows.

3 Universal $\mathcal{F}$-Fibrations I

From this point on we assume - except where we specify otherwise - that F is a given $\mathcal{E}$-space and $\mathcal{F}$ is the associated category of fibres.

Let $q: Y \to C$ be an $\mathcal{F}$-overspace. Then the map $Prin_F(Y) \to Y$ that evaluates at an arbitrarily chosen point of F is continuous. Let $prin_F q: Prin_F(Y) \to C$ denote the obvious projection function. Then $prin_F q$ is the composite of this evaluation map with q and so is continuous.

Definitions 3.1 *The space S will be said to be weakly contractible if $\pi_n(S) = 0$, for all non-negative integers n. The $\mathcal{F}$-fibration $q: Y \to C$ will be said to be weakly contractible universal if $Prin_F(Y)$ is weakly contractible.*

The validity of the above property of q depends on $\mathcal{F}$, and is clearly independent of our choice of F in $\mathcal{F}$.

Lemma 3.2 *Let $p: X \to B$ is an $\mathcal{F}$-fibration over a weak Hausdorff space B and $q: Y \to B$ be a weakly contractible universal $\mathcal{F}$-fibration. Then $p\Box_1 q$ is a weak homotopy equivalence.*

Proof. It follows from proposition 2.5 that $p\Box_1 q$ satifies the WCHP. If $b \in B$ the fibre of $p\Box_1 q$ over b is $Prin_{X|b}(Y)$ (corollary 2.4) which has the homotopy type of $Prin_F(Y)$, a weakly contractible space. Hence the fibres $Prin_{X|b}(Y)$ are weakly contractible.

Let $\xi, \xi' \in X\Box Y$, with $(p\Box_1 q)(\xi) = b$ and $(p\Box_1 q)(\xi') = b'$ lying in the same path component of B. Applying the $WCHP$ to a path from b' to b that is stationary on $[0, \frac{1}{2}]$, we see that there is a path in $X\Box Y$ from ξ' to ξ'', where $\xi'' \in Prin_{X|b}Y$. Clearly there is a path from ξ'' to ξ in $Prin_{X|b}Y$, and so there is a path from ξ' to ξ. Thus $p\Box_1 q$ induces an injection between the path components of its domain and range spaces.

It follows from the definition of $\mathcal{F}$-overspace that p and q are surjective; hence so also is $p\Box_1 q$. Thus $p\Box_1 q$ induces a bijection between path components. In fact considering the exact homotopy sequence of $p\Box_1 q$ we see that this map is a weak homotopy equivalence.

We now recall some concepts from [B2]. An *F-grounded $\mathcal{F}$-fibration* (p, h) consists of an $\mathcal{F}$-fibration $p: X \to B$ over a pointed space B, and an $\mathcal{F}$-homotopy equivalence $h: F \to X|*$, where $*$ denotes the base point of B.

Let $(p_\lambda : X_\lambda \to B,\ h_\lambda : F \to X_\lambda|*)$ be F-grounded $\mathcal{F}$-fibrations, for $\lambda = 0$ and 1, and $g: X_0 \to X_1$ be an $\mathcal{F}FHE$. Then g will be said to be an *F-grounded $\mathcal{F}FHE$ from (p_0, h_0) to (p_1, h_1)* if $h_1 \simeq_{\mathcal{F}} (g|(X_0|*))h_0$, where the range of this homotopy is restricted to $X_1|*$. In this case we write $(p_0, h_0) \equiv_F (p_1, h_1)$.

Let B be a pointed space. The class $\mathcal{F}FHE^F(B)$ will consist of all F-grounded $\mathcal{F}$FHE-classes of F-grounded $\mathcal{F}$-fibrations over B. Then $\mathcal{F}$ will be said to be *$\mathcal{F}FHE^F$ set-valued* if, for all choices of a pointed CW-complex B, $\mathcal{F}FHE^F(B)$ is a set.

Let $(p: X \to B,\ h: F \to X|*)$ be an F-grounded $\mathcal{F}$-fibration and $f: B' \to B$ be a pointed map. Identifying the distinguished fibre of p_f, i.e. $(X|*)\times\{*\}$, with $X|*$, we see that there is an *induced F-grounded $\mathcal{F}$-fibration* (p_f, h).

Let $\mathcal{F}$ be $\mathcal{F}FHE^F$ set-valued and $(r: Z \to D,\ \ell: F \to Z|*)$ be an

F-grounded $\mathcal{F}$-fibration. We will use $\mathcal{W}^0$ to denote the class of all pointed spaces that have the pointed homotopy type of a pointed CW-complex. Then $(r,\ \ell)$ will be said to be *grounded universal amongst F-grounded $\mathcal{F}$-fibrations* if, for all choices of a space $B \in \mathcal{W}^0$,

$$\phi = \phi(r,\ell): [B,\ D]^0 \to \mathcal{F}FHE^F(B),\ \ \phi([f]) = [(r_f, \ell)]$$

is a natural bijection.

Definition 3.3 *Let $\mathcal{F}$ be $\mathcal{F}FHE^F$ set-valued and $r: Z \to D$ be an $\mathcal{F}$-fibration. Then r will be said to be grounded universal amongst $\mathcal{F}$-fibrations if there is a point $* \in D$ and a map $\ell \in \mathcal{F}(F, Z|*)$ such that (r, ℓ) is grounded universal amongst F-grounded $\mathcal{F}$-fibrations.*

Theorem 3.4 *Let $(\mathcal{E}, U)$ be a category of enriched spaces, F be an $\mathcal{E}$-space and $\mathcal{F}$ be the associated category of fibres. We assume that there is a weakly contractible universal $\mathcal{F}$-fibration $q: Y \to C$. Then $\mathcal{F}$ is $\mathcal{F}FHE^F$ set-valued and q is grounded universal amongst $\mathcal{F}$-fibrations.*

Let us make an arbitrary choice of $* \in C$ and $k \in \mathcal{F}(F, Y|*)$. The underlying idea of the proof, which takes up the remainder of this section, is rather easy. Let $(p: X \to B,\ h: F \to X|*)$ be an F-grounded $\mathcal{F}$-fibration over a pointed CW-complex B. We know via lemma 3.2 that the fibration $p\Box_1 q$ is a weak homotopy equivalence and so should have a unique vertical homotopy class of sections. It then follows from corollaries 2.2 and 2.3 that there is a unique F-grounded $\mathcal{F}$-pairwise homotopy class of $\mathcal{F}$-pairwise maps from (p, h) to (q, k), and the result should follow via the universal property of pullbacks.

A similar result for F-grounded $\mathcal{F}$-fibrations over pointed CW-complexes, where $\mathcal{F}$-fibration is understood in the $\mathcal{F}CHP$ sense [B1, ch.3], was given in [BHP, thm.3.3]. However, in the present case, there are complications. To show that we have suitably defined sections to $p\Box_1 q$, we wish to use the relative CHP described in [S, thm.7.8.9] and a direct application would require our Dold fibration $p\Box_1 q$ to be at least a Serre fibration (called a weak fibration in [S]). Another factor is that our base spaces B are only known to be in $\mathcal{W}^0$. So we require some extra technical arguments to deal with these matters. Our proof is given in four parts, the first being a variation on [S, thm.7.6.22] that is tailored to deal with the specific situations that we encounter.

Lemma 3.5 *Let us assume that (K, L) is a relative CW-complex, that $q^\bullet : Y^\bullet \to C^\bullet$ is a Dold fibration, $p^\bullet : C^\bullet \to D^\bullet$ is a Hurewicz fibration, and that $\alpha : L \to Y^\bullet$ and $\beta : K \to D^\bullet$ are such that $p^\bullet q^\bullet \alpha = \beta | L$. We further assume that $p^\bullet q^\bullet$ is a weak homotopy equivalence. Then there is a map $\psi : K \to Y^\bullet$ such that $p^\bullet q^\bullet \psi = \beta$ and $\psi | L = w\alpha$, where $w : Y^\bullet \to Y^\bullet$ is a map over $C^\bullet$ that is homotopic over $C^\bullet$ to the identity on $Y^\bullet$.*

Proof. Let us factorize $q^\bullet$ as $r^\bullet u$, where $u : Y^\bullet \to Z^\bullet$ is a homotopy equivalence and $r^\bullet : Z^\bullet \to C^\bullet$ is a Hurewicz fibration. We know that $p^\bullet r^\bullet u\alpha = p^\bullet q^\bullet \alpha = \beta | L$; also that $p^\bullet r^\bullet$, like $p^\bullet q^\bullet$, is a weak homotopy equivalence. It follows from [S, thm.7.6.22] that there is a map $\sigma : K \to Z^\bullet$ such that $p^\bullet r^\bullet \sigma \simeq^L \beta$ and $\sigma | L = u\alpha$. Now $p^\bullet r^\bullet$ is a Hurewicz fibration: applying the relative CHP of [S, thm.7.8.9] we obtain a map $\tau : K \to Z^\bullet$ such that $p^\bullet r^\bullet \tau = \beta$. and $\tau | L = u\alpha$.

Let $v : Z^\bullet \to Y^\bullet$ be a fibre homotopy inverse of the FHE u (see [D, thm.6.1]). We define $\psi : K \to Y^\bullet$ to be $v\tau$ and $w = vu$. Then w is homotopic over $C^\bullet$ to the identity on $Y^\bullet$ as required. Further $p^\bullet q^\bullet \psi = p^\bullet q^\bullet v\tau = p^\bullet r^\bullet \tau = \beta$, and $\psi | L = v\tau | L = vu\alpha = w\alpha$.

Lemma 3.6 *Let $q : Y \to C$ be a weakly contractible universal $\mathcal{F}$-fibration, $* \in C$ and $k \in \mathcal{F}(F, Y|*)$. If $(p : X \to B,\ h)$ is an F-grounded $\mathcal{F}$-fibration over a space B in $\mathcal{W}^0$, then there exists a pointed map $g : B \to C$ such that $(p, h) \equiv_F (qg, k)$.*

Proof. We first prove the result in the case where B is a pointed CW-complex.

(i) Let $h' : X| * \to F$ be an $\mathcal{F}$-homotopy inverse of h. Then the map $kh' : X| * \to Y|*$ is an $\mathcal{F}$-homotopy equivalence. We now apply lemma 3.5, taking (K, L) to be $(B, *)$, $q^\bullet$ to be $p \Box q$, $p^\bullet$ to be the projection $B \times C \to B$, α to be the map $\{*\} \to X \Box Y$ value kh' and β the identity map on B. Then $p^\bullet q^\bullet = p \Box_1 q$ is a weak homotopy equivalence (lemma 3.2) and we obtain a section ψ to $p \Box_1 q$. Further the $\mathcal{F}$-map $\psi(*) = w\alpha(*) = w(kh') : X| * \to Y|*$, where w is a self-map of $X \Box Y$ that is over $B \times C$ and homotopic over $B \times C$ to the identity on $X \Box Y$. Then the $\mathcal{F}$-maps $\psi(*)$ and $\alpha(*) = kh'$, from $X|*$ to $Y|*$, must $\mathcal{F}$-homotopic.

It follows from corollary 2.2 that there exists an $\mathcal{F}$-pairwise map $<\gamma, g>: p \to q$ with $\gamma|(X|*) = \psi(*) \simeq_{\mathcal{F}} kh'$. Let $x \in X|*$. Then $g(*) = gp(x) = q\gamma(x) = q(\psi(*)(x))$. Now $\psi(*): X|* \to Y|*$, so $\psi(*)(x) \in Y|*$ and $q(\psi(*)(x)) = *$. Hence $g(*) = *$, i.e. g is a pointed map.

There is an $\mathcal{F}$-map $(\gamma, p): X \to Y \sqcap B$ over B; the restrictions of (γ, p) to individual fibres are essentially the restrictions of γ to individual fibres, and so are $\mathcal{F}$-homotopy equivalences. Hence (γ, p) is an $\mathcal{F}FHE$ [B1, thm.5.4]. Now $(\gamma, p)|(X|*)$ agrees with $\gamma|(X|*)$, modulo the identification $(Y|*) \times \{*\} = Y|*$. So we have $\gamma|(X|*) \simeq_{\mathcal{F}} kh'$ and $(\gamma|(X|*))h \simeq_{\mathcal{F}} kh'h \simeq_{\mathcal{F}} k$. Hence (γ, p) is an F-grounded $\mathcal{F}FHE$ from (p, h) to (q_g, k).

(ii) We now deal with the case where $B \in \mathcal{W}^0$. Let $\mu: K \to B$ be a pointed homotopy equivalence, where K is a pointed CW-complex. Let $\nu: B \to K$ be a pointed map that is a homotopy inverse, in the pointed sense, to μ.

We know from (i) that there is a pointed map $g': K \to C$ such that $(q_{g'}, k) \equiv_F (p_\mu, h)$. Then $(q_{g'\nu}, k) = ((q_{g'})_\nu, k) \equiv_F ((p_\mu)_\nu, h) = (p_{\mu\nu}, h)$. It follows from [B2, lemma 4.3(ii)] that $(p_{\mu\nu}, h) \equiv_F (p, h)$. Hence, if we take $g = g'\nu$, then $(p, h) \equiv_F (q_{g'\nu}, k) = (q_g, k)$.

Proposition 3.7 *If there is a weakly contractible universal $\mathcal{F}$-fibration $q: Y \to C$, then $\mathcal{F}$ is set-valued in both the $\mathcal{F}FHE$ and $\mathcal{F}FHE^F$ senses.*

Proof. We first consider the $\mathcal{F}FHE^F$ case. There is only a set of based maps g from B to C, so it follows that there is only a set of possibilities for the grounded $\mathcal{F}$-fibre homotopy type of (q_g, k). If B is a pointed CW-complex, then we see via lemma 3.6 that $\mathcal{F}FHE^F(B)$ is a set. Hence $\mathcal{F}$ is $\mathcal{F}FHE^F$ set-valued.

A similar argument applies in the $\mathcal{F}FHE$ case. Let us, in part (i) of the proof of lemma 3.6, forget about base points and distinguished fibres. We have then shown the following result: for every $\mathcal{F}$-fibration $p: X \to B$ over a CW-complex B, there is a map $g: B \to C$ such that p is $\mathcal{F}FHE$ to q_g. It follows that $\mathcal{F}$ is $\mathcal{F}FHE$ set-valued.

We need to prove, in the proof of theorem 3.4, that the corresponding function ϕ is bijective. The surjectivity property follows easily from lemma 3.6; the following lemma establishes injectivity.

Lemma 3.8 *Let $B \in \mathcal{W}^0$ and $q: Y \to C$ be a weakly contractible universal $\mathcal{F}$-fibration, with $* \in C$ and $k \in \mathcal{F}(F, Y|*)$. If f and g are pointed maps from B to C such that (q_f, k) is F-grounded $\mathcal{F}FHE$ to (q_g, k), then $f \simeq^0 g$.*

Proof. (i) We will first prove the result in the case where B is a pointed CW-complex. Let $Y \sqcap_f B$ and $Y \sqcap_g B$ denote the spaces obtained by pulling Y back over f and g, respectively. Let $f_q: Y \sqcap_f B \to Y$ and $g_q: Y \sqcap_g B \to Y$ denote the projections. Then there are $\mathcal{F}$-pairwise maps $<f_q, f>: q_f \to q$ and $<g_q, g>: q_g \to q$. Let s be an F-grounded $\mathcal{F}FHE$ from (q_f, k) to (q_g, k); then there is also an $\mathcal{F}$-pairwise map $<g_q s, g> : q_f \to q$. Applying corollary 2.2, the pairs $<f_q, f>$ and $<g_q s, g>$ give rise to sections f^0 and g^0, respectively, to $(q_f)\Box_1 q$. Then $f^0(b)(y,b) = f_q(y,b) = y$ and $g^0(b)(y,b) = g_q s(y,b)$, where $q(y) = f(b)$. Hence $f^0(*) = 1_{(Y|*)}$ and $g^0(*) = s|(Y|*): Y|* \to Y|*$. Now the $\mathcal{F}FHE$ s is F-grounded, so there is an $\mathcal{F}$-homotopy $G: (Y|*) \times I \to Y|*$ from $1_{(Y|*)}$ to $s|(Y|*)$. It follows that there is a map $G^0: I \to \mathcal{F}(Y|*, Y|*)$, associated with G via the rule $G^0(t)(y) = G(y,t)$, where $y \in Y|*$ and $t \in I$. Hence G^0 is a path from $1_{(Y|*)}$ to $s|(Y|*)$ in $\mathcal{F}(Y|*, Y|*)$ (see [B1, 0.1]).

If $p^\bullet$ is the projection $B \times C \to B$ and $q^\bullet$ is $(q_f)\Box q$, then $p^\bullet q^\bullet$ is the weak homotopy equivalence $(q_f)\Box_1 q$ (lemma 3.2). As in lemma 3.6, this fact will enable us to apply lemma 3.5. We define the map $\alpha: (B \times \dot{I}) \cup (* \times I) \to (Y \sqcap_f B) \Box Y$ by

$$\alpha(b,t) = \begin{cases} f^0(b) & \text{if } b \in B \text{ and } t = 0 \\ g^0(b) & \text{if } b \in B \text{ and } t = 1 \\ G^0(t) & \text{if } b = * \text{ and } t \in I. \end{cases}$$

Let $(K, L) = (B \times I, (B \times \dot{I}) \cup (\{*\} \times I))$ and $\beta: B \times I \to B$ be the projection. Then, by lemma 3.5, there is a map $\Gamma^0 = \psi: B \times I \to (Y \sqcap_f B) \Box Y$ with $((q_f)\Box_1 q)\Gamma^0 = \beta$ and $\Gamma^0|(B \times \dot{I}) \cup (\{*\} \times I) = w\alpha$. Here w is a self-map of $(Y \sqcap_f B) \Box Y$ over $B \times C$ that is homotopic over $B \times C$ to the identity on $(Y \sqcap_f B) \Box Y$.

So we have $\Gamma^0(b,0) = wf^0(b)$ and $\Gamma^0(b,1) = wg^0(b)$, where $b \in B$, and $\Gamma^0(*,t) = w(G^0(t))$, where $t \in I$. In particular Γ^0 is a vertical homotopy from wf^0 to wg^0.

Applying corollary 2.3(a) to Γ^0, the rule $\Gamma(y,b,t) = \Gamma^0(b,t)(y,b)$ determines an $\mathcal{F}$-pairwise map $<\Gamma,\Delta>: q_f \times 1_I \to q$, where $q(y) = f(b)$. We will view $<\Gamma,\Delta>$ as being an $\mathcal{F}$-pairwise homotopy between two $\mathcal{F}$-pairwise maps $q_f \to q$, i.e from $<\gamma,\delta>$ to $<\zeta,\eta>$. Then, according to corollary 2.3(b), wf^0 and wg^0 correspond in the sense of corollary 2.2 to $<\gamma,\delta>$ and $<\zeta,\eta>$, respectively. Recalling that π_C denotes the projection $B \times C \to C$, we have:

δ	$=$	$((q_f)\Box_2 q)wf^0$	by corollary 2.2
	$=$	$\pi_C((q_f)\Box q)wf^0$	see the definition of $p\Box_2 q$
	$=$	$\pi_C((q_f)\Box q)f^0$	since w is a map over $B\times C$
	$=$	$((q_f)\Box_2 q)f^0$	see the definition of $p\Box_2 q$
	$=$	f	by corollary 2.2.

In a similar way $\eta = g$. Thus Δ is a homotopy from f to g.

Now $\Delta = ((q_f)\Box_2 q)\Gamma^0$ by corollary 2.3(a), it follows that $\Delta(*,t) = ((q_f)\Box_2 q)\Gamma^0(*,t) = ((q_f)\Box_2 q)w\alpha(*,t) = \pi_C((q_f)\Box q)w(G^0(t)) = \pi_C((q_f)\Box q)(G^0(t)) = ((q_f)\Box_2 q)(G^0(t))$. Also we have that $G^0(t): Y|* \to Y|*$, so $\Delta(*,t) = ((q_f)\Box_2 q)(G^0(t)) = *$. Hence Δ is a based homotopy, and so $f = \delta \simeq^0 \eta = g$.

(ii) Let us weaken our assumption on B, requiring only that $B \in \mathcal{W}^0$. We will assume that $\mu: K \to B$ is a pointed homotopy equivalence from a pointed CW-complex K into B. If $(q_f,k) \equiv_F (q_g,k)$, it follows that

$$(q_{f\mu},k) \equiv_F ((q_f)\mu,k) \equiv_F ((q_g)\mu,k) \equiv_F (q_{g\mu},k).$$

We see from (i) that $f\mu \simeq^0 g\mu$, and hence that $f \simeq^0 g$.

Proof of theorem 3.4. The first part is given in proposition 3.7, the main part follows from lemmas 3.6 and 3.8.

4 Universal $\mathcal{F}$-Fibrations II

We assume, throughout this section, that $\mathcal{F}$ is $\mathcal{F}FHE$ set-

valued and that the double retraction property [B2, def.3.2] **holds for $\mathcal{F}$-fibrations.** It follows that the subfibration replacement property holds for $\mathcal{F}$-fibrations [B2, thm.4.8], and hence that $\mathcal{F}$ is $\mathcal{F}FHE^F$ set-valued [B2, lem.6.2].

Theorem 4.1 *Let $r: Z \to D$ be an $\mathcal{F}$-fibration that is grounded universal amongst $\mathcal{F}$-fibrations. Then r is also weakly contractible universal.*

It follows from the theorem data that there is a point $* \in D$ and an $\ell \in \mathcal{F}(F, Z|*)$, such that $(r: Z \to D, \ell)$ is F-grounded universal amongst F-grounded $\mathcal{F}$-fibrations. Let $d \in D$. Then i_d will denote the inclusion $Z|d \subset Z$.

The proof of the theorem requires the following three lemmas. Their proofs require that we make careful distinctions between maps into the fibres of fibrations and the corresponding maps into the total spaces of the same fibrations. We will use the following alternative view of $Prin_F Z$.

Let $\{*\}$ denote a one point space and $e: F \to \{*\}$ denote the constant map. Then e carries the structure of an $\mathcal{F}$-overspace, in an obvious way. We can identify any $\xi \in \mathcal{F}(F, Z|d)$ with the $\mathcal{F}$-pairwise map $<(i_d)\xi, \delta>: e \to r$, where $d \in D$ and $\delta(*) = d$. On this view the underlying set of $Prin_F Z$ consists of all $\mathcal{F}$-pairwise maps from e to r.

Any path in $Prin_F Z$, i.e. a map $f^0: I \to Prin_F Z$, can be viewed as a path in $\mathcal{T}(F, Z)$. It hence determines a homotopy $f: F \times I \to Z$ by $f(x,t) = f^0(t)(x)$, where $x \in F$ and $t \in I$. Then $<f, (prin_F r)f^0>$ is an $\mathcal{F}$-pairwise homotopy between the $\mathcal{F}$-pairwise maps, from e to r, that correspond to $f^0(0)$ and $f^0(1)$. So we can view paths in $Prin_F Z$ as $\mathcal{F}$-pairwise homotopies, i.e. between the $\mathcal{F}$-pairwise maps determined by the endpoints of the paths.

Lemma 4.2 $\pi_0(Prin_F Z) = 0$.

Proof. There is just a single $\mathcal{F}FHE^F$-class of F-grounded $\mathcal{F}$-fibrations over a discrete pointed space with just two elements. It follows, from the grounded universality of (r, ℓ), that there is just a single pointed homotopy class of pointed maps from such a two-point space to D. Hence D is path connected.

Let $\xi \in Prin_F Z$. We can now apply the the $\mathcal{F}$WCHP property of r, using the $\mathcal{F}$-overspace e, the map $(i_d)\xi : F \to Z$, where $d = (prin_F r)\xi$, and a path in D from $(prin_F r)(\xi)$ to $*$. This path should be stationary on the interval $[0, \frac{1}{2}]$. It follows that every element of $Prin_F Z$ is $\mathcal{F}$-pairwise homotopic to a member of $\mathcal{F}(F, Z|*)$. Hence it is sufficient to show that $\ell \in \mathcal{F}(F, Z|*)$ and an arbitrarily chosen member of $\mathcal{F}(F, Z|*)$, viewed as $\mathcal{F}$-pairwise maps from e to r, must necessarily be $\mathcal{F}$-pairwise homotopic to each other. For then all elements of $Prin_F Z$ are $\mathcal{F}$-pairwise homotopic to each other, and hence in the same path component of $Prin_F Z$.

We will now define an $\mathcal{F}$-fibration over S^1, and use it to establish the above sufficient condition. If $h \in \mathcal{F}(F, F)$, then $(e : F \to \{*\},\ h)$ must be an F-grounded $\mathcal{F}$-fibration. It follows, using the double retraction property for $\mathcal{F}$-fibrations, that there exists an $\mathcal{F}$-space $Z(h)$, together with $\mathcal{F}$-maps $\imath : F \to Z(h)$, $\jmath : F \to Z(h)$ and $\sigma : Z(h) \to F$. These satisfy $\jmath\sigma \simeq_{\mathcal{F}} 1_{Z(h)}$, $\sigma\imath = h$ and other conditions. We notice that $\imath \simeq_{\mathcal{F}} \jmath\sigma\imath = \jmath h$. In practise $Z(h)$ is likely to be an appropriately defined mapping cylinder for h.

We define $F \triangle F$ to be the quotient space of

$$(F \times [0, \frac{1}{3}]) \sqcup (Z(h) \times [\frac{1}{3}, \frac{2}{3}]) \sqcup (F \times [\frac{2}{3}, 1])$$

obtained by identifying $(x, \frac{1}{3})$ with $(\imath(x), \frac{1}{3})$, $(x, \frac{2}{3})$ with $(\jmath(x), \frac{2}{3})$ and $(x, 0)$ with $(x, 1)$, where $x \in F$.

Let us view S^1 as being obtained by identifying together the 0 and 1 ends of the unit interval I. The function $e\triangle h : F\triangle F \to S^1$ is defined to be the obvious projection; it clearly is continuous and carries the structure of an $\mathcal{F}$-overspace.

Applying [B2, prop.4.5] with "B" = "C" = $\{*\}$, "p" = "q" = e, and using the above h, we obtain the $\mathcal{F}$-fibration $e \natural h : F \natural F \to I$, since $I \cong MC(\{*\} \to \{*\})$. We notice that $(F\triangle F)|(0,1) = (F\natural F)(0,1)$, and so $(e\triangle h)|(0,1) = (e\natural h)|(0,1)$. Now $e\natural h$ satisfies the $\mathcal{F}WCHP$, hence so also does $(e\triangle h)|(0,1)$. Taking $[0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$ as a subspace of I, the identification $I \to S^1$ determines a corresponding open subspace W of S^1. Further $(F\triangle F)|W = F \times W$. Hence $(e\triangle h)|W : (F\triangle F)|W \to W$ is the projection and trivial $\mathcal{F}$-overspace $F \times W \to W$, and $(e\triangle h)|W$ satisfies the $\mathcal{F}WCHP$. The cover $\{(0,\ 1),\ W\}$ of S^1 is numerable; it follows from [B1, thm.4.7] that $e\triangle h$ is an $\mathcal{F}$-fibration.

We will now introduce two other ways of specifying $\mathcal{F}$-pairwise maps. If $<\gamma,\ \delta>$ is an $\mathcal{F}$-pairwise map, then this same map can alternatively be described by the notations $<\gamma,\ b\to\delta(b)>$ and $<x\to\gamma(x),\ b\to\delta(b)>$, where x and b are typical elements in the domains of γ and δ, respectively.

Let $\chi: F\to F\triangle F$ be the map defined by $\chi(x)=(x,*)$, where $x\in F$. Then χ is a homeomorphism into. The members of the following list of $\mathcal{F}$-pairwise maps from e to $e\triangle h$ are, in turn, $\mathcal{F}$-pairwise homotopic. This is clear for (i) and (ii) from the definition of $F\triangle F$, for (ii) and (iii) using the homotopy $\jmath h\simeq_{\mathcal{F}}\imath$ and for (iii) and (iv) via the definition of $F\triangle F$.

(i) $<\chi,\ *\to *> \ = \ <x\to(x,[0]),\ *\to[0]>$,

(ii) $<x\to([x],\tfrac{1}{3}),\ *\to\tfrac{1}{3}> \ = \ <x\to(\imath[x],\ \tfrac{1}{3}),\ *\to\tfrac{1}{3})>$,

(iii) $<x\to([\jmath h(x)],\ \tfrac{2}{3}),*\to\tfrac{2}{3}> \ = \ <x\to([h(x)],\ \tfrac{2}{3}),\ *\to\tfrac{2}{3}>$, and

(iv) $<x\to(h(x),[1]),\ *\to[1]> \ = \ <\chi h,\ *\to *>$.

Hence the $\mathcal{F}$-pairwise maps $<\chi,\ *\to *>$ and $<\chi h,\ *\to *>$ are $\mathcal{F}$-pairwise homotopic. We notice that the two ranges of this $\mathcal{F}$-pairwise homotopy circle around $F\triangle F$ and S^1, respectively.

Let us take $[0]=[1]$ to be the distinguished point of S^1, and denote it by $*$. Then the distinguished fibre of $e\triangle h$ is $(e\triangle h)^{-1}(*)=F\times\{*\}$. This $\mathcal{F}$-space will be identified with and replaced by F. Then the inclusion of this fibre in $F\triangle F$ is χ and the pair $(e\triangle h, 1_F)$ is an F-grounded $\mathcal{F}$-fibration. There is a pointed classifying map $\eta: S^1\to D$ for $(e\triangle h, 1_F)$, i.e. $(r\eta,\ell)$ is F-grounded $\mathcal{F}FHE$ to $(e\triangle h, 1_F)$. It follows, by composition of this $\mathcal{F}FHE$ with the projection $Z\sqcap_\eta S^1\to Z$, that there is an $\mathcal{F}$-pairwise map $<\zeta,\eta>$ from $e\triangle h$ to r, such that $\ell\simeq_{\mathcal{F}}\zeta|F$.

So there are $\mathcal{F}$-pairwise homotopies from e to r, determined by the following known homotopies between maps from F to Z:

$$i_*\ell\simeq i_*(\zeta|F)=\zeta\chi\simeq\zeta\chi h\simeq i_*\ell h.$$

Hence $<i_*\ell, * \to *>$ and $<i_*\ell h, * \to *>$ are $\mathcal{F}$-pairwise homotopic. If $k \in \mathcal{F}(F, Z|*)$, then we can choose $h \in \mathcal{F}(F, F)$ such that $\ell h \simeq_{\mathcal{F}} k$. Then $<i_*\ell h, * \to *>$ and $<i_*k, * \to *>$ are $\mathcal{F}$-pairwise homotopic, so $<i_*\ell, * \to *>$ and $<i_*k, * \to *>$ are $\mathcal{F}$-pairwise homotopic. The result follows.

Lemma 4.3 *(= a generalization of [A, lem.4.1]). Let j denote the inclusion map $\mathcal{F}(F, Z|*) \subset Prin_F Z$. If K is a pointed CW-complex, and $u: K \to Prin_F Z$ is a map with $u(*) = \ell \in \mathcal{F}(F, Z|*)$, then there is a map $v: K \to \mathcal{F}(F, Z|*)$ such that $u \simeq jv$.*

Proof. We notice that u corresponds, via the exponential law [B1, 0.1], to a map $\gamma: F \times K \to Z$ defined by $\gamma(x, y) = u(y)(x)$, for all $x \in F$ and $y \in K$ Let $\delta: K \to D$ denote the composite $(prin_F r)u$ and π_K denote the projection $F \times K \to K$. Then

$$r\gamma(x, y) = r(u(y)(x)) = (prin_F r)u(y) = \delta(y) = \delta\pi_K(x, y),$$

for $x \in F$ and $y \in K$, i.e. $r\gamma = \delta\pi_K$; it follows that $<\gamma, \delta>$ is an $\mathcal{F}$-pairwise map from π_K to r. We notice that $\gamma|F: F \times \{*\} \to Z|*$ is the $\mathcal{F}$-map $(x, *) \to \ell(x)$, where $x \in F$. Hence, taking $\epsilon: F \to F \times \{*\}$ to be the $\mathcal{F}$-homeomorphism $x \to (x, *)$ where $x \in F$, δ is a classifying map for the trivial grounded $\mathcal{F}$-fibration (π_K, ϵ). Now $c: K \to D$, the constant map to $*$, is also a classifying map for (π_K, ϵ). It follows that $\delta \simeq^0 c$. A homotopy from δ to c that is stationary on the values $[0, \frac{1}{2}]$ can be selected. Then, since r has the $\mathcal{F}WCHP$, c can be lifted to $\zeta: F \times K \to Z$ such that $<\zeta, c>$ is an $\mathcal{F}$-pairwise map from π_K to r that is $\mathcal{F}$-pairwise homotopic to $<\gamma, \delta>$. The range of ζ is $Z|*$; hence ζ determines a map $v: K \to \mathcal{F}(F, Z|*)$ with $v(y)(x) = \zeta(x, y)$, where $y \in K$ and $x \in F$. The homotopy $\gamma \simeq \zeta: F \times K \times I \to Z$ similarly determines a homotopy $K \times I \to Prin_F Z$ from u to jv.

The next result is largely a generalization of [A, lem.4.2]. However, we avoid the quasi-fibration aspect of the argument of [A], as that would lead us into making unwanted assumptions concerning $\mathcal{F}$.

Lemma 4.4 *Let K be a pointed CW-complex and $v: K \to \mathcal{F}(F, Z|*)$ be a map. Then, with j as in the previous lemma, $jv: K \to Prin_F Z$ is freely homotopic to a constant map.*

Proof. The exponential law [B1, 0.1] determines a map $v': F \times K \to Z|*$ by $v'(x,y) = v(y)(x)$, for $x \in F$ and $y \in K$. Let $h: F \times K \to (Z|*) \times K$ denote the map $h(x,y) = (v'(x,y), y)$, where $x \in F$ and $y \in K$. It follows from [B1, thm.5.4] that h is an $\mathcal{F}FHE$ between the projections $p: F \times K \to K$ and $q: (Z|*) \times K \to K$.

We will now use h to define a corresponding $\mathcal{F}$-fibration ρ. It follows, via the double retraction property for $\mathcal{F}$-fibrations, that there is an $\mathcal{F}$-overspace $\mu: Z(h) \to K$. Further there are maps $\imath: F \times K \to Z(h)$, $\jmath: (Z|*) \times K \to Z(h)$ and $\sigma: Z(h) \to (Z|*) \times K$ that are all $\mathcal{F}$-maps over K. These must satisfy $\jmath\sigma \simeq_{\mathcal{F}_K} 1_F$, $\sigma\imath = h$ and other conditions. We notice that $\jmath h = \jmath\sigma\imath \simeq_{\mathcal{F}_K} \imath$. In practise $Z(h)$ is likely to be an appropriately defined mapping cylinder for h.

We define P to be the quotient space of

$$(F \times K \times [0, \tfrac{1}{3}]) \sqcup (Z(h) \times [\tfrac{1}{3}, \tfrac{2}{3}]) \sqcup ((Z|*) \times K \times [\tfrac{2}{3}, 1])$$

under the relations $(x,y,0) \equiv (x,y',0)$, $(x,y,\frac{1}{3}) \equiv (\imath(x,y),\frac{1}{3})$, $(z,y,\frac{2}{3}) \equiv (\jmath(z,y),\frac{2}{3})$, and $(z,y,1) \equiv (z,y',1)$, for all $x \in F$, y, $y' \in K$ and $z \in Z|*$.

Let us define the suspension space SK to be the quotient space of $K \times I$ under the relations $(y,0) \equiv (y',0)$ and $(y,1) \equiv (y',1)$, for all $y, y' \in K$. The "0" and "1" endpoints of SK will be denoted by [0] and [1], respectively.

We define the map $\rho: P \to SK$ to be the obvious projection, i.e. using μ on the "middle section" of P. Then ρ can clearly be taken to be an $\mathcal{F}$-overspace of SK.

Let CK denote the quotient space of $K \times [0, \frac{1}{3})$ under the relation $(y,0) \equiv (y',0)$, for all y, $y' \in K$. Then $\rho|CK : P|CK \to CK$ is the projection and trivial $\mathcal{F}$-fibration $F \times CK \to CK$. Let $C'K$ denote the analagous subspace of SK corresponding to $(\frac{2}{3}, 1]$; then $\rho|C'K$ is the projection and trivial $\mathcal{F}$-fibration $(Z|*) \times C'K \to C'K$.

We now use the $q\natural h$ construction of [B2, section 4] with "B" = "C' = K, thus obtaining an $\mathcal{F}$-overspace $q\natural h: ((Z|*) \times K)\natural (F \times K) \to K \times I$. Let V denote the subspace $K \times (0,1)$ of SK. We know that $q\natural h$ is an $\mathcal{F}$-fibration [B2, prop.4.5], hence so also is $\rho|V = (q\natural h)|(K \times (0,1))$.

Now $\{CK,\ C'K,\ V\}$ is a numerable cover of SK , so ρ must be an $\mathcal{F}$-fibration [B1, prop.6.2].

Let $\chi_0 : F \to P$ be defined by $x \to (x, \{K\}, 0)$, where $x \in F$, and $\chi_1 : Z| * \to P$ be defined by $z \to (z, \{K\}, 1)$, where $z \in Z|*$. Then χ_0 and χ_1 are homeomorphisms into P; their images are the fibres of ρ over $[0]$ and $[1]$, respectively. We now identify F and $Z|*$ with their images under these homeomorphisms. So the fibres of ρ over $[0]$ and $[1]$ are now F and $Z|*$, respectively; their inclusions in P are now χ_0 and χ_1, respectively.

Taking $* = [1] \in SK$ as basepoint, ρ has distinguished fibre $Z|*$ and (ρ, ℓ) is an F-grounded $\mathcal{F}$-fibration. Then (ρ, ℓ) has a classifying map $\delta \in \mathcal{T}^0(SK, D)$, relative to the F-grounded universal $\mathcal{F}$-fibration (r, ℓ). It follows that there is an $\mathcal{F}$-pairwise map $<\gamma, \delta>$ from ρ to r such that $(\gamma|(P|*))\ell \simeq_{\mathcal{F}} \ell$. Hence $(\gamma|(P|*)) \simeq_{\mathcal{F}} 1_{Z|*}$.

The following $\mathcal{F}$-pairwise maps from $p : F \times K \to K$ to $\rho : P \to SK$ are, in turn, $\mathcal{F}$-pairwise homotopic. This is clear for (iii) and (iv) using the homotopy $\imath \simeq_{\mathcal{F}_K} \jmath h$; in all other cases it follows via the definition of P. We assume, throughout this list, that $x \in F$ and $y \in K$.

(i) $<(x,y) \to \chi_0(x), \quad y \to [0]>$,

(ii) $<(x,y) \to (x, y, \frac{1}{6}), \quad y \to (y, \frac{1}{6})>$,

(iii) $<(x,y) \to ((x,y), \frac{1}{3}) = (\imath(x,y), \frac{1}{3}), \quad y \to (y, \frac{1}{3})>$,

(iv) $<(x,y) \to (\jmath h(x,y), \frac{2}{3}) = (h(x,y), \frac{2}{3}), \quad y \to (y, \frac{2}{3})>$,

(v) $<(x,y) \to (h(x,y), \frac{5}{6}), \quad y \to (y, \frac{5}{6})>$ and

(vi) $<(x,y) \to \chi_1 v'(x,y), \quad y \to [1] = *>$.

Let $\pi_F : F \times K \to F$ denote the projection. We have seen that the $\mathcal{F}$-pairwise maps (i) $<\chi_0 \pi_F, y \to [0]>$ and (vi) $<\chi_1 v', y \to *>$, where $y \in K$, are $\mathcal{F}$-pairwise homotopic.

Let c_0 and c_1 denote the constant maps $K \to D$, with values $\delta([0])$ and $\delta[1] = \delta(*) = *$. Composing the last two $\mathcal{F}$-pairwise maps with $<\gamma, \delta>$, we obtain the $\mathcal{F}$-pairwise maps $<\gamma\chi_0\pi_F, c_0>$ and $<\gamma\chi_1 v', c_1>$ from p to r. Recalling that $i_* : Z| * \to Z$ denotes the inclusion and that $\gamma\chi_1 = i_*(\gamma|(P|*))$, we see that $<\gamma\chi_1 v', c_1> = <i_*(\gamma|(P|*))v', c_1>$. Now there is an $\mathcal{F}$-homotopy between $\gamma|(Z|*)$ and the identity on $Z|*$, so $<i_*(\gamma|(P|*))v', c_1>$ is $\mathcal{F}$-pairwise homotopic to $<i_*v', c_1>$. Hence we have an $\mathcal{F}$-pairwise homotopy from $<\gamma\chi_0\pi_F, c_0>$ to $<i_*v', c_1>$.

Applying the exponential law [B1, 0.1] to this homotopy, we see that the constant map to $\gamma\chi_0 \in Prin_F Z$ and jv are freely homotopic maps of K into $Prin_F Z$.

Proof of theorem 4.1. The $n = 0$ case is lemma 4.2. Let us now assume that $n > 0$, (r, ℓ) is grounded universal amongst F-grounded $\mathcal{F}$-fibrations and that the map $u: S^n \to Prin_F Z$ has $u(*) = \ell$. It follows from lemmas 4.3 and 4.4 that u is freely homotopic to a constant map, and therefore (lemma 4.2) freely homotopic to the constant map to ℓ. We see, via [S, thm.1.3.12], that $\pi_n(Prin_F Z, \ell) = 0$.

We now combine the main results of the last two sections.

Theorem 4.5 *Let $(\mathcal{E}, U_{\mathcal{E}}, \{X \times_{\mathcal{E}} I\})$ be a category of well enriched spaces under a space A, F be an $\mathcal{E}$-space, $\mathcal{F}$ be the category of fibres in $\mathcal{E}$ determined by F and $p: X \to B$ be an $\mathcal{F}$-fibration over a CW-complex B. Then p is a weakly contractible universal $\mathcal{F}$-fibration if and only if $\mathcal{F}$ is $\mathcal{F}FHE^F$ set-valued and p is grounded universal amongst $\mathcal{F}$-fibrations.*

Proof. It follows from [B2, lem.2.4] that $\mathcal{E}$ carries the structure of a category of enriched spaces and from [B2, thm.3.7] that the double retraction property holds for $\mathcal{F}$-fibrations. The "only if" part follows from theorem 3.4; the "if" part from theorem 4.1.

Note. A fibration, as in theorem 4.5, is also *free universal amongst $\mathcal{F}$-fibrations* [B2, def.2.2 and prop.7.4].

Definition 4.6 *If an $\mathcal{F}$-fibration is free universal amongst $\mathcal{F}$-fibrations, grounded universal amongst $\mathcal{F}$-fibrations and weakly contractible universal, then it will be said to be a universal $\mathcal{F}$-fibration.*

5 The Set-theoretical Difficulty and the Main Result

We assume, throughout this section, that $(\mathcal{E}, U_{\mathcal{E}}, \{X \times_{\mathcal{E}} I\})$ is a category of well enriched spaces under a space A. Taking $\mathcal{A}$ to denote the category of spaces under A, we recall that $U_{\mathcal{E}}$ is a functor from $\mathcal{E}$ to $\mathcal{A}$, and that there is also an underlying space functor $U_{\mathcal{A}} : \mathcal{A} \to \mathcal{T}$ (see [B2, section 2]). In the following discussion we refer to *the underlying set of an $\mathcal{E}$-space X*. This should be interpreted to mean the underlying set of the underlying topological space $U_{\mathcal{A}} U_{\mathcal{E}}(X)$.

Definitions 5.1 *(i) Let κ be a cardinal number. If the underlying set of an $\mathcal{E}$-space has cardinality less than or equal to κ, then it will be said to be a $\kappa\mathcal{E}$-space. Let $\kappa\mathcal{E}$ be the full subcategory of $\mathcal{E}$ containing all such $\kappa\mathcal{E}$-spaces. Then the functor $U_{\kappa\mathcal{E}} : \kappa\mathcal{E} \to \mathcal{A}$ will be defined to be the restriction to $\kappa\mathcal{E}$ of the functor $U_{\mathcal{E}} : \mathcal{E} \to \mathcal{A}$. If X is a $\kappa\mathcal{E}$-space, then we define the cylinder $X \times_{\kappa\mathcal{E}} I$ to be the cylinder $X \times_{\mathcal{E}} I$.*

(ii) The category $\mathcal{E}$ will be said to be proper if, for every choice of a category of fibres $\mathcal{F}$ in $\mathcal{E}$ and of space X under A, the class of all associated $\mathcal{F}$-space structures on X is a set.

Proposition 5.2 *Let κ be a cardinal number greater than or equal to the cardinality of the continuum. Then:*

(i) $(\kappa\mathcal{E}, U_{\kappa\mathcal{E}}, \{X \times_{\kappa\mathcal{E}} I\})$ is a (possibly empty) category of well enriched spaces under A, and

(ii) $\kappa\mathcal{E}$ is $\kappa\mathcal{E}FHE$ set-valued.

Proof. The functor $U_{\kappa\mathcal{E}}$ is clearly faithful. If S is an $\mathcal{E}$-space, then $card(S)$ will denote the cardinality of the underlying set of S. Then, if X is an object of $\kappa\mathcal{E}$, we have

$$card(X \times_{\mathcal{E}} I) = card(X \times_{\mathcal{A}} I) \leq card(X) \times card(I) \leq \kappa.$$

Hence, for such X, the $\mathcal{E}$-cylinder $X \times_{\mathcal{E}} I$ is also in $\kappa\mathcal{E}$. Given a morphism f of $\kappa\mathcal{E}$, a similar argument can be applied to $\kappa\mathcal{E}MC(f) = \mathcal{E}MC(f)$; the rest is easy.

(ii) We see from the definition of $\kappa\mathcal{E}$ that, for every $\kappa\mathcal{E}$-overspace $p: X \to B$, $card(X) \leq \kappa \times card(B)$. Hence, if $\kappa\mathcal{E}$-overspaces of B are classified up to homeomorphism over B , i.e. $\mathcal{T}$-homeomorphism over B, then there is just a set of such types.

We notice that categories of fibres in $\kappa\mathcal{E}$ take the form $\kappa\mathcal{F}$, where $\mathcal{F}$ is a category of fibres in $\mathcal{E}$. Then $\kappa\mathcal{F} = \mathcal{F} \cap \kappa\mathcal{E}$. We know that every $\kappa\mathcal{F}$-fibration is a $\kappa\mathcal{E}$-overspace, so there is just a set of homeomorphism over B types of $\kappa\mathcal{F}$-fibrations over B. Also $\mathcal{E}$ is proper, so there is just a set of $\kappa\mathcal{F}$-homeomorphism over B types of $\kappa\mathcal{F}$-fibrations over B. Now any two $\kappa\mathcal{F}$-fibrations over B, that are $\kappa\mathcal{F}$-homeomorphic over B, are necessarily $\kappa\mathcal{F}FHE$. It follows that all such $\kappa\mathcal{F}FHE(B)$ are sets, and so $\kappa\mathcal{E}$ is $\kappa\mathcal{E}FHE$ set-valued.

Theorem 5.3 : Main Result. *Let $(\mathcal{E}, U_{\mathcal{E}}, \{X \times_{\mathcal{E}} I\})$ be a proper category of well enriched spaces under a space A, F be an $\mathcal{E}$-space and $\mathcal{F}$ be the category of fibres determined by F. Then there is a universal $\mathcal{F}$-fibration over a path connected CW-complex.*

Proof. Let κ be the greater of the cardinalities of the underlying set of F and of the continuum. Proposition 5.2 then allows us to apply [B2, thm.8.1((a) $\Rightarrow$ (d))] and [B2, prop.7.3] in the $\kappa\mathcal{F}$ context. Thus we take the "$\mathcal{F}$" of that proposition 7.3 and the "$\mathcal{E}$" of that theorem 8.1 to be our present $\kappa\mathcal{F}$. It then follows that there exists a grounded universal $\kappa\mathcal{F}$-fibration $p_{\kappa\mathcal{F}}: E_{\kappa\mathcal{F}} \to B_{\kappa\mathcal{F}}$ over a path connected CW-complex $B_{\kappa\mathcal{F}}$. Applying theorem 4.1 in the $\kappa\mathcal{F}$ context, we see that $p_{\kappa\mathcal{F}}$ is weakly contractible universal.

Now $\kappa\mathcal{F} \subset \mathcal{F}$, so every $\kappa\mathcal{F}$-fibration is an $\mathcal{F}$-fibration. Hence $p_{\kappa\mathcal{F}}$ is a weakly contractible universal $\mathcal{F}$-fibration. Applying theorem 4.5 in the $\mathcal{F}$ context, we see that $\mathcal{F}$ is $\mathcal{F}FHE^F$ set-valued, and that $p_{\kappa\mathcal{F}}$ is universal amongst $\mathcal{F}$-fibrations.

6 Examples

The following brief comments are intended to make our discussion of proper categories of well enriched spaces more concrete. We show that this concept does indeed apply to the three "classical" theories of fibrations. It follows that theorem 5.3 holds for each of these cases. **A more extensive and detailed discussion of these and other examples will be given in [B3].**

(i) Dold Fibrations and Hurewicz Fibrations. Let $\mathcal{E}$ be the category $\mathcal{T}$ of all (of course cg-) spaces and maps, A be the empty space and $U_{\mathcal{A}} = U_{\mathcal{E}} = 1_{\mathcal{T}}$, the identity functor on $\mathcal{T}$. The cylinder $X \times_{\mathcal{E}} I$ is just the cartesian product space $X \times I$. It may be seen that $(\mathcal{T}, 1_{\mathcal{T}}, \{X \times I\})$ is a category of well enriched spaces.

The additional structure required to make a space into an $\mathcal{E}$-space is in this case empty, so the class of associated $\mathcal{E}$-structures on any given space is just a singleton set. Thus this category of well enriched spaces is proper.

Given a space F, we define $\mathcal{F}$ to consist of all spaces that are homotopy equivalent to F and all homotopy equivalences between such spaces. Then theorem 5.3 applies to Dold fibrations.

Any Dold fibration, like any other map, can be factorized as the composite of a Hurewicz fibration and a homotopy equivalence. It follows from [D, thm.6.1] that the homotopy equivalence is a fibre homotopy equivalence. This result enables us to derive a classification theory for Hurewicz fibrations from that for Dold fibrations.

(ii) Principal Fibrations. Let G be a topological monoid. We take both $\mathcal{E}$ and $\mathcal{F}$ to be the category with G-spaces that are G-homotopy equivalent to G as objects, and G-homotopy equivalences between such G-spaces as morphisms. Then A will again be the empty space, and $U_{\mathcal{E}} : \mathcal{E} \to \mathcal{T}$ will be the functor that forgets G-actions. We notice that $U_{\mathcal{A}}$ is again the identity functor on $\mathcal{T}$. If X is a G-space, then we define $X \times_{\mathcal{E}} I$ to be the space $X \times I$, with the action $(g,(x,t)) \to (gx,t)$, where $g \in G, x \in X$, and $t \in I$. Then it may be seen that $(\mathcal{E}, U_{\mathcal{E}}, \{X \times I\})$ is a category of well enriched spaces.

Let X be a space. Then the actions $G \times X \to X$ correspond to the continuous homomorphisms from G into the group of self-homeomorphisms of X. Hence there is only a set of such actions, and our category of well enriched spaces is proper. So theorem 5.3 applies to principal fibrations.

(iii) Sectioned Fibrations. Let us take $\mathcal{E}$ to be $\mathcal{T}^0$, the category of pointed spaces and pointed maps, A to be a singleton space and $U_{\mathcal{E}}$ to be the identity functor 1^0 on $\mathcal{T}^0$. Then $U_{\mathcal{A}} : \mathcal{T}^0 \to \mathcal{T}$ is the functor that forgets base points. If $(X, *)$ is a pointed space, then the associated cylinder is defined to be the quotient space $X \times I / \{*\} \times I$. It can be seen that $(\mathcal{T}^0, 1^0, \{X \times I/\{*\} \times I\})$ is a category of well enriched spaces.

The additional structure required to make a pointed space into an $\mathcal{E}$-space is empty, so the class of associated $\mathcal{E}$-structures on any given space is just a singleton set. Hence this category of well enriched spaces is proper.

Let F be a given pointed space. We define the corresponding $\mathcal{F}$ to consist of all pointed spaces that are pointed homotopy equivalent to F, and all pointed homotopy equivalences between such pointed spaces. Then theorem 5.3 applies to sectioned fibrations.

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