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TOPOLOGICAL QUASITOPOS HULLS OF CATEGORIES CONTAINING TOPOLOGICAL AND METRIC OBJECTS

by E. and R. LOWEN

RÉSUMÉ. La catégorie des espaces topologiques et celle des espaces métriques généralisés se plongent "bien" dans la catégorie *AP* des espaces d'approximation (approach spaces). Dans cet article, on construit un quasitopos topologique enveloppe de *AP*. La construction se fait en deux étapes: on construit d'abord une enveloppe topologique extensionnelle, *PRAP*, de *AP* ; puis une enveloppe cartésienne fermée topologique de *PRAP*. Les objets de ces catégories sont caractérisées en utilisant la notion de fonctions limites.

1. INTRODUCTION.

We would like to motivate the contents of this paper on two fundamentally different levels.

First there is the general motivation for the theory of approach spaces as developed in this paper and its forerunners [12, 13, 14], namely to introduce a unification of the theories of convergence and topological spaces on the one hand and of metric spaces on the other hand. *AP*, as introduced in [13], is a topological category with *TOP* and *pq-MET*$^\infty$ (extended pseudo-quasi metric spaces and non-expansive maps) embedded as full subcategories, and *CAP* is a topological quasitopos with *Conv*, *pqs-MET*$^\infty$ (extended pseudo-quasi-semimetric spaces and non-expansive maps) and *AP* embedded as full subcategories. These "unification-categories" turn out to be advantageous in several respects.

(1) They permit new viewpoints concerning the metrizability of topological spaces and give more faithful relations between metrics and their associated topologies. Thus e.g. the natural functor *pq-MET*$^\infty \rightarrow$ *TOP* does not preserve products. Embedding *pq-MET*$^\infty$ and *TOP* in *AP* this functor extends to the *TOP*-coreflector which does preserve products. However the product of *pq-MET*$^\infty$ objects in general is not a *pq-MET*$^\infty$ object.

(2) They permit unification of topological and metric concepts. Thus e.g. connectedness and Cantor's Kettenzussamenhang (or uniform connectedness) are special cases of a categorical concept of connectedness in *AP* in the sense of Preuss [19], and compactness and total boundedness both are special instances of a generalization in *AP* of Kuratowski's measure of non-compactness in complete metric spaces [11].

(3) There are examples of topological and convergence spaces prominently used in Probability Theory and Statistics, such as spaces of measures with the weak topology and spaces of random variables with the topology of convergence in measure or the convergence structure of convergence a.e., which can more naturally be equipped with structures in *AP* and *CAP* such that the classical structures are obtained as certain coreflections.

Second there is the specific motivation for the present paper. It has been recognized by an increasing number of mathematicians that a category is no longer nice merely because it is topological. For several reasons (see e.g. [6,16]) it is desirable to have more convenience properties. In particular, with respect to function spaces or final sinks it is extremely nice to work in a quasitopos in the sense of Penon [18], i.e. an extensional cartesian closed topological category. (See also [6]; note that in [6] "extensional" is called "hereditary".) *AP* is topological but not extensional, nor cartesian closed. The bigger category *CAP* on the other hand is a topological quasitopos [4]. In this paper we show that *AP* can be embedded in an even smaller topological quasitopos. Taking into account that the smaller the extension of a category *A* is, the more structure of *A* it will preserve, it is interesting to find the topological quasitopos hull TQ(*A*) of *A* (its Wyler completion) which is characterized as its smallest finally dense topological quasitopos extension [6]. Important topological quasitopos hulls which have been found in the last decade are e.g. TQ(*TOP*) = the category of pseudotopological spaces (O. Wyler [20]), TQ(*UNIF*) = the category of submetrizable bornological merotopic spaces (J. Adamek and J. Reiterman [1]). It is the purpose of this paper to describe TQ(*AP*), the topological quasitopos hull of the category of approach spaces.

2. PRELIMINARIES.

In this paper all subcategories are supposed to be full and isomorphism closed. We denote $\mathbf{P} := [0,\infty]$. Subtraction in $\mathbf{P}$ is defined as the usual operation on $\mathbf{P}\setminus\{\infty\}$, if $a \in \mathbf{P}\setminus\{\infty\}$ then

$$\infty - a := \infty \text{ and } \infty - \infty := 0.$$

Negative values are truncated at 0, i.e., if a and b are variables in $\mathbf{P}$ and possibly $a < b$ then we always use and write $(a-b)\vee 0$.

An *approach space* is a pair (X,δ) where $X \in |SET|$ and where $\delta : X \times 2^X \to \mathbf{P}$ is a map which has the properties:

(D1) $\forall A \in 2^X \; \forall x \in A : \delta(x,A) = 0$.

(D2) $\forall x \in X : \delta(x,\emptyset) = \infty$.

(D3) $\forall A, B \in 2^X \; \forall x \in X : \delta(x, A \cup B) = \delta(x,A) \wedge \delta(x,B)$.

(D4) $\forall A \in 2^X \; \forall x \in X \; \forall \varepsilon \in \mathbf{P} : \delta(x,A) \leq \delta(x,A^{(\varepsilon)}) + \varepsilon$ where

$$A^{(\varepsilon)} := \{x \in X \mid \delta(x,A) \leq \varepsilon\}.$$

The map δ is called a *distance*. For $A \subset X$ we put $\delta_A := \delta(\cdot, A)$.

Given approach spaces (X,δ) et (X',δ') a map $f : X \to X'$ is called a *contraction* if it fulfills the property that

$$\text{for all } A \subset X \text{ and } x \in X : \delta'(f(x), f(A)) \leq \delta(x,A).$$

The category *AP* of approach spaces and contractions is a topological category in the sense of Herrlich [5] and it was studied extensively in [13].

In [12] the authors showed that *AP* could also be described by means of a concept of limits. First some notations. Given a set X, $\mathbf{F}(X)$ stands for the set of all filters on X; if $F \in \mathbf{F}(X)$, then $\mathbf{U}(F)$ stands for the set of all ultrafilters finer than F;

$$\sec F := \{A \subset X \mid \forall F \in F : A \cap F \neq \emptyset\}, \text{ i.e., } \sec F = \cup\{U \mid U \in \mathbf{U}(F)\};$$

if $F = \{X\}$ then we write $\mathbf{U}(X)$ instead of $\mathbf{U}(\{X\})$. If $A \subset 2^X$ then

$$\text{stack}_X A := \{B \subset X \mid \exists A \in A : A \subset B\},$$

if A consists of a single set A we write $\text{stack}_X A$ and if moreover A consists of a single point a, we write $\text{stack}_X a$ for short; moreover if no confusion can occur we drop the subscript X.

If $(S_j)_{j \in J}$ is a family of sets then elements of their product are often denoted functionally, i.e., $s \in \Pi_{j \in J} S_j$ means $s(j) \in S_j$ for all $j \in J$. If $(F_j)_{j \in J}$ is a family of filters on X and A is a filter on J then

$$D((F_j)_{j \in J}, A) := \bigvee_{F \in A} \bigcap_{j \in J} F_j$$

is the Kowalsky diagonal filter [10] (with respect to $(F_j)_{j \in J}$ and A). If the family of filters is a so-called selection, i.e., a family $(S(x))_{x \in X}$ of filters indexed by X, we write shortly $D(S,A)$ for $D((S(x))_{x \in X}, A)$.

A map $\lambda : \mathbf{F}(X) \to \mathbf{P}^X$ is called an *approach limit* if it fulfills the properties:

(CAL1) $\forall x \in X : \lambda(\text{stack}\, x)(x) = 0$.

(CAL2) $F \subset G \Rightarrow \lambda(G) \leq \lambda(F)$.

(PRAL) For any family $(F_j)_{j \in J}$ of filters on X:

$$\lambda(\bigcap_{j \in J} F_j) = \sup_{j \in J} \lambda(F_j).$$

(AL) For any $F \in \mathbf{F}(X)$ and any selection S on X:

$$\lambda(D(S,F)) \leq \lambda(F) + \sup_{x \in X} \lambda(S(x))(x).$$

Notice that by (PRAL) an approach limit is completely determined by its restriction to ultrafilters.

2.1. THEOREM [12]. (1) *If* $X \in |SET|$ *and* δ *is a distance on* X *then* λ_δ *defined by*

$$\lambda_\delta(F) := \sup_{\mathcal{U} \in \sec(F)} \delta_{\mathcal{U}}$$

is an approach limit on X*: and vice versa if* λ *is an approach limit on* X *then* δ_λ *defined by*

$$\delta_\lambda(x,A) := \inf_{U \in \mathbf{U}(\text{stack}\, A)} \lambda(U)(x)$$

is a distance on X.

(2) $\lambda_{\delta_\lambda} = \lambda$ *and* $\delta_{\lambda_\delta} = \delta$.

(3) *If* $(X,\delta), (X',\delta') \in |AP|$ *and if* $f: X \to X'$ *then the following are equivalent:*

(i) *f is a contraction.*

(ii) $\forall F \in \mathbf{F}(X) : \lambda'(\text{stack}\, f(F)) \circ f \leq \lambda(F)$.

(iii) $\forall F \in \mathbf{U}(X) : \lambda'(\text{stack}\, f(F)) \circ f \leq \lambda(F)$.

Unless confusion can occur we write λ for λ_δ and δ for δ_λ.

The following alternative formulation of (D4) will sometimes be useful.

2.2. PROPOSITION. *If* $X \in |SET|$ *and* $\delta : X \times 2^X \to \mathbf{P}$ *fulfills* (D1)-(D3) *then* (D4) *is equivalent to the property*

(D4)' $\quad \forall A, B \subset X,\ x \in X : \delta(x,A) \leq \delta(x,B) + \sup_{b \in B} \delta(b,A)$.

PROOF. That (D4)' implies (D4) follows from letting $B := A^{(\varepsilon)}$, and that (D4) implies (D4)' follows from letting

$$\varepsilon := \inf\{\vartheta \in \mathbf{P} \mid B \subset A^{(\vartheta)}\}.$$

Our main goal being the construction of quasitopos hulls, we require a basic supercategory which is a quasitopos. Such a category was introduced in [12]. For $X \in |SET|$ a map $\lambda : \mathbf{F}(X) \to \mathbf{P}^X$ is called a *convergence-approach limit* (or shortly limit) if it fulfills the properties (CAL1), (CAL2) and

(CAL3) $\quad \forall\, F, G \in \mathbf{F}(X) : \lambda(F \cap G) = \lambda(F) \vee \lambda(G).$

The pair (X,λ) is called a *convergence-approach space*. Given two convergence-approach spaces (X,λ) and (X',λ'), a function $f: X \to X'$ is called a *contraction* if it fulfills the property

(c) $\quad \forall\, F \in \mathbf{F}(X) : \lambda'(\text{stack}\, f(F)) \circ f \leq \lambda(F).$

The category *CAP* of convergence-approach spaces and contractions is a topological category. We describe its initial and final structures, leaving verifications to the reader.

2.3. PROPOSITION. *Let $(X_j)_{j \in J}$ be a class of CAP objects. If $(f_j : X \longrightarrow (X_j, \lambda_j))_{j \in J}$ is a source then the initial limit on X is given by*

$$\lambda(F) := \sup_{j \in J} \lambda_j(\text{stack}\, f_j(F)) \circ f_j$$

and if $(f_j : (X_j, \lambda_j) \longrightarrow X)_{j \in J}$ is a sink then the final limit on X is given by

$$\lambda(F)(x) := \begin{cases} 0 & \text{if } F = \text{stack}\, x \\ \inf\limits_{j \in J} \inf\limits_{z \in f_j^{-1}(x)} \inf\limits_{\substack{G \in \mathbf{F}(X_j) \\ \text{stack}\, f_j(G) \subset F}} \lambda_j(G)(z) & \text{otherwise} \end{cases}$$ ∎

It was shown in [12] that *CAP* is moreover a quasitopos (or a topological universe in the sense of Nel [17]). Since we shall need its Hom-objects and what we shall call ∗-objects (see Herrlich [7]), we now recall these.

Given $(X,\lambda_X), (Y,\lambda_Y) \in |CAP|$ the limit on Hom(X,Y) is determined by

$$\lambda\Psi(f) := \inf\{\alpha \in \mathbf{P} \mid \forall\, F \in \mathbf{F}(X) : \lambda_Y(\text{stack}\,\Psi(F)) \circ f \leq \lambda_X(F) \vee \alpha\}$$

where $\Psi \in \mathbf{F}(\text{Hom}(X,Y))$, $f \in \text{Hom}(X,Y)$.

$$\Psi(F) := \{\psi(F) \mid \psi \in \Psi,\ F \in F\} \text{ and } \psi(F) := \{g(y) \mid g \in \psi,\ y \in F\}.$$

By ∗*-objects* we mean the following: If $(X,\lambda) \in |CAP|$ then $X^* := X \cup \{x\}$ where $x \notin X$ and $\lambda^* : \mathbf{F}(X^*) \to \mathbf{P}^{X^*}$ is given by $\lambda^*(\text{stack}_{X^*} x) = 0$ and if $F \neq \text{stack}_{X^*} x$ then

$$\lambda^*(F)(y) := \begin{cases} \lambda(F|_X)(y) & \text{if } y \in X \\ 0 & \text{if } y = x. \end{cases}$$

where $F|_X$ stands for the trace of F on X.

3. FINAL DENSITY IN *CAP* AND INITIAL DENSITY IN *AP*.

It was shown in [12] that by means of the equivalence between distances and approach limits given in 2.1, *AP* is embedded as a bireflective subcategory of *CAP*. We now show that moreover *AP* is finally dense in *CAP*. Hereto we need to introduce a special class of *AP* objects. For any set X, any $F \in \mathbf{F}(X)$ and any $f \in \mathbf{P}^X$ we define $\lambda_{(F,f)} : \mathbf{F}(X) \to \mathbf{P}^X$ by

$$\lambda_{(F,f)}G(x) := \begin{cases} f(x) & \text{if } F \cap \text{stack}\, x \subset G,\ G \neq \text{stack}\, x \\ \infty & \text{if } F \cap \text{stack}\, x \not\subset G \\ 0 & \text{if } G = \text{stack}\, x \end{cases}$$

3.1. PROPOSITION. $(X, \lambda_{(F,f)}) \in |AP|$.

PROOF. (CAL1) and (CAL2) follow at once from the definition. To verify (PRAL) let $(F_j)_{j \in J} \subset \mathbf{F}(X)$; then one inequality follows at once from (CAL2). To prove the other one let $x \in X$, then the only case where $\lambda_{(F,f)}(\bigcap_{j \in J} F_j)(x)$ is not necessarily equal to 0 occurs when $\bigcap_{j \in J} F_j \neq \text{stack}\, x$. Under this assumption we consider the following two cases. Either $F \cap \text{stack}\, x \subset \bigcap_{j \in J} F_j$ and then

$$\lambda_{(F,f)}(\bigcap_{j \in J} F_j)(x) = f(x) = \sup_{j \in J} \lambda_{(F,f)}(F_j)(x);$$

or $F \cap \text{stack}\, x \not\subset \bigcap_{j \in J} F_j$ and then there exists a $j \in J$ such that we have $F \cap \text{stack}\, x \not\subset F_j$. So in this case we clearly have

$$\lambda_{(F,f)}(\bigcap_{j \in J} F_j)(x) = \infty = \sup_{j \in J} \lambda_{(F,f)}(F_j)(x).$$

In order to verify (AL) let $(S(y))_{y \in X}$ be a selection of filters on X. First we suppose $U \in \mathbf{U}(X)$. Given $x \in X$ the only case in which

$$\lambda_{(F,f)}(U)(x) + \sup_{y \in X} \lambda_{(F,f)} S(y)(y)$$

is not necessarily equal to ∞ is when

$$F \cap \text{stack}\, x \subset U \text{ and } F \cap \text{stack}\, y \subset S(y) \text{ for every } y \in X.$$

Under these two assumptions we consider the following two cases. Either $U = \text{stack}\, x$ and then $D(S, U) = S(x)$ from which (AL) follows at once; or $U \neq \text{stack}\, x$ and then we have $F \subset U$ and

$$D(S, U) \supset \bigvee_{U \in U} \bigcap_{y \in U} (F \cap \text{stack}\, y) \supset F$$

So in this case we have

$$\lambda_{(F,f)} D(S,U)(x) \le \lambda_{(F,f)}(F)(x) = \lambda_{(F,f)}(U)(x)$$
$$\le \lambda_{(F,f)}(U)(x) + \sup_{y\in X} \lambda_{(F,f)} S(y)(y).$$

Consequently (AL) is proved in the ultrafilter case.

Second let $H \in \mathbf{F}(X)$. Since

$$D(S,H) = \bigcap_{U\in \mathbf{U}(H)} D(S,U)$$

it follows from the ultrafilter case and upon applying (PRAL) that

$$\begin{aligned} \lambda_{(F,f)} D(S,H) &= \sup_{U\in \mathbf{U}(H)} \lambda_{(F,f)} D(S,U) \\ &\le \sup_{U\in \mathbf{U}(H)} \lambda_{(F,f)} U + \sup_{y\in X} \lambda_{(F,f)} S(y)(y) \\ &= \lambda_{(F,f)} H + \sup_{y\in X} \lambda_{(F,f)} S(y)(y) \end{aligned}$$

which proves (AL) in general, and we are done. ■

It is easily seen and worthwile to notice that in general the spaces $(X,\lambda_{(F,f)})$ are "genuine" approach spaces in the sense that they are neither topological nor metric.

3.2. THEOREM. *AP is finally dense in CAP.*

PROOF. Let $(X,\lambda) \in |CAP|$. For each $F \in \mathbf{F}(X)$, by 3.1 we can consider the approach space $(X,\lambda_{(F,\lambda F)})$ and the sink

$$(\mathrm{id}_X : (X,\lambda_{(F,\lambda F)}) \longrightarrow X)_{F\in \mathbf{F}(X)}$$

It is easily verified from 2.3 that the final limit μ on X is determined by

$$\mu G(x) := \begin{cases} 0 & \text{if } G = \text{stack}\, x \\ \inf_{F\in \mathbf{F}(X)} \lambda_{(F,\lambda F)} G(x) & \text{otherwise} \end{cases}$$

from which it follows at once that $\mu = \lambda$. ■

We now introduce another special *AP* object which together with its #-object we shall need repeatedly in the sequel. The underlying set is **P** and for $x \in \mathbf{P}$ and $A \subset \mathbf{P}$ we define

$$\delta^{\mathbf{P}}(x,A) := \begin{cases} (x - \sup A) \vee 0 & \text{if } A \neq \emptyset \\ \infty & \text{if } A = \emptyset \end{cases}$$

3.3. PROPOSITION. $(\mathbf{P}, \delta^{\mathbf{P}}) \in |AP|$.

PROOF. (D1), (D2) and (D3) are quite trivial. As for (D4) we use

2.2. If $x \in \mathbf{P}$ and $A, B \subset \mathbf{P}$ then the only case where

$$\delta^{\mathbf{P}}(x,A) \neq 0 \quad \text{and} \quad \delta^{\mathbf{P}}(x,B) + \sup_{x \in B} \delta^{\mathbf{P}}(b,A) < \infty$$

occurs when $x < \infty$ and $\infty \notin B$. Then the inequality we have to verify reduces to

$$(x - \sup A) \vee 0 \leq (x - \sup B) \vee 0 + (\sup B - \sup A) \vee 0$$

which is clearly correct. ∎

We shall need the approach limit associated with $\delta^{\mathbf{P}}$. In order to describe this explicitly we recall the definition of the right order topology T_r on $\mathbf{P}$, i.e.,

$$T_r := \{\mathbf{P}\} \cup \{]\alpha, \infty] \mid \alpha \in \mathbf{P}\}.$$

The closure operator in this topology is denoted $-^r$.

For $F \in \mathbf{F}(\mathbf{P})$ and $\beta \in \mathbf{P}$, $F \xrightarrow{r} \beta$ means that F converges to β in this topology. Notice that the set $\lim_r F$ of all limit points of F is a closed interval $[0, \vartheta]$ for some $\vartheta \in \mathbf{P}$. To be precise we shall put

$$\pounds(F) := \sup \lim_r F, \quad \text{i.e.,} \quad \lim_r F = [0, \pounds(F)].$$

The following expression for $\pounds(F)$ will be useful.

3.4. LEMMA.

$$\pounds(F) = \inf_{U \in \sec(F)} \sup U.$$

PROOF. If for some $U \in \sec(F)$ we have $\sup U < \pounds(F)$ then $\pounds(F) \notin \overline{U}^r$ contradicting that $F \xrightarrow{r} \pounds(F)$.

On the other hand, first if $\pounds(F) = \infty$ then $F \xrightarrow{r} \infty$ and thus for any $U \in \sec(F)$ we have $\sup U = \infty$; second if $\pounds(F) < \infty$ then for any $\gamma > \pounds(F)$, we have $F \not\xrightarrow{r} \gamma$ and thus there exists $U \in \sec(F)$ such that $\sup U \leq \gamma$. ∎

3.5. PROPOSITION. *The approach limit* $\lambda^{\mathbf{P}}$ *associated with* $\delta^{\mathbf{P}}$ *is given by*

$$\lambda^{\mathbf{P}}(F)(\beta) = (\beta - \pounds(F)) \vee 0, \quad F \in \mathbf{F}(X), \ \beta \in \mathbf{P}.$$

PROOF. Immediate from the definition of $\delta^{\mathbf{P}}$, 2.1 and 3.4. ∎

3.6. PROPOSITION. *If* $(X, \delta) \in |AP|$ *and* $A \subset X$, *then the map* δ_A: $(X, \delta) \to (\mathbf{P}, \delta^{\mathbf{P}})$ *is a contraction.*

PROOF. Using the alternative formulation of (D4) which is given in 2.2 this follows from an easy verification of cases. ∎

3.7. THEOREM. $(\mathbf{P},\delta^{\mathbf{P}})$ *is initially dense in AP.*

PROOF. Let $(X,\delta)\in|AP|$, let λ be its limit and consider the source $(\delta_A : X\to(\mathbf{P},\lambda^{\mathbf{P}}))_{A\subset X}$. From 3.6 it already follows that if μ stands for the initial limit on X then $\mu\le\lambda$.

Conversely let $\mathcal{U}\in \mathbf{U}(X)$ and $x\in X$. For each $U\in\mathcal{U}$ we have

$$\delta_U(x) < \delta^{\mathbf{P}}(\delta_U(x),\delta_U(U))$$

as it is easily seen verifying cases. Consequently from 2.3 we obtain

$$\mu(\mathcal{U})(x) = \sup_{A\subset X}\lambda^{\mathbf{P}}(\delta_A(\mathcal{U}))(\delta_A(x)) = \sup_{A\subset X}\sup_{U\in\mathcal{U}}\delta^{\mathbf{P}}(\delta_A(x),\delta_A(U))$$

$$\ge \sup_{U\in\mathcal{U}}\delta^{\mathbf{P}}(\delta_U(x),\delta_U(U)) \ge \sup_{U\in\mathcal{U}}\delta_U(x) = \lambda(\mathcal{U})(x).$$

By the arbitrariness of $\mathcal{U}$ and x and the fact that both μ and λ are approach limits we are done. ∎

4. THE EXTENSIONAL TOPOLOGICAL HULL OF *AP*.

Given a set X, a map $\lambda : \mathbf{F}(X)\to \mathbf{P}^X$ is called a *pre-approach limit* if it fulfills (CAL1), (CAL2) and (PRAL). The pair (X,λ) is then called a *pre-approach space*. Each pre-approach space is a convergence-approach space, and we denote *PRAP* the full subcategory of *CAP* with objects all pre-approach spaces.

A map $\delta : X\times 2^X\to \mathbf{P}^X$ which fulfills (D1), (D2) and (D3) is called a *pre-distance*. In [12] it was shown that the same equivalence which holds between distances and approach limits in *AP* is valid between pre-distances and pre-approach limits in *PRAP*. Thus the formulas (2.1) and (2.2) turn a pre-distance into a pre-approach limit and vice versa, giving equivalent characterizations of *PRAP* objects. We shall use whichever description is more convenient. In [12] it was also shown that *PRAP* is an extensional bireflective subcategory of *CAP* containing *AP*. Note that in [12] "extensional" is called "hereditary".

As a first step towards the construction of the topological quasitopos hull of *AP* we now show that *PRAP* is actually the extensional topological hull of *AP*. It is well known that *PRTOP* is the extensional topological hull of *TOP*. One way of seeing this was given in Herrlich [7] where it was shown that the #-object in *PRTOP* of the Sierpinski space in *TOP* is an initially dense object in *PRTOP*. An analogous situation presents itself here.

Let $\mathbf{P}^{\#} := \mathbf{P} \cup \{p\}$ and $(\lambda^{\mathbf{P}})^{\#}$ be as defined in the foregoing section. $(\mathbf{P}^{\#}, (\lambda^{\mathbf{P}})^{\#})$ being a *PRAP* object (see [12]) it will be useful also to describe its pre-distance $(\delta^{\mathbf{P}})^{\#}: \mathbf{P}^{\#} \times 2^{\mathbf{P}^{\#}} \to \mathbf{P}$ which is fully determined as the unique distance extending $\delta^{\mathbf{P}}$ and which fulfills

$$(\delta^{\mathbf{P}})^{\#}(x, A) := 0 \text{ if } p \in \{x\} \cup A \text{ and } A \neq \emptyset.$$

We leave it to the reader to verify that $(\lambda^{\mathbf{P}})^{\#}$ and $(\delta^{\mathbf{P}})^{\#}$ are indeed equivalent in the sense of 2.1.

Again, when no confusion can occur, we often denote $(\mathbf{P}^{\#}, (\delta^{\mathbf{P}})^{\#})$ simply by $\mathbf{P}^{\#}$. In analogy to 3.7 we then have the following result.

4.1. THEOREM. $(\mathbf{P}^{\#}, (\delta^{\mathbf{P}})^{\#})$ *is initially dense in PRAP.*

PROOF. Let $(X, \delta_X), (Y, \delta_Y) \in |PRAP|$ and let $f: (X, \delta_X) \to (Y, \delta_Y)$ not be a contraction. Thus there exist $x \in X$, $A \subset X$ such that $\delta_X(x, A) < \delta_Y(f(x), f(A))$. We define $g: (Y, \delta_Y) \to (\mathbf{P}^{\#}, (\delta^{\mathbf{P}})^{\#})$ as follows:

$$g(y) := \begin{cases} \delta_Y(f(x), f(A)) & y = f(x) \\ p & y \notin f(A),\ y \neq f(x) \\ 0 & y \in f(A) \end{cases}$$

To see that g is a contraction notice that for $y \in Y$ and $\emptyset \neq B \subset Y$, $(\delta^{\mathbf{P}})^{\#}(g(y), g(B))$ is non zero only if

$$y = f(x),\ p \notin g(B) \text{ and } \delta_Y(f(x), f(A))) \notin g(B).$$

In that case however we have

$$\begin{aligned}(\delta^{\mathbf{P}})^{\#}(g(y), g(B)) &= \delta^{\mathbf{P}}(\delta_Y(f(x), f(A)), \{0\}) \\ &= \delta_Y(f(x), f(A)) \leq \delta_Y(y, B).\end{aligned}$$

Finally that $g \circ f$ is not a contraction follows from

$$\begin{aligned}\delta_X(x, A) < \delta_Y(f(x), f(A)) &= \delta^{\mathbf{P}}(\delta_Y(f(x), f(A)), \{0\}) \\ &= (\delta^{\mathbf{P}})^{\#}\ (g(f(x)), g(f(A))),\end{aligned}$$

and this completes the proof. ∎

4.2. THEOREM. *PRAP is the extensional topological hull of AP.*

PROOF. Since *PRAP* is extensional [12], since by 3.2 *AP* is finally dense in *PRAP* and since by 4.1 the class

$$\{(X^{\#}, \lambda^{\#}) \mid (X, \lambda) \in |AP|\}$$

is initially dense in *PRAP*, this follows immediately from Theorem 4 in Herrlich [7].

5. THE CARTESIAN CLOSED TOPOLOGICAL HULL OF *PRAP*.

We use the notation and general construction from Bourdaud [2,3].

Let $C(\mathbf{P}^*)$ be the full subcategory of *CAP* with objects those spaces X which carry the initial structure of the source

$$j: \mathrm{X} \longrightarrow \mathrm{Hom}(\mathrm{Hom}(\mathrm{X}, \mathbf{P}^*), \mathbf{P}^*): x \longrightarrow (f \to f(x)).$$

5.1. PROPOSITION. *$C(\mathbf{P}^*)$ is the cartesian closed hull of PRAP.*

PROOF. By Bourdaud [3], $C(\mathbf{P}^*)$ is cartesian closed topological with Hom-objects formed as in *CAP*, $C(\mathbf{P}^*)$ is bireflective in *CAP* and $\mathbf{P}^* \in |C(\mathbf{P}^*)|$ and by [12] *PRAP* is bireflective in *CAP*. So it follows from 4.1 that *PRAP* is a subcategory of $C(\mathbf{P}^*)$ closed under the formation of finite products in $C(\mathbf{P}^*)$. That *PRAP* is finally dense in $C(\mathbf{P}^*)$ follows at once from 3.2. Moreover since the functor $\mathrm{Hom}(-,\mathbf{P}^*)$: $CAP \to CAP$ transforms final epi-sinks into initial sources, 3.2 also implies that powers of *PRAP* objects are initially dense in $C(\mathbf{P}^*)$.

Consequently, following Herrlich and Nel [8], $C(\mathbf{P}^*)$ is the cartesian closed topological hull of *PRAP*. ■

In order to give an internal characterization of $C(\mathbf{P}^*)$ we need an explicit formulation of the initial limit determined by the source $j: \mathrm{X} \longrightarrow \mathrm{Hom}(\mathrm{Hom}(\mathrm{X},\mathbf{P}^*), \mathbf{P}^*)$ where $\mathrm{X} \in |CAP|$.

We shall use the following notations:

- λ^P_{H} for the Hom limit on $\mathrm{Hom}(\mathrm{X},\mathbf{P}^*)$,
- λ^P_{HH} for the Hom limit on $\mathrm{Hom}(\mathrm{Hom}(\mathrm{X},\mathbf{P}^*),\mathbf{P}^*)$.

Further for $\mathrm{A} \subset \mathrm{X}$ and $\beta \in \mathbf{P}$ we put

$$\mathrm{A}^{<\beta>} := \{k \in \mathrm{Hom}(\mathrm{X},\mathbf{P}^*) \mid \{x \in \mathrm{X} \mid k(x) \in [0,\beta]\} \cap \mathrm{A} = \emptyset\}.$$

5.2. PROPOSITION. *If μ stands for the initial limit on X determined by the source $j: \mathrm{X} \longrightarrow \mathrm{Hom}(\mathrm{Hom}(\mathrm{X},\mathbf{P}^*),\mathbf{P}^*)$, $H \in \mathbf{F}(\mathrm{X})$, $a \in \mathrm{X}$ and $\alpha \in \mathbf{P}$ then we have $\mu(H)(a) \le \alpha$ iff the following condition (∗) is fulfilled:*

(∗) *For every $\Psi \in \mathbf{F}(\mathrm{Hom}(\mathrm{X},\mathbf{P}^*))$, for every $f \in \mathrm{Hom}(\mathrm{X},\mathbf{P}^*)$ such that $f(a) \neq p$, stack $\Psi(H) \neq$ stack p and $\lambda^P_{\mathrm{H}}(\Psi)(f) < \infty$, and for every $\beta < f(a) - (\lambda^P_{\mathrm{H}}(\Psi)(f) \vee \alpha)$ there exists $\mathrm{H} \in H$ such that $\mathrm{H}^{<\beta>} \in \Psi$.*

PROOF. For simplicity in notation let us denote by D the set of those pairs $(\Psi, f) \in \mathbf{F}(\mathrm{Hom}(\mathrm{X},\mathbf{P}^*)) \times \mathrm{Hom}(\mathrm{X},\mathbf{P}^*)$ which satisfy

$$f(a) \neq p, \ \text{stack}\,\Psi(H) \neq \text{stack}\,p \ \text{and} \ \lambda^P_{\mathrm{H}}(\Psi)(f) < \infty,$$

i.e., precisely the conditions mentioned in (*). Then

$$\mu(H)(a) = \lambda^{P}_{\mathbf{HH}}(\text{stack}\, j(H))(j(a)) \leq \alpha$$

$$\Leftrightarrow \quad \forall\, \Psi \in \mathbf{F}(\text{Hom}(X,\mathbf{P}^{*})),\ \forall\, f \in \text{Hom}(X,\mathbf{P}^{*}): \quad \lambda^{\mathbf{P}^{*}}(\text{stack}\,\Psi(H))(f(a)) \leq \lambda^{P}_{\mathbf{H}}(\Psi)(f) \vee \alpha$$

$$\Leftrightarrow \quad \forall\, (\Psi,f) \in D : \lambda^{\mathbf{P}}(\text{stack}\,\Psi(H)|_{\mathbf{P}})(f(a)) \leq \lambda_{\mathbf{H}}(\Psi)(f) \vee \alpha$$

$$\Leftrightarrow \quad \forall\, (\Psi,f) \in D : (f(a) - \pounds(\text{stack}\,\Psi(H)|_{\mathbf{P}})) \vee 0 \leq \lambda^{P}_{\mathbf{H}}(\Psi)(f) \vee \alpha$$

$$\Leftrightarrow \quad \forall\, (\Psi,f) \in D\ \forall\, \beta \in [0, f(a) - \lambda^{P}_{\mathbf{H}}(\Psi)(f) \vee \alpha[\ :\ ]\beta,\infty] \in \text{stack}\,\Psi(H)|_{\mathbf{P}}$$

$$\Leftrightarrow \quad \forall\, (\Psi,f) \in D\ \forall\, \beta < f(a) - \lambda^{P}_{\mathbf{H}}(\Psi)(f) \vee \alpha[\ \exists\, \mathrm{H} \in H : \mathrm{H}^{<\beta>} \in \Psi. \quad \blacksquare$$

5.3. DEFINITION. We define and denote by *PSAP* the full subcategory of *CAP* the objects (X,λ) of which satisfy the following supplementary condition:

(PSAL) For all $F \in \mathbf{F}(X)$: $\lambda(F) = \sup_{U \in \mathbf{U}(F)} \lambda(U)$.

An object $(X,\lambda) \in |PSAP|$ is called a *pseudo-approach space* and λ is called a *pseudo-approach limit.*

In order to study *PSAP* in greater detail we need some preliminary results.

5.4. LEMMA. *For* (X,λ_X), $(Y,\lambda_Y) \in |CAP|$, $\Psi \in \mathbf{F}(\text{Hom}(X,Y))$, $F \in \mathbf{F}(X)$ *and* $W \in \mathbf{U}(\text{stack}\,\Psi(F))$ *we have:*

(1) $\exists\, U \in \mathbf{U}(F)$: $\text{stack}\,\Psi(U) \subset W$.

(2) $\exists\, \Phi \in \mathbf{U}(\Psi)$: $\text{stack}\,\Phi(F) \subset W$.

PROOF. Since (2) is analogous to (1), we only prove (1). Suppose that for every $U \in \mathbf{U}(F)$ there exists $\psi \in \Psi$ and $\mathrm{U} \in U$ such that $\psi(\mathrm{U}) \not\in W$. Apply Proposition 2.1 in [12] to select a finite collection $U_1, \dots, U_n$ in $\mathbf{U}(F)$ and corresponding sets $\psi_1, \dots, \psi_n$ in Ψ and $\mathrm{U}_i \in U_i$ for $i \in \{1, \dots, n\}$ such that

$$\bigcup_{i=1}^{n} \mathrm{U}_i \in F \text{ and } \psi_i(\mathrm{U}_i) \not\in W \text{ for all } i \in \{1, \dots, n\}.$$

Then

$$(\bigcap_{i=1}^{n} \psi_i)(\bigcup_{i=1}^{n} \mathrm{U}_i) \in W$$

which is a contradiction. $\blacksquare$

5.5. PROPOSITION. *For* $(X,\lambda_X) \in |CAP|$, $(Y,\lambda_Y) \in |PSAP|$, $\Psi \in \mathbf{F}(\text{Hom}(X,Y))$, $f \in \text{Hom}(X,Y)$ *and* $\alpha \in \mathbf{P}$

the following are equivalent:

(1) $\forall\, F \in \mathbf{F}(X)$: $\lambda_Y(\text{stack}\,\Psi(F)) \circ f \leq \lambda_X(F) \vee \alpha$.

(2) $\forall\, U \in \mathbf{U}(X)$: $\lambda_Y(\text{stack}\,\Psi(U)) \circ f \leq \lambda_X(U) \vee \alpha$.

(3) $\forall\, F \in \mathbf{F}(X),\ \forall\, \Phi \in \mathbf{U}(\Psi)$: $\lambda_Y(\text{stack}\,\Phi(F)) \circ f \leq \lambda_X(F) \vee \alpha$.

(4) $\forall\, U \in \mathbf{U}(X),\ \forall\, \Phi \in \mathbf{U}(\Psi)$: $\lambda_Y(\text{stack}\,\Phi(U)) \circ f \leq \lambda_X(U) \vee \alpha$.

PROOF. The implications (1)⇒(2)⇒(4) and (1)⇒(3)⇒(4) are evident. So the only implication we have to prove is (4)⇒(1). From 5.4 (1) and (2) and (PSAL) we obtain

$$\begin{aligned}\lambda_Y(\text{stack}\,\Psi(F)) \circ f &= \sup_{W \in \mathbf{U}(\Psi(F))} \lambda_Y(W) \circ f \\ &\leq \sup_{U \in \mathbf{U}(F)} \sup_{\Phi \in \mathbf{U}(\Psi)} \lambda_Y(\text{stack}\,\Phi(U)) \circ f \\ &\leq \sup_{U \in \mathbf{U}(F)} \lambda_X(U) \vee \alpha \leq \lambda_X(F) \vee \alpha. \qquad \blacksquare\end{aligned}$$

5.6. COROLLARY. *If* $(X, \lambda_X) \in |CAP|$, $(Y, \lambda_Y) \in |PSAP|$ *and* $f: X \to Y$, *then the following are equivalent:*

(1) *f is a contraction.*

(2) $\forall\, U \in \mathbf{U}(X)$: $\lambda_Y(\text{stack}\, f(U)) \circ f \leq \lambda_X(U)$.

PROOF. Apply 5.5 for $\Psi := \text{stack}\, f$ and $\alpha := 0$.

5.7. PROPOSITION. *PSAP is a bireflective subcategory of CAP.*

PROOF. Using 5.6 it is immediately verified that if (X, λ) is in $|CAP|$ its *PSAP* reflection is given by $id_X: (X, \lambda) \to (X, \tilde{\lambda})$ where for all $F \in \mathbf{F}(X)$:

$$\tilde{\lambda}(F) := \sup_{U \in \mathbf{U}(F)} \lambda(U). \qquad \blacksquare$$

5.8. PROPOSITION. *If* $(X, \lambda_X) \in |CAP|$ *and* $(Y, \lambda_Y) \in |PSAP|$ *then* Hom(X, Y) *equipped with its natural limit is a PSAP object.*

PROOF. Let λ stand for the natural limit on Hom(X, Y) as defined in [12] (see also §3). Let $\Psi \in \mathbf{F}(\text{Hom}(X,Y))$, $f \in \text{Hom}(X,Y)$ and put

$$\alpha := \sup_{\Phi \in \mathbf{U}(\Psi)} \lambda(\Phi)(f).$$

Then for all $f \in \mathbf{U}(\Psi)$ and $F \in \mathbf{F}(X)$ we have

$$\lambda_Y(\Phi(F)) \circ f \leq \lambda_X(F) \vee \alpha,$$

which by 5.5 implies that for all $F \in \mathbf{F}(X)$

$$\lambda_Y(\Psi(F)) \circ f \leq \lambda_X(F) \vee \alpha$$

and thus $\lambda(\Psi(F)) \leq \alpha$. $\blacksquare$

We have now gathered sufficient material to prove our main result of this section, i.e. the internal characterization of the cartesian closed topological hull of *PRAP*. Notice that this characterization is similar to that of the cartesian closed hull of the category of pre-topological spaces as in Bourdaud [2].

5.9. THEOREM. $C(\mathbf{P}^{*}) = PSAP$.

PROOF. If $(X,\lambda_X) \in |C(\mathbf{P}^{*})|$ then it follows from 5.8 that

$$\mathrm{Hom}(\mathrm{Hom}(X,\mathbf{P}^{*}),\mathbf{P}^{*}) \in |PSAP|.$$

By 5.7 it then follows that also $(X,\lambda_X) \in |PSAP|$. Conversely let $(X,\lambda_X) \in |PSAP|$ and let μ stand for the initial limit on X determined by the source $j: X \longrightarrow \mathrm{Hom}(\mathrm{Hom}(X,\mathbf{P}^{*}),\mathbf{P}^{*})$. Then $\mu \leq \lambda_X$. In order that μ and λ_X coincide, since they both satisfy (PSAL), it suffices to show that they are equal on ultrafilters. Suppose on the contrary that there exist $a \in X$ and $H \in \mathbf{U}(X)$ such that $\mu(H)(a) < \lambda_X(H)(a)$. For all $W \in H$ put

$$\tilde{W} := \left\{ k \in \mathrm{Hom}((X,\lambda_X),\mathbf{P}^{*}) \mid k(X\setminus W) = \{p\} \right\}.$$

In particular, for every $\gamma \in \mathbf{P}$, the two-valued function

$$(X,\lambda_X) \xrightarrow{k_\gamma^W} \mathbf{P}^{*} : x \longrightarrow \begin{cases} p & \text{if } x \in X\setminus W \\ \gamma & \text{if } x \in W \end{cases}$$

is a contraction and belongs to $\tilde{W}$. It follows that $\{\tilde{W} \mid W \in H\}$ is a filter base on $\mathrm{Hom}(X,\mathbf{P}^{*})$. Let Ψ be the filter generated by this base. Further consider the two valued contraction

$$(X,\lambda_X) \xrightarrow{\ f\ } \mathbf{P}^{*} : x \longrightarrow \begin{cases} p & \text{if } x \neq a \\ \lambda_X H(a) & \text{if } x = a. \end{cases}$$

CLAIM. $\lambda_H^P(\Psi)(f) = 0.$

Indeed, let $U \in \mathbf{U}(X)$. If $U \neq H$ then $\mathrm{stack}\,\Psi(U) = \mathrm{stack}\,p$, therefore

$$\lambda^{\mathbf{P}^{*}}(\mathrm{stack}\,\Psi(U))(f(a)) = 0.$$

If $U = H$ then $\mathrm{stack}\,\Psi(U) \neq \mathrm{stack}\,p$ and then

$$\lambda^{\mathbf{P}^{*}}(\mathrm{stack}\,\Psi(U))(f(a)) = \lambda^{\mathbf{P}}(\mathrm{stack}\,\Psi(U)|_{\mathbf{P}})(f(a)) \leq f(a).$$

Thus it follows that for all $U \in \mathbf{U}(X)$:

$$\lambda^{\mathbf{P}^{*}}(\mathrm{stack}\,\Psi(U))(f(a)) \leq \lambda_X(U)(a).$$

Moreover since $f(x) = p$ for $x \neq a$ we finally have

$$\lambda^{\mathbf{P}^{*}}(\mathrm{stack}\,\Psi(U)) \circ f \leq \lambda_X(U).$$

Our claim now follows from the arbitrariness of U and upon applying 5.5.

If we now put $\alpha := \mu(H)(a)$ then it is clear that Ψ and f satisfy all the conditions in the characterization (*) of 5.2. Since $\alpha < f(a)$ we can choose $\beta \in [0.f(a)-\alpha[$ and it follows from 5.2 that there exists $H \in \mathcal{H}$ such that $H^{<\beta>} \in \Psi$. Then let $W \in \mathcal{H}$ be such that $\tilde{W} \subset H^{<\beta>}$. Since $k_\beta^W \in \tilde{W}$ it follows that

$$W \cap H = \{x \mid k_\beta^W(x) \in [0,\beta]\} \cap H = \emptyset$$

which is a contradiction, and we are done. ■

5.10. COROLLARY. *PSAP is the cartesian closed topological hull of PRAP.*

PROOF. From 5.1 and 5.9. ■

6. THE TOPOLOGICAL QUASITOPOS HULLS OF *PRAP* AND *AP*.

6.1. THEOREM. *PSAP is extensional.*

PROOF. The proof is analogous to the proof of the fact that *CAP* is extensional (see Theorem 4.3 in [12]). The only difference is that now the spaces $(X.\lambda_X)$ and $(Y.\lambda)$ are in $|PSAP|$ and one has to show that the $\#$-object $(Y^\#,\lambda^\#)$ too is in $|PSAP|$. This is an easy verification which we leave to the reader. ■

6.2. THEOREM. *PSAP is the quasitopos hull of PRAP.*

PROOF. By 6.1 and 5.10 *PSAP* is a quasitopos. Consequently it follows from 5.10 again that *PSAP* is the topological quasitopos hull of *PRAP*. ■

6.3. THEOREM. *PSAP is the quasitopos hull of AP.*

PROOF. By 3.2 *AP* is finally dense in *PSAP* and by 4.2 *PRAP* is the extensional hull of *AP*. Consequently it follows from 6.2 that *PSAP* is the topological quasitopos hull of *AP*. ■

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