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Rank zero quadratic twists of modular elliptic curves

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Abstract. In [11] L. Mai and M. R. Murty proved that if E is a modular elliptic curve with conductor N, then there exists infinitely many square-free integers $D \equiv 1 \bmod 4N$ such that E_D, the D-quadratic twist of E, has rank 0. Moreover, assuming the Birch and Swinnerton–Dyer Conjecture, they obtain analytic estimates on the lower bounds for the orders of their Tate–Shafarevich groups. However regarding ranks, simply by the sign of functional equations, it is not expected that there will be infinitely many square-free D in every arithmetic progression $r \pmod t$ where $\gcd(r,t)$ is square-free such that E_D has rank zero. Given a square-free positive integer r, under mild conditions we show that there exists an integer t_r and a positive integer N where $t_r \equiv r \bmod \mathbb{Q}_p^{\times 2}$ for all $p \mid N$, such that there are infinitely many positive square-free integers $D \equiv t_r \bmod N'$ where E_D has rank zero. The modulus N' is defined by $N' := 8\prod_{p|N} p$ where the product is over odd primes p dividing N. As an example of this theorem, let E be defined by

$$y^2 = x^3 + 15x^2 + 36x.$$

If $r = 1, 3, 5, 9, 11, 13, 17, 19,$ or 21, then there exists infinitely many positive square-free integers $D \equiv r \bmod 24$ such that E_D has rank 0. As another example, let E be defined by

$$y^2 = x^3 - 1.$$

If $r = 1, 2, 5, 10, 13, 14, 17,$ or 22, then there exists infinitely many positive square-free integers $D \equiv r \bmod 24$ such that E_D has rank 0. More examples are given at the end of the paper. The proof is elementary and uses standard facts about modular forms.

Key words: elliptic curves, modular forms.

1. Main theorem

Let E denote the $\mathbb{Q}$-rational points of the elliptic curve given by the equation

$$y^2 = x^3 + Ax^2 + Bx + C,$$

where $A, B, C \in \mathbb{Z}$. By Mordell's Theorem, which was subsequently generalized by Weil, E forms a finitely generated abelian group which therefore satisfies

$$E \tilde{=} E_{\text{tor}} \times \mathbb{Z}^r,$$

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where E_{tor}, the *torsion subgroup* of E, is a finite abelian group and the rank r is a nonnegative integer. Moreover by Mazur's theorem E_{tor} satisfies

$$E_{\text{tor}} \in \begin{cases} \mathbb{Z}m & | \text{ where } 1 \le m \le 10, \quad \text{or } m = 12, \\ \mathbb{Z}2 \times \mathbb{Z}2m & | \text{ where } 1 \le m \le 4. \end{cases}$$

It is conjectured that the rank r of E is uniquely determined by the analytic behavior of the Hasse–Weil L-function $L(E, s)$ at $s = 1$. To define $L(E, s)$, for every prime p let $N(p)$ denote the number of points (including the point at infinity) on E_p, the reduction of E modulo p and then define $A(p)$ by $A(p) := p + 1 - N(p)$. Then the L-function $L(E, s)$ is defined by the Euler product

$$L(E, s) := \sum_{n=1}^{\infty} \frac{A(n)}{n^s} = \prod_{p|\Delta} \frac{1}{1 - A(p)p^{-s}} \prod_{p\nmid\Delta} \frac{1}{1 - A(p)p^{-s} + p^{1-2s}}.$$

The conjectured connection between $L(E, 1)$ and the rank of E is illustrated by the following weak version of the Birch and Swinnerton–Dyer Conjecture:

CONJECTURE (B-SD). *Let E be an elliptic curve over $\mathbb{Q}$ and let $L(E, s) = \sum_{n=1}^{\infty} \frac{A(n)}{n^s}$ be its Hasse–Weil L-function. Then $L(E, s)$ has an analytic continuation to the entire complex plane and the rank of E is positive if and only if $L(E, 1) = 0$.*

Significant progress has been made towards a proof of this conjecture. In particular by the works of Kolyvagin, Murty, Murty, Bump, Friedberg, and Hoffstein [1, 9, 12] we now know a lot about *modular elliptic curves,* those curves for which $L(E, s)$ is the Mellin transform of a weight 2 newform. The main theorem we use in this paper is:

THEOREM 1. *If E is a modular elliptic curve where $L(E, 1) \neq 0$, then E has rank* 0.

Let ψ be a Dirichlet character mod M and let $f(z)$ be a cusp form in $S_k(N, \chi)$ with Fourier expansion $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ ($q := \mathrm{e}^{2\pi iz}$ throughout). Then the function $f_\psi(z)$, the *ψ-twist* of $f(z)$, defined by

$$f_\psi(z) = \sum_{n=1}^{\infty} \psi(n)a(n)q^n \tag{1}$$

is a modular form in $S_k(NM^2, \chi\psi^2)$. Now we relate these twists when ψ is a quadratic character to the twists of an elliptic curve.

If E is given by the equation $y^2 = x^3 + Ax^2 + Bx + C$ and D is a square-free integer, then the equation of its *D-quadratic twist* is

$$Dy^2 = x^3 + Ax^2 + Bx + C$$

and its group of $\mathbb{Q}$-rational points is denoted by E_D. If $L(E,s) = \sum_{n=1}^{\infty} \frac{A(n)}{n^s}$ is the L-function for E, then the L-function for its quadratic twist $L(E_D,s) = \sum_{n=1}^{\infty} \frac{A_D(n)}{n^s}$ satisfies

$$L(E_D,s) = \sum_{n=1}^{\infty} \frac{\chi_D(n)A(n)}{n^s},$$

where $\chi_D(n)$ is the quadratic character for $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. Specifically this means that for primes p that $A_D(p) = \left(\frac{D}{p}\right) A(p)$ where, $\left(\frac{D}{p}\right)$ is the usual Kronecker–Legendre symbol. Hence if E is a modular elliptic curve, then by (1) E_D is also modular.

Now we briefly describe the theory of half-integral weight modular forms as developed by Shimura [15]. Let N be a positive integer that is divisible by 4 and define $\left(\frac{c}{d}\right)$ and ϵ_d by

$$\left(\frac{c}{d}\right) := \begin{cases} -\left(\frac{c}{|d|}\right) & \text{if } c,d<0 \\ \left(\frac{c}{|d|}\right) & \text{otherwise.} \end{cases}$$

$$\epsilon_d := \begin{cases} 1 & d \equiv 1 \mod 4 \\ i & d \equiv 3 \mod 4. \end{cases}$$

Also let $(cz+d)^{1/2}$ be the principal square root of $(cz+d)$ (i.e. with positive imaginary part). Let χ be a Dirichlet character mod N. Then a meromorphic function $f(z)$ on $\mathfrak{H}$ is called a *half integer weight modular form* if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)\left(\frac{c}{d}\right)^{2\lambda+1} \epsilon_d^{-1-2\lambda}(cz+d)^{\lambda+1/2} f(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Such a form is called a form with weight $\lambda + \frac{1}{2}$ and character χ. The set of all such forms that are holomorphic on $\mathfrak{H}$ as well as at the cusps is denoted by $M_{\lambda+(1/2)}(N,\chi)$ and is a finite dimensional $\mathbb{C}$-vector space. The set of those $f(z)$ in $M_{\lambda+(1/2)}(N,\chi)$ that also vanish at the cusps, the cusp forms, is denoted by $S_{\lambda+(1/2)}(N,\chi)$.

As in the integer weight case, there are Hecke operators that preserve $M_{\lambda+(1/2)}(N,\chi)$ and $S_{\lambda+(1/2)}(N,\chi)$. However for these forms the Hecke operators act on Fourier expansions in square towers; specifically if p is a prime, then the Hecke operator T_{p^2} acts on $f(z) \in M_{\lambda+(1/2)}(N,\chi)$ by

$$f(z) \mid T_{p^2} := \sum_{n=0}^{\infty} \left(a(p^2n) + \chi(p)\left(\frac{(-1)^\lambda n}{p}\right) p^{\lambda-1} a(n) \right.$$

$$\left. + \chi(p^2) p^{2\lambda-1} a(n/p^2) \right) q^n. \tag{2}$$

As in the integer weight case, a form $f(z)$ is called an eigenform if for every prime p there exists a complex number λ_p such that

$$f(z) \mid T_{p^2} = \lambda_p f(z).$$

The critical link between half integer weight forms and the integer weight modular forms are the *Shimura lifts,* a family of maps which *essentially* takes the L-function of a half integer weight cusp form and returns the L-function of an integer weight modular form. More precisely let $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+(1/2)}(N, \chi)$ where $\lambda \geq 1$. Let t be a positive square-free integer and define the Dirichlet character ψ_t by $\psi_t(n) = \chi(n)\left(\frac{-1}{n}\right)^{\lambda}\left(\frac{t}{n}\right)$. Now define $A_t(n)$ by the formal product of L-functions

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} := L(s - \lambda + 1, \psi_t) \sum_{n=1}^{\infty} \frac{a(tn^2)}{n^s}.$$

In particular, if p is prime, then $A_t(p)$ is defined by

$$A_t(p) := a(tp^2) + \psi_t(p)p^{\lambda-1}a(t). \tag{3}$$

Then Shimura proved that the Mellin transform of this product, which we denote by $S_t(f(z)) = \sum_{n=1}^{\infty} A_t(n)q^n$ is a weight 2λ modular form in $M_{2\lambda}(\frac{N}{2}, \chi^2)$. Furthermore if $\lambda \geq 2$, then $S_t(f(z))$ happens to be a cusp form. If $f(z)$ is an eigenform of all the Hecke operators T_{p^2} and t_1 and t_2 are two square-free positive integers, then

$$a(t_1)S_{t_2}(f(z)) = a(t_2)S_{t_1}(f(z)). \tag{4}$$

If $F(z) = \sum_{n=1}^{\infty} A(n)q^n$ is a modular form, then its L-function $L(F, s)$ is defined by

$$L(F, s) = \sum_{n=1}^{\infty} \frac{A(n)}{n^s}.$$

If ψ is a Dirichlet character and $F_\psi(z) = \sum_{n=1}^{\infty} \psi(n)A(n)q^n$ is the ψ-twist of $F(z)$, then we denote the modular L-function for $F_\psi(z)$ by $L(F, \psi, s)$.

In [19] Waldspurger proved beautiful formulas which connect certain special values of modular L-functions to the Fourier coefficients of half-integer weight cusp forms. A special case of his theorem is:

THEOREM (Waldspurger). *Let $f(z) \in S_{\lambda+(1/2)}(N, \chi)$ be an eigenform of the Hecke operators T_{p^2} such that $S_1(f(z)) = F(z) \in S_{2\lambda}^{\text{new}}(M, \chi^2)$ for an appropriate positive integer M. Denote their respective Fourier expansions by*

$f(z) = \sum_{n=1}^{\infty} a(n)q^n$ and $F(z) = \sum_{n=1}^{\infty} A(n)q^n$. Let n_1 and n_2 be two positive square-free integers such that $\frac{n_1}{n_2} \in \mathbb{Q}_p^{\times 2}$ for all $p \mid N$. Then

$$a^2(n_1)L\left(F, \left(\frac{-1}{n}\right)^{\lambda} \chi^{-1}\chi_{n_2}, \lambda\right) \chi(n_2/n_1)n_2^{\lambda-1/2}$$

$$= a^2(n_2)L\left(F, \left(\frac{-1}{n}\right)^{\lambda} \chi^{-1}\chi_{n_1}, \lambda\right) n_1^{\lambda-1/2}. \tag{5}$$

By combining Waldspurger's Theorem, B-SD, and Theorem 1 we find that we can determine whether or not the ranks of certain quadratic twists of an elliptic curve over $\mathbb{Q}$ are 0.

THEOREM 2. *Let E be a modular elliptic curve with $L(E, s) = \sum_{n=1}^{\infty} \frac{A(n)}{n^s}$ and let $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{(3/2)}(N, \left(\frac{d}{n}\right))$ be an eigenform of the Hecke operators T_{p^2} such that $S_1(f(z)) = F(z) = \sum_{n=1}^{\infty} A(n)q^n$. Now let n_1 be a positive square-free integer such that $a(n_1) \neq 0$ and such that $L(E_{-dn_1}, 1) \neq 0$. Suppose that n_2 is a positive square-free integer such that $\frac{n_1}{n_2} \in \mathbb{Q}_p^{\times 2}$ for every prime $p \mid N$. If $a(n_2) \neq 0$, then the rank of E_{-dn_2} is unconditionally 0. If $a(n_2) = 0$, then assuming B-SD the rank of E_{-dn_2} is positive.*

Proof. In (5) substitute 1 for λ and replace χ by $\left(\frac{d}{n}\right)$. Then by solving for $L(F, \left(\frac{-dn_2}{n}\right), 1)$ we obtain

$$L\left(F, \left(\frac{-dn_2}{n}\right), 1\right) = \frac{a^2(n_2)L(F, \left(\frac{-dn_1}{n}\right), 1)\sqrt{n_1}}{\sqrt{n_2}a^2(n_1)}.$$

Since E is a modular elliptic curve corresponding to the weight 2 newform $F(z)$, it turns out that for any square-free integer d that $L(E_d, s) = L(F, \left(\frac{d}{n}\right), s)$. Hence we find that

$$L(E_{-dn_2}, 1) = \frac{a^2(n_2)L(E_{-dn_1}, 1)\sqrt{n_1}}{\sqrt{n_2}a^2(n_1)}.$$

So by hypothesis we find that $L(E_{-dn_2}, 1) = 0$ if and only if $a(n_2) = 0$. If $a(n_2) \neq 0$, then by Theorem 1 we find that the rank of E_{-dn_2} is unconditionally 0. If $a(n_2) = 0$, then by B-SD the rank of E_{-dn_2} is positive. □

One may ask how often such a weight $\frac{3}{2}$ eigenform exists. As a consequence of Kohnen's main result in [8], one can deduce that if E is a modular curve with odd conductor N, then such a weight $\frac{3}{2}$ eigenform exists.

We now show how one may use results like Theorem 2 to establish the existence of infinitely many rank 0 quadratic twists of certain curves by fundamental discriminants in arithmetic progressions. In the case of *cubic twists*, Lieman (see [10]) has shown that if $p \neq 3$ is prime and r is any integer, then there exists infinitely many cube-free integers $D \equiv r \bmod p$ such that the elliptic curve

$$y^2 = x^3 - 432D^2$$

has rank 0. These are the D-cubic twists of the curve

$$y^2 = x^3 - 432.$$

In the case of quadratic twists, Gouvêa and Mazur (see [4]) have shown that there are infinitely many quadratic twists with analytic rank ≥ 2. More recently Stewart and Top have proved an unconditional result (see [16]). Using impressive analytic estimates on certain special values of modular L-functions, L. Mai and M. R. Murty [11] proved that a modular elliptic curve E has infinitely many rank 0 and rank 1 quadratic twists by $D \equiv 1 \bmod 4N$, where N is the conductor of E. In fact they also obtain conditional lower bounds on the orders of their Tate–Shafarevich groups. Here we show that one can often determine the existence of infinitely many square-free integers D in certain arithmetic progressions for which the D-quadratic twist of a modular elliptic curve E has rank 0. This is of interest since one does not expect to find infinitely many square-free integers D in any given arithmetic progression such that E_D has rank 0. For example, if $D \equiv 5, 6, 7 \bmod 8$ is a square-free integer, then it is conjectured that the elliptic curves $E(D)$ defined by

$$y^2 = x^3 - D^2x,$$

the D-quadratic twist of $E(1)$ by D, has positive (in fact odd) rank. We shall return to this example later and show that if $r = 1, 2$, or 3, then there are infinitely many positive square-free integers $D \equiv r \bmod 8$ such that $E(D)$ has rank 0.

First we recall the following fact concerning the restriction of the Fourier expansion of a half-integer weight modular form to an arithmetic progression.

LEMMA 1. *Let* $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ *be a modular form in* $M_{\lambda+(1/2)}(N, \chi)$. *If* $0 \leq r < t$, *then*

$$f_{r,t}(z) = \sum_{n \equiv r \bmod t} a(n)q^n$$

is the Fourier expansion of a modular form of weight $\lambda + \frac{1}{2}$ *with respect to the congruence subgroup* $\Gamma_1(Nt^2)$.

The proof of this statement may be deduced from Lemmas 2 and 3 in [18]. Now we state our result.

MAIN THEOREM. *Let E be a modular elliptic curve where $L(E,s) = \sum_{n=1}^{\infty} \frac{A(n)}{n^s}$ and $F(z) = \sum_{n=1}^{\infty} A(n)q^n$ is a weight 2 newform. Suppose that $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{3/2}(N, \left(\frac{d}{n}\right))$ is an eigenform of the Hecke operators T_{p^2} such that $S_1(f(z)) = F(z)$. Let $N' := 8\prod_{p|N} p$ where the product is over the odd prime divisors of N, and let r be a positive square-free integer. Then if $a(r) \neq 0$ and $L(E_{-dr},1) \neq 0$, then there exists an integer $t_r \equiv r \bmod \mathbb{Q}_p^{\times 2}$ for every prime $p \mid N$ such that there are infinitely many positive square-free integers $D \equiv t_r \bmod N'$ where E_{-dD} has rank 0.*

Proof. Recall that since $f(z)$ is an eigenform of the Hecke operators T_{p^2} we know by (2) that if n is a positive square-free integer where $a(n) = 0$, then $a(nm^2) = 0$ for all m.

Let $1 \leq r_1, r_2, \ldots, r_k < N'$ be integers that are congruent to $r \bmod \mathbb{Q}_p^{\times 2}$ for every $p \mid N$ such that $4 \nmid r_i$ for any $1 \leq i \leq k$. For example this means that for all $1 \leq i \leq k$ that $r_i \equiv r \bmod 8$. Now suppose that there exist only finitely many square-free integers, say $d_1, d_2, \ldots, d_s$ such that for every $1 \leq i \leq s$ there exists a $1 \leq j \leq k$ such that $d_i \equiv r_j \bmod N'$. Now if $n \equiv r_j \bmod N'$ for some j and $n = n'm^2$ where n' is square-free, then n' must be one of $d_1, d_2, \ldots, d_s$.

When $f_r(z) = \sum_{n \equiv r_1, r_2, \ldots, r_k \bmod N'} a(n)q^n$, it follows that

$$f_r(z) = \sum_{i=1}^{s} \sum_{\substack{m=1 \\ \text{odd}}}^{\infty} a(d_i m^2) q^{d_i m^2};$$

and by Lemma 1 it is easy to see that $f_r(z)$ is a weight $\frac{3}{2}$ modular form on some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

However $f_r(z)$ is a very special type of modular form. In [18] Vignéras proved that if $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ is a weight $\frac{3}{2}$ modular form on some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ such that there exists a finite set of positive square-free integers $d_1, d_2, \ldots, d_s$ such that $a(n) = 0$ unless $n = d_i m^2$ for some i and some integer m, then there exists a positive integer M such that $f(z)$ is a finite linear combination of theta series $\theta_{a,M}(dz)$ where $d \in \{d_1, d_2, \ldots, d_s\}$ and where

$$\theta_{a,M}(z) = \sum_{\substack{\in \mathbb{Z} \\ n \equiv a \bmod M}} nq^{n^2}.$$

Hence $f_r(z)$ is such a finite linear combination of theta series by Vignéras' theorem. Since $a(d_1) \neq 0$, we consider $S_{d_1}(f(z))$, the image of $f(z)$ under the Shimura map S_{d_1}. By (4) we see that

$$S_{d_1}(f(z)) = \frac{a(d_1)S_1(f(z))}{a(1)} = a(d_1)S_1(f(z))$$

since $a(1) = 1$. Hence $S_{d_1}(f(z))$ is simply a scalar multiple of the weight 2 newform corresponding to $L(E, s)$.

Define $f_{r,1}(z)$ by

$$f_{r,1}(z) := \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} a(d_1 n^2) q^{d_1 n^2};$$

therefore it follows that $S_{d_1}(f(z))$ is *essentially* determined by the Fourier expansion of $f_{r,1}(z)$. Now decompose $f_{r,1}(z)$ as

$$f_{r,1}(z) := \alpha_1 \theta_{a_1,M}(d_1 z) + \alpha_2 \theta_{a_2,M}(d_1 z) + \cdots + \alpha_l \theta_{a_l,M}(d_1 z)$$

for certain nonzero α_i.

So if p is a prime, then by (3) we find that $A_{d_1}(p)$, the coefficient of q^p in $S_{d_1}(f(z))$ is

$$A_{d_1}(p) = a(d_1 p^2) + \left(\frac{-dd_1}{p}\right) a(d_1).$$

It must be the case that the $\gcd(a_i, M) = 1$ for at least one i. For if that were not the case, then for all but finitely many primes p that $A_{d_1}(p) = \pm a(d_1)$ which is absurd since this would imply then that for all but finitely many primes, that the coefficient of q^p in the weight 2 newform is $\pm a(1) = \pm 1$. By the theory of modular Galois representations as developed by Deligne, for every prime ℓ there exists a positive density of primes p for which the coefficients of q^p in the weight 2 newform are multiples of ℓ which contradicts the assertion that all but finitely many of these coefficients are ± 1.

Therefore we may assume that for at least one i that $\gcd(a_i, M)$=1. So if $p \equiv a_i \bmod M$ is prime, then we find that

$$A_{d_1}(p) = \alpha_i p + \alpha_i \left(\frac{-dd_1}{p}\right)$$

by the definition of $\theta_{a,M}(z)$. However by Deligne's theorem [2] for weight 2 cusp forms, it is known that there exists a constant C such that for all primes p

$$| A_{d_1}(p) | \leq C\sqrt{p}.$$

However it is clear that for those $p \equiv a_i \bmod M$ that such an estimate cannot hold. Therefore it is not the case that there are only finitely many square-free integers $n \equiv r_j \bmod N'$ for some j such that $a(n) \neq 0$. Since there are only finitely many relevant arithmetic progressions $r_1, r_2, \ldots, r_k \bmod N'$, at least one of them, say $r_j \bmod N'$ has the property that there are infinitely many positive square-free integers $D \equiv r_j \bmod N'$ such that $a(D) \neq 0$. The result now follows from Theorem 2. □

2. Examples

In this section we work out a number of examples of this theorem. The first three examples mentioned here are also given at the end of [13]. In that paper these results were proved using the theory of modular forms with complex multiplication. In the first three examples let $E_{\mathbb{Q}}(M,N)$ denote the elliptic curve defined by

$$y^2 = x^3 + (M+N)x^2 + MNx.$$

We note that due to recent developments arising from the work of Wiles [20], Diamond and Kramer [3] have proved that if M and N are distinct rational numbers, then $E(M,N)$ is modular. If D is a positive square-free integer, then $E_{\mathbb{Q}}(DM,DN)$ is the D-quadratic twist of $E_{\mathbb{Q}}(M,N)$.

(1) Let $1 \leq r \leq 23$ be an odd integer. Then there are infinitely many positive square-free integers $D \equiv r \bmod 24$ such that for

$$(M,N) \in \left\{ \begin{array}{lll} (24D,18D), & (6D,-18D), & (-6D,-24D) \\ (6D,54D), & (48D,-6D), & (-48D,-54D) \end{array} \right\}$$

the rank of $E_{\mathbb{Q}}(M,N)$ is 0. In this case the weight $\frac{3}{2}$ eigenform

$$f(z) = \frac{1}{2}\left(\sum q_{x,y,z\in\mathbb{Z}} q^{x^2+2y^2+12z^2} - q^{2x^2+3y^2+4z^2}\right) = \sum_{n=1}^{\infty} a(n)q^n$$

is in $S_{3/2}(48,\left(\frac{6}{n}\right))$. Its image $S_1(f(z)) = \sum_{n=1}^{\infty} A(n)q^n$ is the weight 2 newform whose Mellin transform is $L(E_{\mathbb{Q}}(M,N),s)$ where (M,N) is one of $(-4,-3)$, $(-1,3)$, $(1,4)$, $(-1,-9)$, $(-8,1)$, and $(8,9)$. If n is even, then $a(n) = 0$ and the conditions of the Main Theorem are not satisfied.

(2) If $1 \leq r \leq 40$ is an odd integer, then there are infinitely many positive square-free integers $D \equiv r$ or $9r \bmod 40$ such that for

$$(M,N) \in \{(50D,10D), \quad (40D,-10D), \quad (-40D,-50D)\}$$

the rank of $E_{\mathbb{Q}}(M,N)$ is 0. In this case the weight $\frac{3}{2}$ eigenform

$$f(z) = \frac{1}{2}\left(\sum_{x,y,z\in\mathbb{Z}} q^{x^2+2y^2+20z^2} - q^{2x^2+4y^2+5z^2}\right) = \sum_{n=1}^{\infty} a(n)q^n.$$

is in $S_{3/2}(80,\left(\frac{10}{n}\right))$. Its image $S_1(f(z))$ is the weight 2 newform whose Mellin transform is $L(E_{\mathbb{Q}}(M,N),s)$ where (M,N) is one of $(-5,-1)$, $(-4,1)$, or $(4,5)$. If n is even, then $a(n) = 0$ and the conditions of the Main Theorem are not satisfied.

(3) Let r be one of $1, 3, 5, 9, 11, 13, 17, 19$, or 21. Then there are infinitely many positive square-free integers $D \equiv r \bmod 24$ such that for

$$(M, N) \in \left\{ \begin{array}{lll} (12D, 3D), & (9D, -3D), & (-9D, -12D) \\ (27D, 24D), & (3D, -24D), & (-3D, -27D) \end{array} \right\}$$

the rank of $E_{\mathbb{Q}}(M, N)$ is 0.

Here the weight $\frac{3}{2}$ eigenform

$$f(z) = \frac{1}{2} \left(\sum_{x,y,z \in \mathbb{Z}} q^{x^2+7y^2+7z^2-2yz} - q^{3x^2+4y^2+5z^2-4yz} \right) = \sum_{n=1}^{\infty} a(n) q^n$$

is in $S_{3/2}(192, \left(\frac{3}{n}\right))$. Its image $S_1(f(z))$ is the Mellin transform of $L(E_{\mathbb{Q}}(M, N), s)$ where (M, N) is one of $(-4, -1)$, $(-3, 1)$, $(3, 4)$, $(-9, -8)$, $(-1, 8)$, or $(1, 9)$. Note that if n is even or $n \equiv 7 \bmod 8$, then the coefficient $a(n) = 0$. Hence for all $n \equiv 0, 2, 4, 6, 7 \bmod 8$, the conditions of the Main Theorem are not satisfied.

(4) In this example we consider the elliptic curves $E(N)$ which arise in Tunnell's conditional solution of the *congruent number problem* (see [17]). For any nonzero integer N, $E(N)$ is defined by

$$y^2 = x^3 - N^2 x.$$

These curves have complex multiplication by $\mathbb{Q}(i)$. Note that if D is a square-free integer, then $E(D)$ is the D-quadratic twist of $E(1)$. Assuming B-SD, it is conjectured that for square-free $D \equiv 5, 6, 7 \bmod 8$, that the rank of $E(D)$ is positive. If $r = 1, 2$, or 3, then there are infinitely many positive square-free integers $D \equiv r \bmod 8$ such that

$$y^2 = x^3 - D^2 x$$

has rank 0. The relevant weight $\frac{3}{2}$ eigenforms are

$$F_1(z) = \eta(8z)\eta(16z) \cdot \sum_{n \in \mathbb{Z}} q^{2n^2} = \sum_{n=1}^{\infty} a_1(n) q^n$$

and

$$F_2(z) = \eta(8z)\eta(16z) \cdot \sum_{n \in \mathbb{Z}} q^{4n^2} = \sum_{n=1}^{\infty} a_2(n) q^n.$$

It turns out that $F_1(z) \in S_{3/2}(128, \chi_0)$ and $F_2(z) \in S_{3/2}(128, \left(\frac{2}{n}\right))$ and that $S_1(F_1(z)) = S_1(F_2(z)) = \eta^2(4z)\eta^2(8z)$ which is the weight 2 newform that is

the Mellin transform of $L(E(1), s)$. We note that if $a_1(n) \neq 0$ (resp. $a_2(n) \neq 0$), then $n \equiv 1, 3 \bmod 8$ (resp. $(n \equiv 1, 5 \bmod 8)$).

(5) In this case we examine the elliptic curves $E(N)$ defined by

$$y^2 = x^3 + N.$$

These curves have complex multiplication by $\mathbb{Q}(\sqrt{-3})$. Note that if D is a square-free integer, then the D-quadratic twist of $E(N)$ by D is $E(D^3N)$. When $N = 1$ it turns out that the weight 2 cusp form $\eta^4(6z)$ is the Mellin transform of $L(E(1), s)$. If $r = 1, 2, 5, 10, 13, 14, 17$, or 22, then there exists infinitely many square-free positive integers $D \equiv r \bmod 24$ such that $E(-D)$, the elliptic curve

$$y^2 = x^3 - D^3,$$

(or the D-quadratic twist of $E(-1)$) has rank 0. The weight $\frac{3}{2}$ eigenform in $S_{3/2}(144, \chi_0)$ (χ_0 is the trivial character) is

$$F(z) = \eta^2(12z) \cdot \sum_{n \in \mathbb{Z}} q^{n^2} = \sum_{n=1}^{\infty} a(n) q^n$$

and $S_1(F(z))$ is $\eta^4(6z)$. If $n \not\equiv 1, 2, 5, 10 \bmod 12$, then $a(n) = 0$ and the conditions of the Main Theorem do not apply for these arithmetic progressions.

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