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Dilogarithm, Grassmannian complex and scissors congruence groups of algebraic polyhedra

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Abstract. We study the relationship between the Grassmannian complex of weight two and the scissors congruence group of algebraic polyhedra. They are related by the "motivic" dilogarithm, an abstract version of the classical dilogarithm function.

Introduction

In this paper we will present the relationship between the Grassmannian complex and the scissors congruence groups of algebraic polyhedra. (We restrict ourselves to the case of weight two.) More specifically,

(a) We define a two-term complex (concentrated in degrees one and two) $T(2)$: $T_2(2) \xrightarrow{\partial} T_1(2)$ where $T_2(2)$ (resp. $T_1(2)$) is the scissors congruence group of triangles (resp. line segments) in $\mathbb{P}^2$, and ∂ is the map which takes the boundary.

(b) Let $G(2)$ be the Grassmannian complex of weight two. We will construct maps of complexes $G(2) \to T(2)$. The maps are related to Grassmannian dilogarithms.

(c) Let $A(2)$ be the two-term complex introduced in [1]. We show that there is a natural map $T(2) \to A(2)$ which is an isomorphism.

The ideas in this paper were motivated by the geometric construction of (Grassmannian) polylogarithms [7].

We briefly recall a few definitions from [7] in the case of weight two.

Let $\mathbb{P}^n$ be projective n-space over $\mathbb{C}$. The ith coordinate hyperplane is defined by $L_i = \{x_i = 0\}$ where $(x_0 : \cdots : x_n)$ are the homogeneous coordinates. Define G_1^2 to be the set of lines $\xi \subset \mathbb{P}^3$ which meet any intersection of coordinate hyperplanes transversally. Also let G_0^2 be the set of points $P \notin \cup L_i$. Then one has a map δ_i: $G_1^2 \to G_0^2$ $(i = 0, \ldots, 3)$ which sends ξ to $\xi \cap L_i$.

For a smooth complex variety X, denote by $\widetilde{\Omega}^p X$ the vector space of multi-valued holomorphic differential p-forms on X. By definition, a *Grassmannian*

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dilogarithm is a pair (ψ_0, ψ_1) with $\psi_0 \in \widetilde{\Omega}^1 G_0^2$, $\psi_1 \in \widetilde{\Omega}^0 G_1^2$ which satisfies the following conditions:

$$d\psi_1 = \sum (-1)^i \delta_i^* \psi_0, \quad d\psi_0 = \mathrm{vol}_2.$$

Here $\mathrm{vol}_2 := \mathrm{dlog}(x_1/x_0) \wedge \mathrm{dlog}(x_2/x_0)$ (the complex volume form).

The Grassmannian dilogarithm may be constructed as follows (cf. the introduction of [7]):

(1) Construct a family of (linear combinations of) "triangles" M_1^2 parametrized by G_1^2. Also construct a family of "line segments" M_0^2 parametrized by G_0^2.

(2) The integration along the fiber produces associated differential forms

$$\psi_1 := \int [M_1^2 \mid \mathrm{vol}_2] \in \widetilde{\Omega}^0 G_1^2 \quad \text{and} \quad \psi_0 := \int [M_0^2 \mid \mathrm{vol}_2] \in \widetilde{\Omega}^1 G_0^2.$$

(3) By Stokes' formula, one has $d\psi_1 = \int [\partial M_1^2 \mid \mathrm{vol}_2]$ where ∂M_1^2 is the "boundary" of M_1^2. Using geometric relations between ∂M_1^2 and M_0^2, one can prove that (ψ_0, ψ_1) satisfies the required conditions.

A "triangle" M in $\mathbb{P}^2$ is a triplet of lines (M_0, M_1, M_2). A "line segment" is a triplet $\ell = (H; Q_0, Q_1)$ where H is a line in $\mathbb{P}^2$ and $Q_0, Q_1 \in H$. If one has a family of triangles M parametrized by a smooth variety T, and if M is *admissible*, one may form the integral of vol_2 along the fiber: $\int [M|\mathrm{vol}_2]$; this is a multi-valued holomorphic function on T. (The admissibility is a certain genericity condition which implies the convergence of the integral (cf. (1.2)). Similarly, a family of line segments gives a multi-valued 1-form.

There are three objects involved in this construction: Grassmannians, triangles and line segments, and differential forms. Each of them has a related complex: the Grassmannian complex, the complex $T(2)$, or the (truncated) de Rham complex. The Grassmannian complex of weight two is, by definition, $G(2)$: $\mathbb{Q}G_1^2 \xrightarrow{\delta} \mathbb{Q}G_0^2$ (in degrees one and two) where $\mathbb{Q}G_1^2$, for example, is the free $\mathbb{Q}$-vector space on the set G_1^2, and $\delta = \sum_i (-1)^i \delta_i$. The basic idea of the paper is the following. Define $T(2)$ appropriately, incorporating the relations such as those used in (3). Then one can relate $T(2)$ and $G(2)$ via a map $G(2) \to T(2)$. This map can be thought of a formal version of the dilogarithm. Note, however, that the differential forms have been taken out of the picture.

Throughout this paper we take any infinite field k as the ground field. In the first section, we define the groups $T_2(2)$ and $T_1(2)$. $T_2(2)$ is generated by admissible triangles, which are subject to several relations (1.3). The most important among them are the first and second scissors congruence relations. The first scissors congruence is an obvious one (by cutting and pasting); the second congruence of triangles has to do with the four triangles which a triplet of planes $(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2)$ in $\mathbb{P}^3$ gives rise to as intersections with the coordinate hyperplanes. The group $T_1(2)$ is generated by generic line segments and the boundaries of admissible triangles, which have relations similar to those for triangles (cf. (1.4)).

In Section 3 we construct maps of complexes u and f: $G(2) \to T(2)$. The map u is related to the construction of dilogarithm in the introduction of [7]. The map f is a modification of u, and is directly related to the cross ratio of $\xi \in G_1^2$.

In [1] Beilinson, Goncharov, Schechtman, and Varchenko introduced the group A_n of pairs of triangles (modulo scissors congruence) in $\mathbb{P}^n$, and a related complex $A(n)$ for each n. In the case $n = 2$ it takes the form: $A(2) = [A_2 \xrightarrow{\nu} k^*_{\mathbb{Q}} \otimes k^*_{\mathbb{Q}}]$ (where $k^*_{\mathbb{Q}} := k^* \otimes_{\mathbb{Z}} \mathbb{Q}$). They related the cohomology of this complex to weight two K-groups:

$$H^1(A(2)) \cong K_3^{\text{ind}}(k)_{\mathbb{Q}}; \quad H^2(A(2)) \cong K_2^M(k)_{\mathbb{Q}}.$$

Here $K_3^{\text{ind}}(k)$ is the indecomposable part of $K_3(k)$, and $K_2^M(k)$ is the Milnor K-group. In Section 2 we will construct a map of complexes (the *symbol map*) σ: $T(2) \to A(2)$, and prove that it is an isomorphism.

There are advantages of working with $T(2)$ over $A(2)$. The former allows one to discuss algebraic polyhedra without reference to coordinates. This remark will be more relevant in the case of weight at least 3, where coordinate-free viewpoint can be much more amenable. Part of the content of this paper may be generalized to the case of weight $\geqslant 3$. It will be discussed in a future paper.

Although we omit it, it is possible to define the groups $T_2(2)(V)$ (resp. $T_1(2)(V)$) of triangles (resp. line segments) parametrized by V, and exhibit the relationship between the resulting complex $T(2)(V)$ and the Deligne complex. (One needs to be careful in defining them, taking into account the foundations developed in [7, 8].)

This paper naturally resulted from the work with R. MacPherson, whom the author thanks cordially; the paper [1] motivated us to write up the formal aspects of our construction. We would like to thank also A. Beilinson, A. Goncharov, V. Schechtman, and A. Varchenko for helpful conversations.

1. The complex $T(2)$

We give the definition of the complex $T(2)$ and study its structure. The relations involved in the definition were motivated by the cancellation lemma for differential forms [7]. Thus the integration map $M \mapsto \int[M \mid \text{vol}_2]$ factors through $T(2)$ (if M is parametrized by a complex variety).

(1.1) Throughout this paper we fix an infinite field k. Let $\mathbb{P}^n$ be projective n-space over k, with homogeneous coordinates $(x_0 : \cdots : x_n)$. There are canonical inclusions $\delta_i : \mathbb{P}^{n-1} \to \mathbb{P}^n$, $(x_0 : \cdots : x_{n-1}) \mapsto (x_0 : \cdots x_{i-1} : 0 : x_i : \cdots : x_{n-1})$ $(i = 0, \ldots, n)$; its image is the ith coordinate hyperplane $L_i = \{x_i = 0\}$. For a subset $I \subset \{0, \ldots, n\}$, we set $L_I = \bigcap_{i \in I} L_i$. Note that there is a canonical isomorphism $L_I \cong \mathbb{P}^{n-m}$ where $m = |I|$. The collection $\{L_I\}$ is referred to as the *coordinate simplex*. We also let $e_i = L_I$ with $I = \{0, \ldots, i-1, \hat{i}, i+1, \ldots, n\}$ (ith vertex of the simplex).

For a permutation τ of $\{0,1,\dots,n\}$, there is induced an automorphism of $\mathbb{P}^n$ sending L_i to $L_{\tau(i)}$; this we denote by the same τ.

(1.2) A *triangle* in $\mathbb{P}^2$ is defined to be a triplet of lines $M = (M_0, M_1, M_2)$. It is said to be *nondegenerate* if the lines $\{M_i\}$ are in a general position, namely if $\{M_i\}$ are distinct and $M_1 \cap M_2 \cap M_3 = \emptyset$. For a nondegenerate triangle M and distinct elements $i,j \in \{0,1,2\}$, let $M_{ij} = M_i \cap M_j$, which is a point. A triangle M is said to be *admissible* if $M_i \neq L_j$ for any i,j and if there is no pair $\{i,j\}$ such that $M_i \cap M_j = e_k$. M is *generic* if there is no pair $\{i,j\}$ such that $M_i \cap M_j \notin \cup L_i$. Admissible or generic triangles may be degenerate.

Let $T_2'(2)$ be the $\mathbb{Q}$-vector space freely generated by all triangles. Define $T_2^{\mathrm{ad}}(2)$ to be the subspace of $T_2'(2)$ generated by all admissible triangles.

A *line segment* in $\mathbb{P}^2$ is, by definition, a triplet $\ell = (H; Q_0, Q_1)$ where H is a line in $\mathbb{P}^2$ and Q_0, Q_1 are points on H. We say ℓ is *nondegenerate* if $Q_0 \neq Q_1$; ℓ is *generic* if $Q_0, Q_1 \notin \cup L_i$. Let $T_1'(2)$ be the $\mathbb{Q}$-vector space freely generated by all line segments.

One defines the boundary map $\partial\colon T_2'(2) \to T_1'(2)$ as follows on generators: if M is a degenerate triangle, $\partial M = 0$; if M is nondegenerate,

$$\partial M = \sum_{i=0}^{2} (-1)^i (M_i; M_{i0}, \dots, \widehat{M_{ii}}, \dots, M_{i2}).$$

Define $T_1^{\mathrm{ad}}(2) \subset T_1'(2)$ to be the subspace generated by generic line segments and the image of $\partial\colon T_2^{\mathrm{ad}}(2) \to T_1'(2)$; there is the induced map $\partial\colon T_2^{\mathrm{ad}}(2) \to T_1^{\mathrm{ad}}(2)$.

(1.3) Define $T_2(2)$ to be the quotient $\mathbb{Q}$-vector space of $T_2^{\mathrm{ad}}(2)$ with respect to the following relations (0)–(iv).

(0) $M = 0$ if M is degenerate.

(i) For a permutation α of $\{0,1,2\}$,

$$(M_{\alpha(0)}, M_{\alpha(1)}, M_{\alpha(2)}) = \mathrm{sgn}(\alpha)(M_0, M_1, M_2).$$

(ii) For a permutation τ of $\{0,1,2\}$, and the induced automorphism τ of $\mathbb{P}^2$,

$$(\tau(M_0), \tau(M_1), \tau(M_2)) = \mathrm{sgn}(\tau)(M_0, M_1, M_2).$$

(iii) (*First scissors congruence*) For a quadruple of lines $(M_0, \dots, M_3)$ such that $M_i \neq L_j$ and $M_i \cap M_j \neq e_k$,

$$\sum_{i=0}^{3} (-1)^i (M_0, \dots, \widehat{M_i}, \dots, M_3) = 0.$$

(iv) (*Second scissors congruence*) Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2)$ be a triplet of planes in $\mathbb{P}^3$ such that $\mathcal{M}_i \neq L_j$ for any i,j, and for each i,

$$\delta_i^* \mathcal{M} = \mathcal{M} \cap L_i := (\mathcal{M}_0 \cap L_i, \mathcal{M}_1 \cap L_i, \mathcal{M}_2 \cap L_i)$$

is an admissible triangle on $L_i \cong \mathbb{P}^2$. Then one has

$$\sum_{i=0}^{3}(-1)^i \delta_i^* \mathcal{M} = 0.$$

(1.4) For an analogous construction for lines segments, we consider the relations (where $(H; Q_0, Q_1)$, etc. are generic line segments and the triangles are all admissible):

(0) $(H; Q_0, Q_1) = 0$ if it is degenerate.

(i) $(H; Q_0, Q_1) = -(H; Q_1, Q_0)$;
For a permutation α of $\{0, 1, 2\}$, $\partial(M_{\alpha(0)}, M_{\alpha(1)}, M_{\alpha(2)}) = \mathrm{sgn}(\alpha)\partial(M_0, M_1, M_2)$.

(ii) For a permutation τ of $\{0, 1, 2\}$ and the induced automorphism τ of $\mathbb{P}^2$,
$$(\tau(H); \tau(Q_0), \tau(Q_1)) = \mathrm{sgn}(\tau)(H; Q_0, Q_1); \partial(\tau M) = \mathrm{sgn}(\tau)\partial M.$$

(iii) (*First scissors congruence*) (a) For three points Q_0, Q_1, Q_2 on a line H,
$$(H; Q_1, Q_2) - (H; Q_0, Q_2) + (H; Q_0, Q_1) = 0.$$

(b) For $M_0, \dots, M_3$ as in (1.3)(iii),
$$\sum_{i=0}^{3}(-1)^i \partial(M_0, \dots, \widehat{M_i}, \dots, M_3) = 0.$$

(iv) (*Second scissors congruence*) (a) Let $\mathcal{H}$ be a plane in $\mathbb{P}^3$ with $\mathcal{H} \not\subset \bigcup_i L_i$, and $\mathcal{Q}_0$ and $\mathcal{Q}_1$ be lines on $\mathcal{H}$ such that for each i,
$$\delta_i^*(\mathcal{H}; \mathcal{Q}_0, \mathcal{Q}_1) = (\mathcal{H} \cap L_i; \mathcal{Q}_0 \cap L_i, \mathcal{Q}_1 \cap L_i)$$
is a generic line segment on L_i. Then one has
$$\sum_{i=0}^{3}(-1)^i \delta_i^*(\mathcal{H}; \mathcal{Q}_0, \mathcal{Q}_1) = 0.$$

(b) For an $\mathcal{M}$ as in (1.3)(iv),
$$\sum_{i=0}^{3}(-1)^i \partial(\delta_i^* \mathcal{M}) = 0.$$

We let $T_1(2)$ to be the quotient $\mathbb{Q}$-vector space of $T_1^{\mathrm{ad}}(2)$ with respect to the relations (0)–(iv). One has the induced boundary map ∂: $T_2(2) \to T_1(2)$.

The constructions so far are summarized in the following commutative diagram:

$$\begin{array}{ccccc}
T_2'(2) & \longleftarrow & T_2^{\mathrm{ad}}(2) & \longrightarrow & T_2(2) \\
\partial\big\downarrow & & \partial\big\downarrow & & \partial\big\downarrow \\
T_1'(2) & \longleftarrow & T_1^{\mathrm{ad}}(2) & \longrightarrow & T_1(2)
\end{array}$$

Each column is a two-term complex concentrated in degrees one and two. The complex on the extreme right we will denote by $T(2)$.

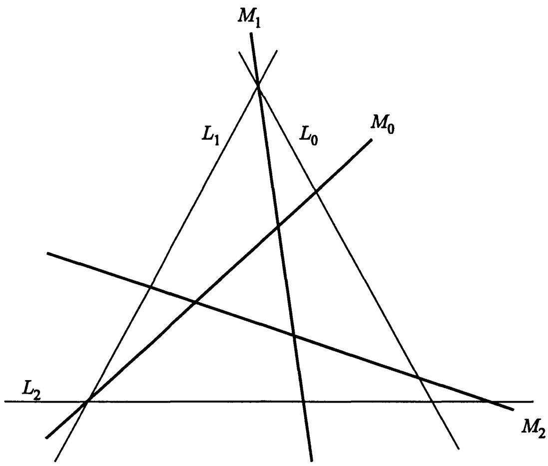

Fig. 1.6.1.

(1.5) For a quadruple of points $P_0, \ldots, P_3 \in \mathbb{P}^1_k$ with $\{P_0, P_1, P_2\}$ distinct, define the cross ratio $r(P_0, P_1; P_2, P_3) = \lambda \in k$ if a (projective) automorphism of $\mathbb{P}^1$ takes $(P_0, P_1; P_2, P_3)$ to $(0, \infty; 1, \lambda)$.

For a point $P = (x_0 : x_1) \in \mathbb{P}^1_k$ with $x_0 x_1 \neq 0$, define its *coordinate* $\mathrm{crd}(P) = -x_1/x_0 \in k^*$. If $I \subset \{0, \ldots, n\}$ with $|L| = n-1$, and $P \in L_I - \{e_j\}$, one has $\mathrm{crd}_{L_I}(P) \in k^*$.

With regard to projective space $\mathbb{P}^n$, one defines the hyperplanes:

$$\begin{aligned} C_n &= \{x_0 + \cdots + x_n = 0\}; \\ K_{ij} &= \{x_i + x_j = 0\} \quad \{i, j\} \subset \{0, \ldots, n\}. \end{aligned}$$

(1.6) It is easy to see that the group $T_2(2)$ is generated by triangles of the two types in Figures 1.6.1 and 1.6.2.

Let D_a be the triangle in Figure 1.6.3 (where $\mathrm{crd}_{L_1}(P) = a \in k^*$ and $M_0 = K_{01}, M_2 = C_2$); if $a \in k^* - \{1\}$, this is a special case of a triangle of the type in Figure 1.6.2. We call D_a the *dilogarithmic configuration* with invariant a (cf. [1, (1.2)]). Note that $r(P, e_1; R, Q) = 1 - a$.

(1.7) The group $T_1(2)$ is generated by generic line segments and *admissible pairs of line segments* which, by definition, are elements of the form $(H; P, Q) - (H'; P, Q')$ where P lies on $(\bigcup L_i) - (\bigcup e_i)$ and $Q, Q' \notin \bigcup L_i$, Figure 1.7.1.

Define $\ell_{a,b} \in T_1(2)$ $(a, b \in k^*)$ to be the class of $(H; Q_0, Q_1)$ in Figure 1.7.2, where $\mathrm{crd}_{L_1}(H \cap L_1) = a$, and $r(e_1, H \cap L_1; Q_0, Q_1) = b$.

PROPOSITION 1.8. (1) *The class of the line segment* $(H; Q_0, Q_1)$ *in Figure* 1.8.1 *is equal to* $\ell_{a,b}$ *with* $a = \mathrm{crd}_{L_1}(H \cap L_1)$, $b = r(e_1, H \cap L_1; Q_0, Q_1)$.

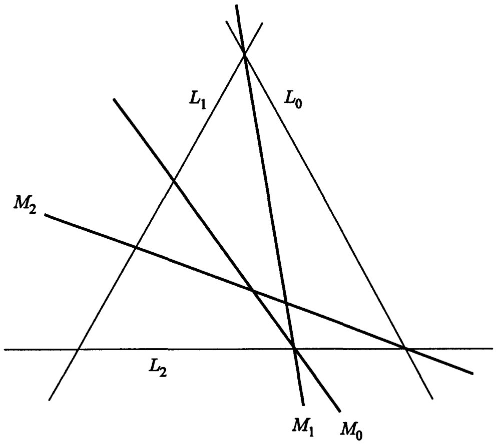

Fig. 1.6.2.

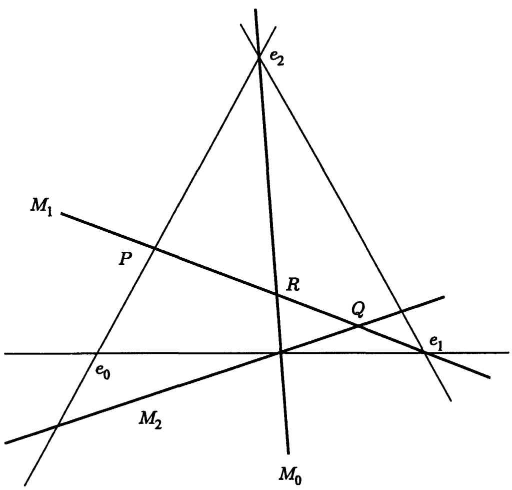

Fig. 1.6.3.

(2) *The class of a generic line segment on K_{ij} or on C_2 is zero.*

(3) *In the group $T_1(2)$ we have the following identities:*

$$\ell_{a,b} + \ell_{a,b'} = \ell_{a,bb'} ;$$
$$\ell_{a,b} + \ell_{a',b} = \ell_{aa',b}.$$

(4) *The class of the admissible pair of line segments $(C_2; e_{01}, P) - (K_{01}; e_{01}, Q)$, where $P \in C_2$, $Q \in K_{01}$, is zero.*

(5) *One has $\partial D_a = \ell_{a,1-a}$.*

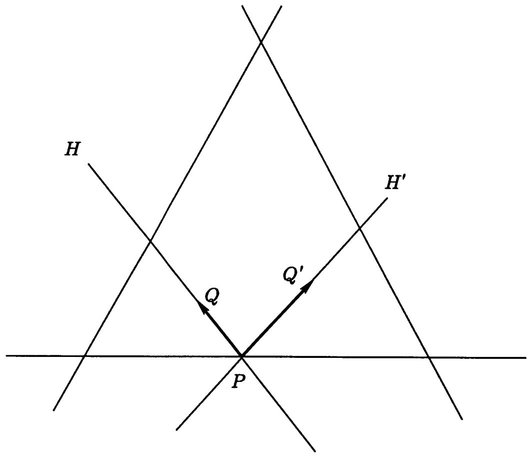

Fig. 1.7.1.

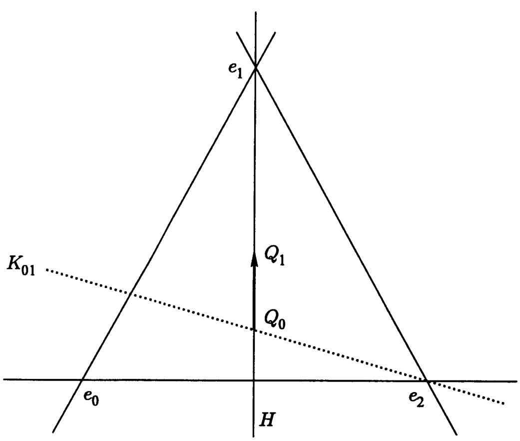

Fig. 1.7.2.

Proof. (1) Let $\mathcal{H} = \{ax_2 + x_3 = 0\} \subset \mathbb{P}^3$, $\mathcal{Q}_0, \mathcal{Q}_1$ be lines $\subset \mathcal{H}$ such that $\mathcal{Q}_0 \cap \mathcal{Q}_1 \subset L_{23}$, $\delta_1^*(\mathcal{H}; \mathcal{Q}_0, \mathcal{Q}_1) = (H; Q_0, Q_1)$ and $\delta_0^*(\mathcal{H}; \mathcal{Q}_0, \mathcal{Q}_1)$ be the line segment in Figure 1.7.2. One applies (1.4)(iv)(a).

(2) Let $(K_{01}; Q_0, Q_1)$ be a generic line segment. One takes $\mathcal{H} = \{x_0 + x_1 + x_2 = 0\} \subset \mathbb{P}^3$, $\mathcal{Q}_0, \mathcal{Q}_1$ be lines $\subset \mathcal{H}$ such that $\mathcal{Q}_0 \cap \mathcal{Q}_1$ is a point in L_3 and $\delta_0^*(\mathcal{H}; \mathcal{Q}_0, \mathcal{Q}_1) = (K_{01}; Q_0, Q_1)$, Figure 1.8.2. By (1), $\delta_i^*(\mathcal{H}; \mathcal{Q}_0, \mathcal{Q}_1)$, $i = 0, 1, 2$, are all equal in $T_1(2)$. Applying the relation (1.4)(iv)(a), one obtains $(K_{01}; Q_0, Q_1) = 0$.

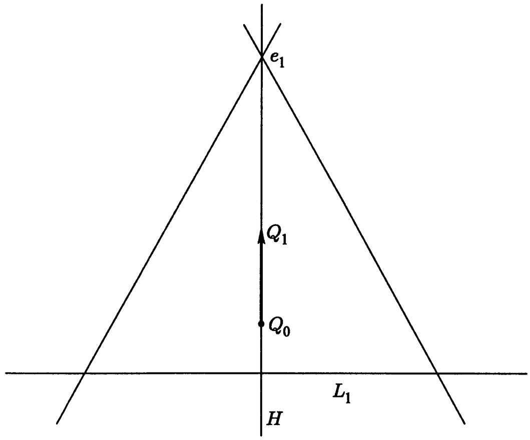

Fig. 1.8.1.

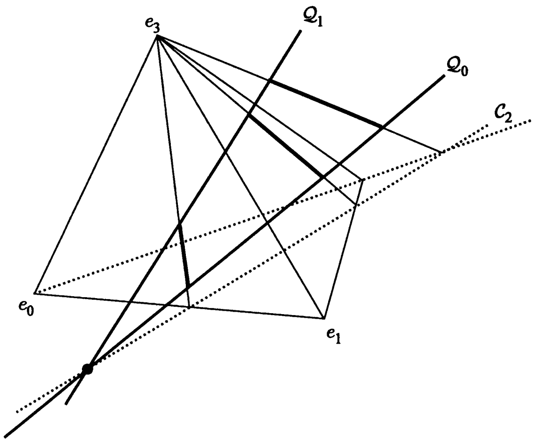

Fig. 1.8.2.

The argument for a line segment on $\mathcal{C}_2$ is as follows. Let $\mathcal{H} = \{x_0+x_1+x_2 = 0\} \subset \mathbb{P}^3$, $\mathcal{Q}_0, \mathcal{Q}_1$ be lines $\subset \mathcal{H}$ such that $\delta_i^*(\mathcal{H}; \mathcal{Q}_0, \mathcal{Q}_1)$ are generic for $i = 0, 1, 2$ and $\delta_3^*(\mathcal{H}; \mathcal{Q}_0, \mathcal{Q}_1) = (\mathcal{C}_2; Q_0, Q_1)$. Note $\delta_i^*(\mathcal{H}; \mathcal{Q}_0, \mathcal{Q}_1) = 0$ for $i = 0, 1, 2$ since they are supported on K_{01}. One applies (1.4)(iv)(a).

(3) For the first relation, let $H = \{ax_0 + x_1 = 0\} \subset \mathbb{P}^2$, $R = H \cap L_2$, and $P_0, P_1, P_2 \in H$ such that $r(R, e_2; P_0, P_1) = b$, $r(R, e_2; P_1, P_2) = b'$, and $r(R, e_2; P_0, P_2) = bb'$. Apply (1.4)(iii)(a).

For the second, take $\mathcal{H} = \{ax_0 + aa'x_1 + x_2 = 0\} \subset \mathbb{P}^3$; then $\mathrm{crd}_{L_{23}}(\mathcal{H} \cap L_{23}) = a'^{-1}$, $\mathrm{crd}_{L_{13}}(\mathcal{H} \cap L_{13}) = a$ and $\mathrm{crd}_{L_{03}}(\mathcal{H} \cap L_{03}) = aa'$. Take $\mathcal{Q}_0, \mathcal{Q}_1$ be lines $\subset \mathcal{H}$ so that $\mathcal{Q}_0 \cap \mathcal{Q}_1 \in \mathcal{H} \cap L_3$, $r(\mathcal{H} \cap L_{i3}, e_3; \mathcal{Q}_0 \cap L_i, \mathcal{Q}_1 \cap L_i) = b$ $(i = 0, 1, 2)$. Then apply (1.4)(iv)(a).

(4) By adding generic line segments on $\mathcal{C}_2$ and K_{01}, one is reduced to the case: $P = K_{12} \cap \mathcal{C}_2, Q = K_{12} \cap K_{01}$. Since $(K_{12}; Q, P) = 0$, this class equals $\partial(\mathcal{C}_2, K_{01}, K_{12})$. This is seen to be zero by applying (1.4)(iii)(b) to $(\mathcal{C}_2, K_{01}, K_{02}, K_{12})$ and using $\partial(K_{01}, K_{02}, K_{12}) = 0$, (2).

(5) Follows from (4) and (1).

2. The complex $A(2)$ and the symbol map σ: $T(2) \to A(2)$

(2.1) *The group A_n* [1]. We keep the notation from Section 1 except that L is not exclusively used for coordinate hyperplanes.

An ordered collection of $(n+1)$ hyperplanes in $\mathbb{P}^n$ is called an $(n+1)$-simplex: $L = (L_0, \cdots, L_n)$. For $I \subset \{0, \ldots, n\}$, let $L_I = \bigcap_{i \in I} L_i$. L is *nondegenerate* if $\{L_i\}$ are in general position, namely if for any subset $I \subset \{0, \ldots, n\}$, dim $L_I = n - |I|$. Let $A_n (n \geqslant 1)$ be the $\mathbb{Q}$-vector space defined by the following generators and relations.

Generators: pairs of ordered $(n+1)$-tuples of hyperplanes in $\mathbb{P}^n$,

$$(L; M) = (L_0, \ldots, L_n; M_0, \cdots M_n),$$

which are admissible; by definition $(L; M)$ is *admissible* if the sets $\{M_I\}$, $\{L_I\}$ are disjoint.

Relations: (i) If L or M is degenerate, $(L; M) = 0$.
(ii) For a permutation α of $\{0, 1, \ldots, n\}$,

$$\begin{aligned}&(L_{\alpha(0)}, \ldots, L_{\alpha(n)}; M_0, \ldots M_n)\\ &\quad = (L_0, \ldots, L_n; M_{\alpha(0)}, \ldots M_{\alpha(n)}) = \operatorname{sgn}(\alpha)(L; M).\end{aligned}$$

(iii) For $(n+2)$ hyperplanes $(L_0, \ldots L_{n+1})$,

$$\sum_{j=0}^{n+1} (-1)^j (L_0, \ldots, \widehat{L_j}, \ldots, L_{n+1}; M_0, \ldots M_n) = 0.$$

Similarly,

$$\sum_{j=0}^{n+1} (-1)^j (L_0, \ldots, L_n; M_0, \ldots, \widehat{M_j}, \ldots, M_{n+1}) = 0.$$

(iv) For $g \in PGL(n+1)$,

$$(gL; gM) = (L; M).$$

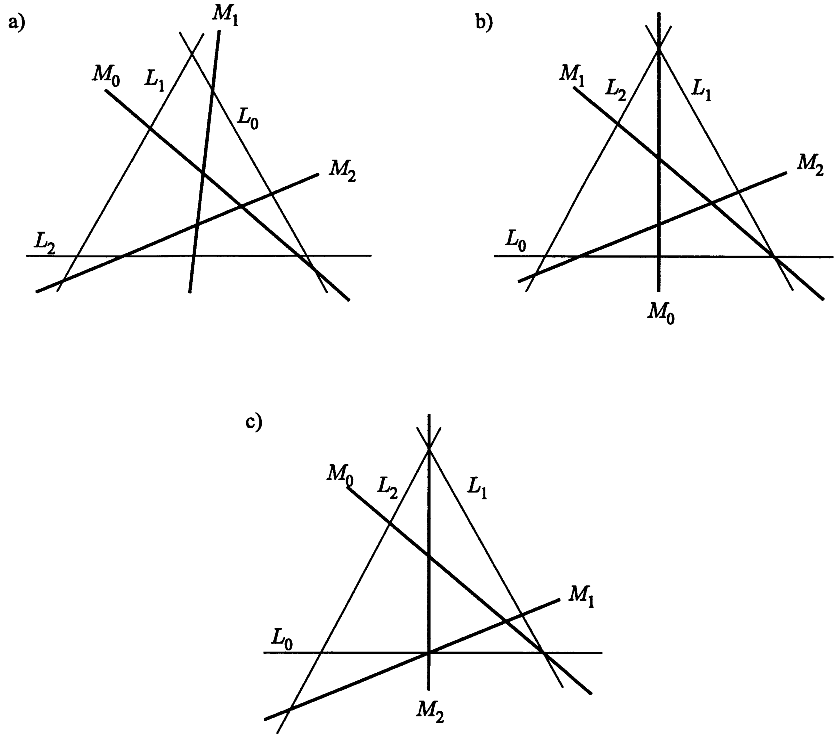

Fig. 2.2.

The group A_n is generated by $(L; M)$ which are nondegenerate admissible. We say $(L; M)$ is *generic* if for any I, $M_I \not\subset \bigcup_j L_j$.

The group A_1 is generated by $(L_0, L_1; M_0, M_1)$, quadruples of distinct points in $\mathbb{P}^1$. One has an isomorphism

$$r\colon\ A_1 \xrightarrow{\sim} k^*_{\mathbb{Q}},$$

where r is the cross-ratio of a quadruple.

(2.2) There is a comultiplication map ν: $A_2 \to A_1 \otimes A_1$ as defined in [1]. Let $A(2)$ be the complex with two terms A_2 and $A_1 \otimes A_1$, placed in degrees one and two respectively, and with the map ν as the differential. The map ν has categorical meanings, *loc. cit.*

The map ν is given explicitly on generators $(L; M)$ as follows.

(a) If $(L; M)$ is nondegenerate and no $L_{ij} \notin M_k$, $M_{ij} \notin L_k$, Figure 2.2.a, then

$$\nu((L; M)) \\ = L_2 \cap (L_0, L_1; M_0, M_2) \otimes M_2 \cap (L_0, L_2; M_0, M_1) -$$

$$\begin{aligned}&-L_2\cap(L_0,L_1;M_0,M_1)\otimes M_1\cap(L_0,L_2;M_0,M_2)-\\&-L_1\cap(L_0,L_2;M_0,M_2)\otimes M_2\cap(L_0,L_1;M_0,M_1)+\\&+L_1\cap(L_0,L_2;M_0,M_1)\otimes M_1\cap(L_0,L_1;M_0,M_2).\end{aligned}$$

Here $L_2 \cap (L_0, L_1; M_0, M_2)$, for example, stands for $(L_2 \cap L_0, L_2 \cap L_1; L_2 \cap M_0, L_2 \cap M_2) \in A_1$.

(b) If $(L; M)$ is of the type as in Figure 2.2.b,

$$\begin{aligned}&\nu((L;M))\\&\quad=L_0\cap(L_1,L_2;M_0,M_2)\otimes M_2\cap(L_0,L_1;M_0,M_2)+\\&\qquad+L_2\cap(L_0,L_1;M_1,M_2)\otimes M_2\cap(L_0,L_2;M_0,M_2).\end{aligned}$$

(c) If $(L; M)$ is as in Figure 2.2.c,

$$\begin{aligned}&\nu((L;M))\\&\quad=L_2\cap(L_0,L_1;M_0,M_1)\otimes M_0\cap(e_2,L_2;M_1,M_2)\\&\quad=L_2\cap(L_0,L_1;M_0,M_1)\otimes L_2\cap(L_0,M_0;M_1,L_1).\end{aligned}$$

In particular, if $D_a = (L_0, L_1, L_2; M_0, M_1, M_2)$ in Figure 1.6.3, one has

$$\nu(D_a) = a \otimes (1-a).$$

(2.3) We will define a map of complexes

$$\sigma\colon\ T(2) \to A(2)$$

called the *symbol map* in the following. On the degree 1 level, $\sigma\colon T_2(2) \to A_2$ is to be the map which sends an admissible triangle $M = (M_0, M_1, M_2) \in T_2(2)$ to $(L_0, L_1, L_2; M_0, M_1, M_2)$, where (L_0, L_1, L_2) is the coordinate simplex.

PROPOSITION. *The map $\sigma\colon T_2(2) \to A_2$ is well-defined.*

Proof. The relation (1.3)(0) (resp. (i)) is taken under σ to (2.1) (i) (resp. the second equality in (2.1)(ii)). The relation (1.3)(ii) is taken to

$$(L; \tau M_0, \tau M_1, \tau M_2) = \operatorname{sgn}(\tau)\,(L; M_0, M_1, M_2)$$

which is a consequence of (2.1)(iv), (ii) as follows:

$$\begin{aligned}&(L;\tau M_0,\tau M_1,\tau M_2)\\&\quad=(\tau^{-1}L_0,\tau^{-1}L_1,\tau^{-1}L_2;M_0,M_1,M_2)&&\text{by (2.1)(iv)}\\&\quad=(L_{\tau^{-1}(0)},L_{\tau^{-1}(1)},L_{\tau^{-1}(2)};M_0,M_1,M_2)\\&\quad=\operatorname{sgn}(\tau)\,(L;M_0,M_1,M_2)&&\text{by (2.1)(ii).}\end{aligned}$$

The relation (1.3)(iii) is taken to the second equality of (2.1)(iii).

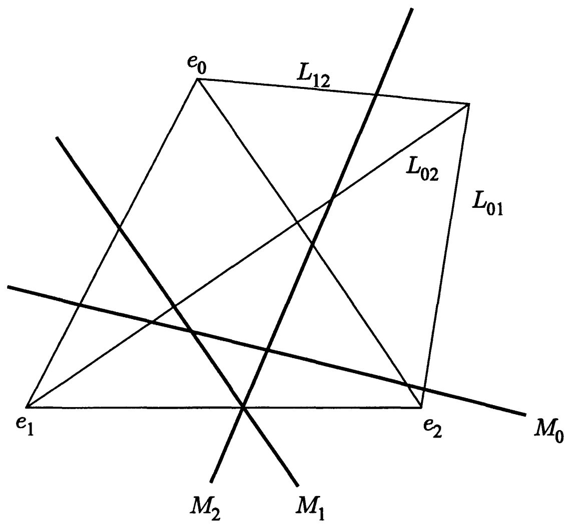

Fig. 2.3.

Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2)$ be a triplet of planes in $\mathbb{P}^3$ as in Definition (1.3)(iv), and put $v = \mathcal{M}_0 \cap \mathcal{M}_1 \cap \mathcal{M}_2$. Note $v \neq e_k$. Project $\mathbb{P}^3$ from v to any plane not containing v. We have a configuration of lines in the plane, Figure 2.3. Here by abuse of the notation, we denote by L_{ij}, and e_i the images under the projection of the respective subspaces. Let M_i be the image of $\mathcal{M}_i$. Since

$$\sigma(\mathcal{M} \cap L_i) = (L_{i0}, \cdots, \widehat{L_{ii}}, \ldots, L_{i3}; M_0, M_1, M_2),$$

using (2.1)(iii), one shows

$$\sum_{i=0}^{3}(-1)^i \sigma(\mathcal{M} \cap L_i) = 0.$$

(2.4) To define the map σ: $T_1(2) \to k^*_{\mathbb{Q}} \otimes k^*_{\mathbb{Q}}$ in degree two, we first treat generic line segments.

(a) If $\ell = (H; Q_0, Q_1)$ is generic and H does not contain any e_i, one takes any point $v \in H - \{Q_0, Q_1\}$ and let

$$\sigma((H; Q_0, Q_1)) = \sum_{i=0}^{2}(-1)^i \mathrm{crd}_{L_i}(H \cap L_i) \otimes (v, H \cap L_i; Q_0, Q_1).$$

Note that this is well-defined independent of the choice of v. In fact, if we take another point $v' \in H - \{Q_0, Q_1\}$, the result will differ by

$$\sum_{i=0}^{2}(-1)^i \mathrm{crd}_{L_i}(H \cap L_i) \otimes (v, v'; Q_0, Q_1)$$

which equals zero since $\sum_{i=0}^{2}(-1)^i \mathrm{crd}_{L_i}(H \cap L_i) = 0$ by the lemma below. Taking $v = H \cap L_o$ for example, one has

$$\sigma((H; Q_0, Q_1)) = \sum_{i=1,2} (-1)^i \mathrm{crd}_{L_i}(H \cap L_i) \otimes (H \cap L_0, H \cap L_i; Q_0, Q_1).$$

(b) If $H \ni e_i$, then we let

$$\sigma((H; Q_0, Q_1)) = (-1)^i \mathrm{crd}_{L_i}(H \cap L_i) \otimes (e_i, H \cap L_i, ; Q_0, Q_1).$$

LEMMA. *Let $\xi \subset \mathbb{P}^2$ be a line which meets the coordinate simplex transversally, and $P_i = \xi \cap L_i$. Then one has*

$$\sum_{i=0}^{2} \mathrm{crd}_{L_i}(P_i) = 0 \quad in \quad k^*_{\mathbb{Q}}.$$

(2.5) Let $\overline{T_1^{\mathrm{ad}}}(2) := T_1^{\mathrm{ad}}(2)/\sim$ where $\sim$ denotes the relation (subspace) generated by the relations (1.4)(0), (i), (iii).

To any element of $\overline{T_1^{\mathrm{ad}}}(2)$ one can associate its *support* as follows. For a non-dgenerate "line segment" $\ell = (H; Q_0, Q_1)$, let $\mid \ell \mid := \overline{Q_0 Q_1}$. An element $n \in \overline{T_1^{\mathrm{ad}}}(2)$ can be written $n = \sum b_i \, \ell_i$, $b_i \in \mathbb{Q}$, where

(2.5.1) Each $b_i \neq 0$. Each ℓ_i is a nondegenerate "line segment" and if $i \neq i'$, $\mid \ell_i \mid \cap \mid \ell_{i'} \mid$ is either empty or a point.

Define the support of n by $\mid n \mid = \cup \mid \ell_i \mid$; this is well-defined independent of the expression of n. Note $n = 0$ in $\overline{T_1^{\mathrm{ad}}}(2)$ iff $\mid n \mid = \emptyset$.

We will use the following alternative description of the group $\overline{T_1^{\mathrm{ad}}}(2)$. Consider the $\mathbb{Q}$-vector space with generic line segments $\ell \in T_1'(2)$ and formal symbols ∂M, one for each admissible triangle M, as generators, and with the following relations (denoted $\approx$):

(1.4)(0), (i), (iii) ((iii)(b) should be regarded as a relation between the four formal symbols ∂M), and: for a *generic* triangle M,

$$\partial M \approx \sum (-1)^i (M_i; M_{i0}, \ldots, \widehat{M_{ii}}, \ldots, M_{i2}). \tag{$\dagger$}$$

CLAIM. *The natural map $\mathbb{Q}\{\partial M, \ell\}/\approx \to T_1^{\mathrm{ad}}(2)/\sim$ is an isomorphism.*

Proof. The assignment $\partial M \mapsto \partial M \in T_1^{\mathrm{ad}}(2)$ induces the map in the statement since $\approx$ is sent to $\sim$. This map is clearly surjective. To show the injectivity, take an expression

$$n = \sum a_r \partial M_r + \sum b_j \ell_j$$

with $a_r, b_j \in \mathbb{Q}$ such that $n \sim 0$ (so $\mid n \mid = \emptyset$).

Say that an admissible triangle $M = (M_0, M_1, M_2)$ has a singular point P if $P \in M_j \cap M_k$ and $P \in \cup L_i$. We will show the above claim by induction on the cardinality of the set $\{P \mid P$ is a singular point of some $M_r\}$.

If there is no singular point, using (†) to each M_r, $n \approx \sum b'_j \ell'_j$ (ℓ'_j is generic), which is, using (1.4)(0), (i), (iii), $\approx \sum b''_j \ell''_j$ where the condition (2.5.1) is satisfied. Since $\mid n \mid = \emptyset$, all $b''_j = 0$.

If there is a singular point P, let $\mathcal{I} = \{I_k\}$ be the set of lines I_k such that there exists an M_r with I_k as one of the edges and $P \in I_k$. Take a generic line $I \ni P$, and a generic line $J (\not\ni P)$. One may assume that the triangles (I_k, I, J) have no singularities other than P.

Then for each M_r with P as a singular point, one has unique $I_k, I_\ell \in \mathcal{I}$ such that

$$M_r \text{ is equivalent, by (1.3)(0), (i), (iii), to } \quad (I_k, I, J) - (I_\ell, I, J) + N_r$$

where N_r is a triangle without P as a singular point. Taking linear combinations and then taking the boundary, one obtains

$$\sum a_r \partial M_r \approx \sum c_k \partial (I_k, I, J) + \sum a_r \partial N_r.$$

Since $\mid n \mid = \emptyset$, the support of $\sum a_r \partial M_r$ is empty in a neighborhood of P, hence so is the support of the right hand side of the above. This implies all $c_k = 0$. Thus $\sum a_r \partial M_r + \sum b_j \ell_j \approx \sum a_r \, \partial N_r + \sum b_j \ell_j$ and the induction prodeeds.

For $n \in \overline{T}_1^{\text{ad}}(2)$, write $n = \sum a_r \partial M_r + \sum b_j \ell_j$ and define

$$\sigma(n) = \sum a_r \nu(\sigma(M_r)) + \sum b_j \sigma(\ell_j) \in k^*_{\mathbb{Q}} \otimes k^*_{\mathbb{Q}}. \qquad (*)$$

PROPOSITION. *For a generic triangle M, the following identity holds: $\nu(\sigma(M)) = \sigma(\partial M)$. The formula $(*)$ gives a well-defined map σ: $\overline{T}_1^{\text{ad}}(2) \to k^*_{\mathbb{Q}} \otimes k^*_{\mathbb{Q}}$.*

Proof. For the first assertion, one needs to check this only for M in Figure 2.2.b. One has

$$\begin{aligned}
&\sigma(\partial M)\\
&\quad = \mathrm{crd}_{L_0}(L_0 \cap M_0) \otimes M_0 \cap (L_0, e_0; M_1, M_2) -\\
&\qquad -\mathrm{crd}_{L_2}(L_2 \cap M_1) \otimes M_1 \cap (L_2, e_2; M_0, M_2) +\\
&\qquad +\mathrm{crd}_{L_0}(L_0 \cap M_2) \otimes M_2 \cap (L_0, L_1; M_0, M_1) +\\
&\qquad +\mathrm{crd}_{L_2}(L_2 \cap M_0) \otimes M_2 \cap (L_2, L_1; M_0, M_1).
\end{aligned}$$

Since $M_2 \cap (L_0, L_1; M_0, M_1) = M_0 \cap (L_0, e_0; M_2, M_1)$ and $M_2 \cap (L_2, L_1; M_0, M_1) = M_1 \cap (L_2, e_2; M_0, M_2)$, this equals $\nu((L; M))$.

For the second statement we use the second description of the group: $\overline{T}_1^{ad}(2) = \mathbb{Q}\{\partial M, \ell\}/ \approx$. We have just seen that the relation (†) is taken under σ to zero. That (1.4)(0), (i), (iii)(a) are sent to zero is obvious. As for (1.4) (iii)(b),

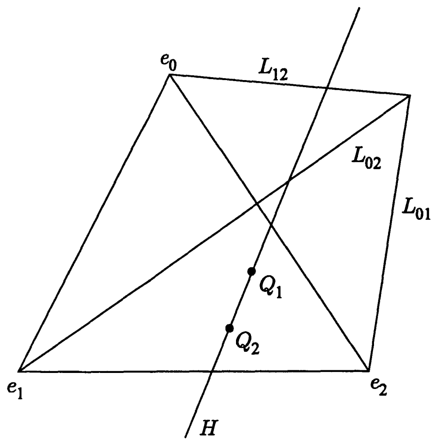

Fig. 2.6.1.

$\sum(-1)^i \nu\sigma(M_0, \cdots, \widehat{M_i}, \cdots, M_3) = 0$ follows from $\sum(-1)^i \sigma(M_0, \ldots, \widehat{M_i}, \ldots, M_3) = 0$.

PROPOSITION 2.6. *The map above* σ: $T_1^{\mathrm{ad}}(2) \to k^*_{\mathbb{Q}} \otimes k^*_{\mathbb{Q}}$ *descends to a map (also denoted* σ*)* $T_1(2) \to k^*_{\mathbb{Q}} \otimes k^*_{\mathbb{Q}}$. *The* σ*'s give a map of complexes* $T(2) \to A(2)$, *namely the following diagram is commutative:*

$$\begin{array}{ccc} T_2(2) & \xrightarrow{\partial} & T_1(2) \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ A_2 & \xrightarrow{\nu} & k^*_{\mathbb{Q}} \otimes k^*_{\mathbb{Q}} \end{array}$$

Proof. That the relation (1.4)(ii) is taken to zero under the map σ is easy to see (left to the reader). We will show that the relations (iv) are also taken to zero.

Let $(\mathcal{H}; \mathcal{Q}_0, \mathcal{Q}_1)$ be as in (1.4)(iv)(a). First we consider the case where $\mathcal{H}$ does not contain any e_k. Let $c = \mathcal{Q}_0 \cap \mathcal{Q}_1$ and pr: $\mathbb{P}^3 - \{c\} \to \mathbb{P}^2$ be the projection with center c. Since $\mathcal{H}$ is projected onto a line H and $\mathcal{Q}_i$ to a point Q_i, one obtains a configuration as in Figure 2.6.1. Here L_{ij} still denotes the image of L_{ij} under the projection. Put $R_{ij} = H \cap L_{ij}$, and take a point $v \in H \backslash \{Q_0, Q_1\}$. One has

$$\begin{aligned} &\sigma(\delta_i^*(\mathcal{H}; \mathcal{Q}_0, \mathcal{Q}_1)) \\ &\quad = \sum_{k<i} (-1)^k \mathrm{crd}_{L_{ik}}(R_{ik}) \otimes (v, R_{ik}; Q_0, Q_1) + \\ &\quad + \sum_{k>i} (-1)^{k-1} \mathrm{crd}_{L_{ik}}(R_{ik}) \otimes (v, R_{ik}; Q_0, Q_1). \end{aligned}$$

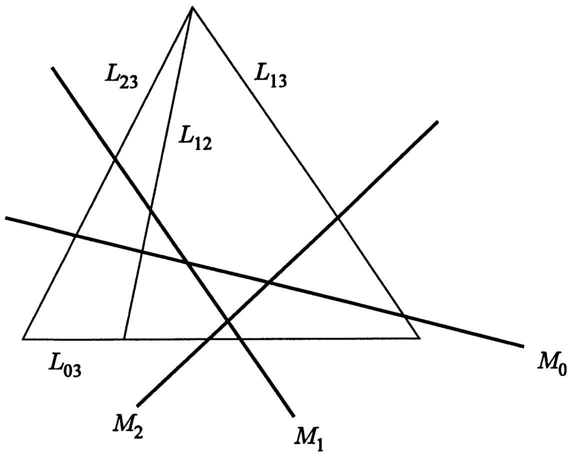

Fig. 2.7.

One hence shows $\sum_{i=0}^{3} \sigma(\delta^*(\mathcal{H}; \mathcal{Q}_0, \mathcal{Q}_1)) = 0$.

The argument for the case that $\mathcal{H}$ contains e_k is almost the same except that one should take $v = e_k$.

Let $\mathcal{M}$ be as in (1.4)(iv)(b). One has

$$\sigma\Big(\sum(-1)^i \partial(\mathcal{M} \cap L_i)\Big) = \nu\Big(\sigma(\sum(-1)^i(\mathcal{M} \cap L_i)\Big) = 0.$$

since $\sum(-1)^i(\mathcal{M} \cap L_i) = 0$ in A_2.

PROPOSITION 2.7. *The symbol map σ: $T(2) \to A(2)$ is an isomorphism of complexes.*

Proof. We first claim: If h is an automorphism of $\mathbb{P}^2$ such that $h(L_i) = L_i$ for each coordinate line L_i, then for any admissible triangle M, $h(M) = M$ in $T_2(2)$.

For the proof of this, note h is of the form $(x_0 : x_1 : x_2) \mapsto (x_0 : a_1x_1 : a_2x_2)$ where $a_1, a_2 \in k^*$. We may assume h: $(x_0 : x_1 : x_2) \mapsto (x_0 : x_1 : ax_2)$.

Let $v = (0 : 0 : 1 : -a) \in L_{01}$, $\mathcal{M}_i = v * M_i$, where $M_i \subset L_3 \cong \mathbb{P}^2$. Then $\delta_i^* M = M$ for $i = 3$; $= h(M)$ for $i = 2$; $= 0$ for $i = 0, 1$. The claim follows from (1.3)(iv).

Define the inverse map σ': $A_2 \to T_2(2)$ by $\sigma'((L; M)) = g(M)$ where g is an automorphism of $\mathbb{P}^2$ such that $g(L_i)$ is the i-th coordinate line. This is well-defined on generators by the claim above. One can show that σ' takes the relations (2.1) to the relations (1.3). The argument for the first relation of (2.1)(iii) goes as follows (the others are easily verified).

Consider projective space $\mathbb{P}^3$ with coordinate hyperplanes L_i (renewing notation). Let $(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2)$ be planes in $\mathbb{P}^3$ such that $v := \mathcal{M}_0 \cap \mathcal{M}_1 \cap \mathcal{M}_2$ is a point $\in L_0 - \cup_j L_{0j}$. Let $(M_0, M_1, M_2) := \delta_3^* \mathcal{M}$. The projection from v onto L_3 gives rise to Figure 2.7. Here L_{ij} also denotes its image. The restriction of the projection gives

$$(L_{01}, L_{12}, L_{13}; \delta_1^* \mathcal{M}) \xrightarrow{\sim} (L_{03}, L_{12}, L_{13}; M).$$

Thus $\sigma'((L_{03}, L_{12}, L_{13}; M)) = \delta_1^* \mathcal{M}$. Similarly,

$$(L_{20}, L_{21}, L_{23}; \delta_2^* \mathcal{M}) \xrightarrow{\sim} (L_{03}, L_{12}, L_{23}; M)$$

and

$$\sigma'((L_{03}, L_{12}, L_{23}; M)) = \delta_2^* \mathcal{M}.$$

Obviously, $\sigma'((L_{30}, L_{31}, L_{32}; M)) = M$ and $\delta_0^* \mathcal{M} = 0$. These imply that the relation

$$(L_{03}, L_{12}, L_{13}; M) - (L_{03}, L_{12}, L_{23}; M) + (L_{03}, L_{13}, L_{23}; M) = 0$$

is carried to $\sum \delta_i^* \mathcal{M} = 0$. In general, a relation (2.1)(iii) is a sum of relations of this type.

As a consequence of $T_2(2) \cong A_2$, note that $T_2(2)$ is generated by triangles of the type Figure 1.6.1 and D_a $(a \neq 1)$. Hence $T_1(2)$ is generated by $\ell_{a,b}$.

The inverse σ': $k_{\mathbb{Q}}^* \otimes k_{\mathbb{Q}}^* \to T_1(2)$ is given by $a \otimes b \mapsto \ell_{a,b}$. Proposition 1.8(3) implies that this map is well-defined. One obviously has $\sigma' \circ \sigma = \mathrm{id}$. To verify $\sigma \circ \sigma' = \mathrm{id}$, we have only to examine the images of line segments $\ell_{a,b}$, which is obvious.

3. The construction of the maps u, f: $G(2) \to T(2)$

In this section we give the constructions of two maps u and f: $G(2) \to T(2)$.

After recalling the definition of the Grassmannian complex, we describe the map u, motivated by the construction of figures in [7, Introduction].

We then discuss the cross ratio of $\xi \in G_1^2$ (which may be thought of a coordinate) and the torus action on G_1^2. The second map f is in terms of the coordinates and directly related to Rogers' modification [9] of the classical dilogarithm function

$$Li_2(x) - Li_2(1-x) \quad \text{where} \quad Li_2(x) = \sum \frac{x^n}{n^2}.$$

In either case, the proof of the compatibility of the map and the boundary maps will follow from facts established in Sections 1 and 2.

(3.1) For $p, q \geqslant 0$, we define the set G_q^p (*the generic part of the Grassmannian*) to be the set of codimension p linear subspaces ξ of $\mathbb{P}^{p+q}$ which are transversal to the coordinate simplex; G_q^p is naturally a variety. The inclusions δ_i: $\mathbb{P}^{p+q-1} \to \mathbb{P}^{p+q}$ (cf. (1.1)) induce the maps

$$\delta_i^*: G_q^p \to G_{q-1}^p, \quad \delta_i^*(\xi) = \xi \cap L_i,$$

and one obtains the (truncated) simplicial variety (without degeneracy maps) :

$$G_{p-1}^p \;\underset{\delta_{2p-1}^*}{\overset{\delta_0^*}{\vdots}}\; G_{p-2}^p \;\vdots\; \cdots \;\underset{\delta_{p+2}^*}{\overset{\delta_0^*}{\vdots}}\; G_1^p \;\underset{\delta_{p+1}^*}{\overset{\delta_0^*}{\vdots}}\; G_0^p$$

Let

$$G(p): \quad \mathbb{Q}G_{p-1}^p \xrightarrow{\delta^*} \cdots \xrightarrow{\delta^*} \mathbb{Q}G_1^p \xrightarrow{\delta^*} \mathbb{Q}G_0^p$$

be the associated complex of $\mathbb{Q}$-vector spaces (the degrees concentrated in $[1,p]$); here $\mathbb{Q}G_{p-1}^p$, for instance, is the $\mathbb{Q}$-vector space with G_{p-1}^p as the basis set, and $\delta^* = \sum(-1)^i\delta_i^*$. $G(p)$ is called the *Grassmannian complex* of weight p.

(3.2) We will construct a map of complexes u: $G(2) \to T(2)$.

In degree two, the map u_2: $\mathbb{Q}G_0^2 \to T_1(2)$ is to be given as follows. For a point $Q \in G_0^2$, we let $H = e_0 * Q$, $Q' = H \cap K$ (where $K := K_{02}$) and define

$$u_2([Q]) = (H; Q', Q).$$

Let us define the map in degree one u_1: $\mathbb{Q}G_1^2 \to T_2(2)$. For a point $\xi \in G_1^2$, which is a line in $\mathbb{P}^3$, take a plane $\Pi \supset \xi$ and put

$$R_i := (\delta_i^*\Pi) \cap C_2.$$

Let $Q_i = \delta_i^*\xi$, $H_i = e_0 * Q_i$, and $Q_i' = H_i \cap K$, Figure 3.2.

We define

$$u_1([\xi]) = \sum_{i=0}^{3}(-1)^i((H_i, K, e_{02} * Q_i) + (e_{02} * Q_i, C_2, \delta_i^*\Pi)) \tag{3.2.1}$$

(the ith term is the "quadrangle" $\square(Q_i'Q_iR_ie_{02})$). Note that

$$\begin{aligned}&\partial((H_i, K, e_{02} * Q_i) + (e_{02} * Q_i, C_2, \delta_i^*\Pi)) \\ &\quad = (H_i; Q_i', Q_i) + (\delta_i^*\Pi; Q_i, R_i) + (C_2; R_i, e_{02}) + (K; e_{02}, Q_i').\end{aligned}$$

CLAIM. *The right hand side of* (3.2.1) *is independent of the choice of* Π.

Proof. If one takes another $\overline{\Pi}$, (3.2.1) will differ by

$$\sum(-1)^i(\delta_i^*\Pi, \delta_i^*\overline{\Pi}, C_2).$$

This equals zero by the relation (1.3)(iv) since $C_2 = \delta_i^*C_3$.

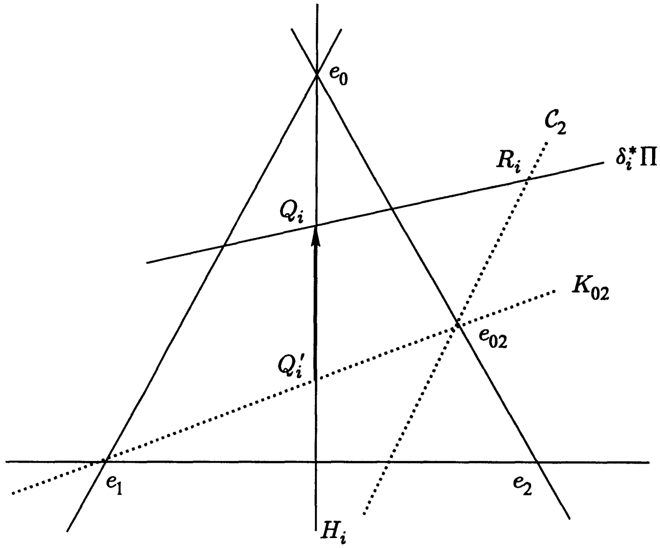

Fig. 3.2.

THEOREM 3.3. *The u's give a map of complexes* $G(2) \to T(2)$.

Proof. This follows from:

$$(H_i; Q_i', Q_i) = u_2([\delta_i^*(\xi)]),$$
$$\textstyle\sum(-1)^i(\delta_i^*\Pi; Q_i, R_i) = 0 \qquad \text{by (1.4)(iv)(a)},$$
$$(C_2; R_i, e_{02}) + (K; R_i, e_{02}) = 0 \qquad \text{by (1.8)(4)}.$$

(3.4) There is a natural bijection between the points of G_1^2 and the set

$$GL(2,k)\backslash\{(v_0, v_1, v_2, v_3) \in \overbrace{k^2 \times \cdots \times k^2}^{4} \mid v_i, v_j, i \neq j \text{ are linearly independent.}\}$$

where the group $GL(2,k)$ acts diagonally on the quadruples of vectors. The quadruple (v_0, v_1, v_2, v_3) can be regarded as a 2 by 4 matrix which gives a linear map $k^4 \to k^2$; the projectivization of the kernel of this map (which is a line in $\mathbb{P}^3$) is to be the point corresponding to (v_0, v_1, v_2, v_3).

Under the bijection above, a point $\xi \in G_1^2$ corresponds to the class of a unique matrix of the form

$$\begin{pmatrix} 1 & 0 & x & z \\ 0 & 1 & y & w \end{pmatrix}$$

where $x, y, z, w \in k^* - \{1\}, xw - yz \neq 0$.

Let the *cross ratio* of ξ be defined by

$$\iota(\xi) = \frac{yz}{xw} \in k^* - \{1\}.$$

For any $a \in k^* - \{1\}$, let $\xi_a \in G_1^2$ be the class of the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & a \end{pmatrix}$$

One has $\iota(\xi_a) = a$.

(3.5) *Tours action.*
The group $(k^*)^4$ $(= \mathbb{G}_m^4(k))$ acts on G_1^2 by

$$(k^*)^4 \times G_1^2 \to G_1^2,$$

$$(s_0, s_1, s_2, s_3).(v_0, v_1, v_2, v_3) = (s_0 v_0, s_1 v_1, s_2 v_2, s_3 v_3);$$

here $(v_0, \ldots, v_3)$ stands for the corresponding point $\in G_1^2$. Note also:

$$(s_0, s_1, s_2, s_3).\begin{pmatrix} 1 & 0 & x & z \\ 0 & 1 & y & w \end{pmatrix} = \begin{pmatrix} 1 & 0 & (s_2/s_0)x & (s_3/s_0)z \\ 0 & 1 & (s_2/s_1)y & (s_3/s_1)w \end{pmatrix}.$$

The following is easy to see:

PROPOSITION 3.6. (1) *The cross ratio $\iota(\xi)$ is invariant under the action of $(k^*)^4$.*
(2) *Each $\xi \in G_1^2$ is conjugate (under the torus action) to ξ_a where $a = \iota(\xi)$.*

(3.7) Define the map f_2: $\mathbb{Q}G_0^2 \to T_1(2)$ by

$$f_2([P]) = \ell_{a_1,a_2} - \ell_{a_2,a_1}$$

if $P = (x_0\colon x_1 : x_2)$ and $a_i = -x_i/x_0$. Also define the map f_1: $\mathbb{Q}G_1^2 \to T_2(2)$ by

$$f_1([\xi]) = -D_{\iota(\xi)} + D_{1-\iota(\xi)}.$$

THEOREM 3.8. *The f's give a map of complexes $G(2) \to T(2)$.*

PROPOSITION. *For $\xi \in G_1^2$, one has*

$$\sum_{i=0}^{3} (-1)^i f_2(\delta_i^* \xi) = \ell_{1-\iota(\xi),\iota(\xi)} - \ell_{\iota(\xi),1-\iota(\xi)}.$$

Proof. One has

$$\delta_i^*(\xi) = \begin{cases} \left(-1\colon \frac{xw-yz}{x} \frac{z}{x}\right) & i=0 \\ \left(-1\colon \frac{-xw+yz}{y} \colon -\frac{w}{y}\right) & i=1 \\ (-1\colon z : w) & i=2 \\ (-1\colon x : y) & i=3. \end{cases}$$

A direct calculation using Proposition 1.8 (3) verifies the claim.

Proof of Theorem 3.8. For a point $\xi \in G_1^2$, let $a = \iota(\xi)$. One has

$$f_2(\delta^*(\xi)) = \ell_{(1-a),a} - \ell_{a,(1-a)}$$

by the proposition. On the other hand, by (5) of this proposition

$$\partial(f_1(\xi)) = \partial(-D_a + D_{1-a}) = \ell_{(1-a),a} - \ell_{a,(1-a)}.$$

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