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Juan J. Nuño Ballesteros

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Counting connected components of the complement of the image of a codimension 1 map

JUAN J. NUÑO BALLESTEROS
Department of Mathematics, Universitat Jaume I, Castello, Spain

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1. Introduction

One of the main tools in the topological classification of continuous maps $f: X \to Y$ between manifolds when $\dim X < \dim Y$, is the study of the topology of the complement of $f(X)$ in Y. In the codimension 1 case, the simplest significant invariant is the number of its connected components.

The first theorem related to this number is the well known Jordan-Brouwer theorem, which asserts that the complement of any embedded S^n in $\mathbb{R}^{n+1}$ has exactly two connected components. This separation property has been recently generalized in many different ways.

In [4], Feighn shows that for any proper C^2 immersion $f: X^n \to Y^{n+1}$ with $H_1(Y; \mathbb{Z}_2) = 0$, the complement of $f(X)$ in Y is disconnected. Here, the C^2 hypothesis is necessary, since there is an example due to Vaccaro [10] (the "house-with-two-rooms") of a PL immersed S^2 in $\mathbb{R}^3$ whose complement is connected.

For the topological case, another approach is given in [7], where the following theorem is proved: let $f: X^n \to Y^{n+1}$ be a proper continuous map with $H_1(Y; \mathbb{Z}_2) = 0$, such that its selfintersection set, $A(f) = \{x \in X : f^{-1}f(x) \neq x\}$, is not dense in any connected component of X; then $Y \backslash f(X)$ is disconnected.

On the other hand, a result by Saeki [8] gives that, under certain restrictions, the number of connected components of $Y \backslash f(X)$ is $\geqslant m + 1$, provided that f has a normal crossing point of multiplicity m. Other results about separation properties of immersions with normal crossings are obtained in [1, 2, 3].

Here, we give a formula for the number of connected components of $Y \backslash f(X)$ in a similar setting to that of [7]: let $f: X^n \to Y^{n+1}$ be a proper continuous map between connected manifolds with $H_1(Y; \mathbb{Z}_2) = 0$, and suppose that $A = \overline{A(f)} \neq X$ and $Y \backslash f(A)$ is connected; then

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$$\beta_0(Y\backslash f(X)) = 2 + \dim_{\mathbb{Z}_2} \operatorname{coker} \lambda,$$

where β_0 denotes the number of connected components and λ is the induced map in the Alexander-Čech cohomology with compact support:

$$\lambda: \bar{H}_c^{n-1}(X; \mathbb{Z}_2) \oplus \bar{H}_c^{n-1}(f(A); \mathbb{Z}_2) \to \bar{H}_c^{n-1}(A; \mathbb{Z}_2).$$

We prove that the imposed conditions, $A \neq X$ and $Y\backslash f(A)$ connected, are generic and some computations are made in the low dimensions $n = 1, 2$ and for $n \geqslant 3$ in the case of an immersion with normal crossings having only double points. Our results are compared with similar results obtained by Izumiya and Marar in [6].

2. Proof of the main theorem

The proof of the main theorem is based on the use of the Alexander-Čech cohomology with compact support and the Alexander duality in its more general version [9]. On the other hand, the algebraic tools we shall need are the five lemma and the following consequence of the ker-coker lemma.

LEMMA 2.1. *Consider the following commutative diagram of R-modules, where the rows are exact and g is an isomorphism*

$$\begin{array}{ccccccc} & & A & \to & B & \to & C & \to 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & \\ D & \xrightarrow{\lambda} & A' & \to & B' & \to & C'. & \end{array}$$

Then, $\ker h \cong \operatorname{coker}(f + \lambda)$, *where* $f + \lambda : A \oplus D \to A'$ *is the induced map.*

Proof. We apply the ker-coker lemma to the following diagram, which is obtained in a natural way from the above

$$\begin{array}{ccccccc} & & A & \to & B & \to & C & \to 0 \\ & & \downarrow \tilde{f} & & \downarrow g & & \downarrow h & \\ 0 & \to & \operatorname{coker} \lambda & \to & B' & \to & C', & \end{array}$$

getting the exact sequence:

$$\ker \tilde{f} \to \ker g \to \ker h \to \operatorname{coker} \tilde{f} \to \operatorname{coker} g \to \operatorname{coker} h.$$

Since $\ker g = \operatorname{coker} g = 0$, we have that $\ker h \cong \operatorname{coker} \tilde{f}$ and the lemma follows from the isomorphism

$$\operatorname{coker} \tilde{f} = \frac{A'/\operatorname{Im} \lambda}{(\operatorname{Im} f + \operatorname{Im} \lambda)/\operatorname{Im} \lambda} \cong \frac{A'}{\operatorname{Im} f + \operatorname{Im} \lambda} = \operatorname{coker}(f + \lambda). \qquad \square$$

THEOREM 2.2. *Let $f: X^n \to Y^{n+1}$ be a proper continuous map between connected manifolds with $H_1(Y;\mathbb{Z}_2) = 0$ and let A be the closure of the selfintersection set $A(f) = \{x \in X : f^{-1}f(x) \neq x\}$. Suppose that $A \neq X$ and $Y \backslash f(A)$ is connected. Then*

$$\beta_0(Y \backslash f(X)) = 2 + \dim_{\mathbb{Z}_2} \operatorname{coker}(i^* + f|_A^*),$$

where

$$i^* + f|_A^* : \bar{H}_c^{n-1}(X;\mathbb{Z}_2) \oplus \bar{H}_c^{n-1}(f(A);\mathbb{Z}_2) \to \bar{H}_c^{n-1}(A;\mathbb{Z}_2)$$

is the induced map.

Proof. To simplify the notation, we shall omit the coefficient group $\mathbb{Z}_2$ in all the homology and cohomology groups.

Since f is proper (and hence closed), $f(A)$, $f(X)$ are closed and we can consider the Alexander-Čech cohomology of the pairs (X, A), $(f(X), f(A))$ and get the following commutative diagram, where the rows are exact:

$$\begin{array}{ccccccc}
\to \bar{H}_c^{n-1}(f(X)) & \to & \bar{H}_c^{n-1}(f(A)) & \to & \bar{H}_c^{n}(f(X), f(A)) & \to \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (3)} & \\
\to \bar{H}_c^{n-1}(X) & \to & \bar{H}_c^{n-1}(A) & \to & \bar{H}_c^{n}(X, A) & \to \\
\to \bar{H}_c^{n}(f(X)) & \to & \bar{H}_c^{n}(f(A)) & \to & \bar{H}_c^{n+1}(f(X), f(A)) & \to \\
\downarrow{\scriptstyle (4)} & & \downarrow{\scriptstyle (5)} & & \downarrow{\scriptstyle (6)} & \\
\to \bar{H}_c^{n}(X) & \to & \bar{H}_c^{n}(A) & \to & \bar{H}_c^{n+1}(X, A) & \to
\end{array}$$

But some of these cohomology groups are computed using the Alexander duality:

$$\bar{H}_c^n(X) \cong H_0(X) \cong \mathbb{Z}_2,$$

$$\bar{H}_c^n(f(X)) \cong H_1(Y, Y \backslash f(X)) \cong \tilde{H}_0(Y \backslash f(X)),$$

where the last isomorphism comes from the exact sequence of the pair $(Y, Y \backslash f(X))$:

$$0 = H_1(Y) \to H_1(Y, Y \backslash f(X)) \to \tilde{H}_0(Y \backslash f(X)) \to \tilde{H}_0(Y) = 0.$$

This gives a formula for the number of connected components of $Y \backslash f(X)$:

$$\beta_0(Y \backslash f(X)) = 1 + \dim_{\mathbb{Z}_2} \bar{H}^n_c(f(X)).$$

We apply also the Alexander duality to A and $f(A)$:

$$\bar{H}^n_c(A) \cong H_0(X, X \backslash A) = 0,$$
$$\bar{H}^n_c(f(A)) \cong H_1(Y, Y \backslash f(A)) = 0,$$

where the last equality comes from the exact sequence of the pair $(Y, Y \backslash f(A))$:

$$0 = H_1(Y) \to H_1(Y, Y \backslash f(A)) \to \tilde{H}_0(Y \backslash f(A)) = 0.$$

On the other hand, note that the maps (3) and (6) in the above diagram are isomorphisms. In fact, in the following commutative diagram

$$\begin{array}{ccc} \bar{H}^*_c(f(X), f(A)) & \to & \bar{H}^*_c(f(X) \backslash f(A)) \\ \downarrow & & \downarrow \\ \bar{H}^*_c(X, A) & \to & \bar{H}^*_c(X \backslash A), \end{array}$$

the horizontal arrows are isomorphisms in general (see [9]), and in this case the map on the right is also an isomorphism because it comes from the homeomorphism

$$f|_{X \backslash A} : X \backslash A \to f(X \backslash A) = f(X) \backslash f(A).$$

Then, we can apply the five lemma to the maps (2), ..., (6) and deduce that $f^* : \bar{H}^n_c(f(X)) \to \bar{H}^n_c(X)$ is an epimorphism. Therefore

$$\dim_{\mathbb{Z}_2} \bar{H}^n_c(f(X)) = 1 + \dim_{\mathbb{Z}_2} \ker(f^*).$$

But the above lemma implies that $\ker(f^*) \cong \operatorname{coker}(i^* + f|_A^*)$, where

$$i^* + f|_A^* : \bar{H}^{n-1}_c(X) \oplus \bar{H}^{n-1}_c(f(A)) \to \bar{H}^{n-1}_c(A)$$

is the induced map. □

COROLLARY 2.3. *In the conditions of the above theorem, the number of connected components of the complement of the image only depends on the behavior of the map on the selfintersection set. That is, let $f, g : X \to Y$ be* *maps satisfying all the hypothesis of Theorem* 2.2 *and suppose that* $A(f) = A(g)$ *and* $f = g$ *on this set*; *then we have that*

$$\beta_0(Y\backslash f(X)) = \beta_0(Y\backslash g(X)).$$

REMARKS. (1) An interesting special case of Theorem 2.2 is when $H_1(X;\mathbb{Z}_2) = 0$ (e.g., $X = S^n$, with $n \geqslant 2$). Then the Alexander duality gives $\bar{H}_c^{n-1}(X;\mathbb{Z}_2) = 0$ and the statement of the conclusion of the theorem is:

$$\beta_0(Y\backslash f(X)) = 2 + \dim_{\mathbb{Z}_2} \operatorname{coker}(f|_A^*),$$

where

$$f|_A^* : \bar{H}_c^{n-1}(f(A);\ \mathbb{Z}_2) \to \bar{H}_c^{n-1}(A;\mathbb{Z}_2)$$

is the induced map.

(2) If the manifold X is compact, A and $f(A)$ are also compact, in which case $\bar{H}_c^*(U;\mathbb{Z}_2) = \bar{H}^*(U;\mathbb{Z}_2)$ for $U = A, f(A), X$. Moreover, if A and $f(A)$ are taut (for instance, they are ANR [9]), we have $\bar{H}^*(U;\mathbb{Z}_2) = H^*(U;\mathbb{Z}_2)$ for $U = A, f(A), X$. Therefore, in this case it is enough to study the coker of the map

$$i^* + f|_A^* : H^{n-1}(X;\ \mathbb{Z}_2) \oplus H^{n-1}(f(A);\ \mathbb{Z}_2) \to H^{n-1}(A;\ \mathbb{Z}_2).$$

But since $\mathbb{Z}_2$ is a field, we can identify the cohomology groups with the dual vector spaces of the corresponding homology groups (by the universal coefficient theorem) and the above map with the dual of the induced map in homology:

$$(i_*, (f|_A)_*) : H_{n-1}(A;\ \mathbb{Z}_2) \to H_{n-1}(X;\ \mathbb{Z}_2) \oplus H_{n-1}(f(A);\ \mathbb{Z}_2),$$

which implies

$$\dim_{\mathbb{Z}_2} \operatorname{coker}(i^* + f|_A^*) = \dim_{\mathbb{Z}_2} \ker(i_*, (f|_A)_*) = \dim_{\mathbb{Z}_2} \ker(i_*) \cap \ker((f|_A)_*).$$

(3) Finally, note that if X and Y are orientable manifolds, then in Theorem 2.2 we can use any field for coefficients of homology and cohomology (e.g., the field of rational numbers). In some cases this would lead to sharper estimates than the use of $\mathbb{Z}_2$ coefficients.

EXAMPLES. We give now some examples in order to show that all the hypothesis in Theorem 2.2 are necessary. The complement of any S^1

embedded into $S^1 \times S^1$ as a meridian or a parallel is connected, which shows that the condition $H_1(Y; \mathbb{Z}_2) = 0$ is essential. The same happens with the condition $A \neq X$: if $n \geqslant 1$, for any constant map $f: X \to Y$ the complement of its image is connected. The following example is constructed to show that the condition that $Y \backslash f(A)$ is connected is essential too.

Let $f: I = [0, 1] \to S^1$ be the immersion given by $f(t) = \exp(4\pi i t)$ and consider the composition of the product $1 \times f: S^1 \times I \to S^1 \times S^1$ with the standard embedding $S^1 \times S^1 \to \mathbb{R}^3$. Then we attach an embedded 2-disk at each boundary curve of the cylinder $S^1 \times I$ as in Fig. 1. The result is an immersion $g: X \to \mathbb{R}^3$, where X is homeomorphic to S^2, and such that $\beta_0(\mathbb{R}^3 \backslash g(X)) = 3$. On the other hand, the selfintersection set A is the cylinder $S^1 \times I$, its image $g(A)$ is the embedded torus in $\mathbb{R}^3$ and the induced map in homology $(g|_A)_*: H_1(S^1 \times I; \mathbb{Z}_2) \to H_1(S^1 \times S^1; \mathbb{Z}_2)$ is injective, which implies that $\ker(i_*) \cap \ker((g|_A)_*) = 0$ and the formula is not true in this case.

3. Genericity of the conditions

In this section we prove that the conditions imposed to f in Theorem 2.2 ($A \neq X$ and $Y \backslash f(A)$ connected) are generic. That is, if X, Y are smooth, they are satisfied for a residual subset in the set of proper C^∞ maps, $\mathrm{Prop}^\infty(X, Y)$, with the Whitney C^∞-topology. We first recall the concept of topological dimension.

DEFINITION 3.1. A topological space X is said to have covering dimension $\leqslant n$, abbreviated $\dim X \leqslant n$, if every open covering of X has an open refinement $\mathscr{W} = \{V_i\}$ such that for any $V_{i_1}, \ldots, V_{i_{n+1}} \in \mathscr{W}$ we have

$$V_{i_1} \cap \cdots \cap V_{i_{n+1}} = \varnothing.$$

We say that X has covering dimension n, denoted by $\dim X = n$, if $\dim X \leqslant n$ but $\dim X \nleqslant n - 1$.

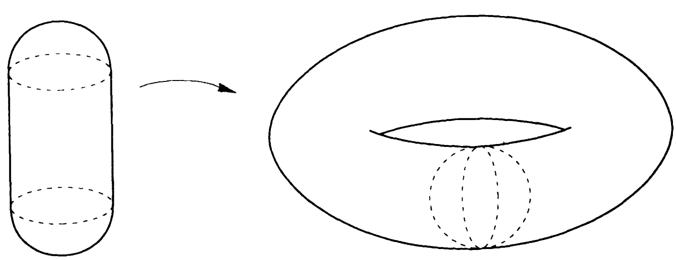

Fig. 1.

We shall use the following property of the covering dimension related to the Alexander-Čech cohomology with compact support [9]: if the space X is locally compact, Haussdorff and $\dim X \leqslant n$, then $\bar{H}_c^q(X;G) = 0$, $\forall q > n$ and for all G. Then, the genericity follows from the two following results.

LEMMA 3.2. *Let $f: X^n \to Y^{n+1}$ be a proper continuous map between connected manifolds with $H_1(Y;\mathbb{Z}_2) = 0$; and let A be the closure of the selfintersection set $A(f) = \{x \in X : f^{-1}f(x) \neq x\}$. Suppose that* $\dim A$, $\dim f(A) \leqslant n-1$. *Then $A \neq X$ and $Y \backslash f(A)$ is connected.*

Proof. Since $\dim X = n$, the first part is obvious. For the second one, we use again the Alexander duality:

$$\tilde{H}_0(Y \backslash f(A)) \cong H_1(Y, Y \backslash f(A)) \cong \bar{H}_c^n(f(A)) = 0,$$

which gives that $Y \backslash f(A)$ is connected. □

LEMMA 3.3. *Let X^n, Y^{n+1} be smooth manifold. Then there is a residual subset of maps $f \in \mathrm{Prop}^\infty(X, Y)$ with the Whitney C^∞-topology, for which* $\dim A$, $\dim f(A) \leqslant n-1$, *where A is the closure of the selfintersection set $A(f)$.*

Proof. The set of Boardman maps satisfying the condition NC is residual in $C^\infty(X, Y)$ with the Whitney C^∞-topology (see [5]). Therefore, its intersection with $\mathrm{Prop}^\infty(X, Y)$ will be residual here. We see that if f is in this set, then $\dim A$, $\dim f(A) \leqslant n-1$.

Since f is proper, we have $A \subset A(f) \cup \Sigma(f)$, where $\Sigma(f)$ is the singular set of f. Moreover, f admits a Whitney regular stratification so that A, $A(f)$, $\Sigma(f)$ and their images by f are union of strata. In this situation, it is enough to show that $A(f)$, $\Sigma(f)$ have dimension $\leqslant n-1$.

Any Boardman maps satisfying the condition NC is in particular a map with normal crossings. This implies that $f^{(2)} \pitchfork \Delta Y$, where $f^{(2)}$ is the restriction of $f^2 : X \times X \to Y \times Y$ to $X^{(2)} = X \times X \backslash \Delta X$. Thus, $B = (f^{(2)})^{-1}(\Delta Y)$ is a submanifold of $X^{(2)}$ of codimension $n+1$, $A(f)$ is the image of the projection

$$\begin{aligned} \pi_1|_B : B &\to X \\ (x, y) &\mapsto x, \end{aligned}$$

and we have $\dim A(f) \leqslant \dim B = n-1$.

For the singular set, note that any Boardman map is also 1-generic, which means that $j^1 f \pitchfork S_r$, for any $r = 0, \dots, n$, where $S_r = \{\sigma \in J^1(X, Y) : \text{corank of } \sigma = r\}$.

In particular, each set $S_r(f) = (j^1 f)^{-1}(S_r)$ is a submanifold of X of codimension $r(r+1)$ and thus $\Sigma(f) = \bigcup_{r=1}^{n} S_r(f)$ has dimension $\leqslant n-1$. $\square$

4. Some computations in special cases

In this section we study generic maps $f: X^n \to Y^{n+1}$, with X compact and $H_1(Y; \mathbb{Z}_2) = 0$, and discuss separately the cases $n = 1$, $n = 2$ and $n \geqslant 3$. The results obtained in each case will be more restrictive than the preceding ones.

A. *The case* $n = 1$. In this case, we must have $X = S^1$ and $Y = \mathbb{R}^2$ or S^2. Since we can embedd $\mathbb{R}^2$ into S^2 through the stereographic projection so that $f(S^1)$ has the same number of connected components in the complement, it is enough to consider the case $Y = S^2$. To guarantee that $A \neq X$ and $Y \backslash f(A)$ is connected, we put the condition that A is finite.

THEOREM 4.1. *Let $f: S^1 \to S^2$ be a continuous map with a finite number of selfintersections $t_1, \ldots, t_m \in S^1$. If $\#\{f(t_1), \ldots, f(t_m)\} = r$, we have*

$$\beta_0(S^2 \backslash f(S^1)) = 2 + m - r.$$

Proof. We must study the kernels of the maps $i_*: H_0(A; \mathbb{Z}_2) \to H_0(S^1; \mathbb{Z}_2)$ and $(f|_A)_*: H_0(A; \mathbb{Z}_2) \to H_0(f(A); \mathbb{Z}_2)$, and compute the dimension of their intersection. But obviously $\ker(f|_A)_* \subset \ker i_*$ and $\dim_{\mathbb{Z}_2} \ker(f|_A)_* = m - r$. $\square$

B. *The case* $n = 2$. Now, we consider topologically stable maps $f: X^2 \to Y^3$, with X compact and $H_1(Y; \mathbb{Z}_2) = 0$. In this case, A and $f(A)$ have a graph structure, the vertices being the triple points and the cross caps, and the edges corresponding to the double points of f. Moreover, the incidence rules in these graphs are as follows: in A, four edges are incident with each triple point and the number of edges incident with a cross cap is equal to two; in $f(A)$, we have six edges incident with each triple point, but only one edge is incident with a cross cap (see Fig. 2).

THEOREM 4.2. *Let $f: X^2 \to Y^3$ be a topologically stable map with X compact and $H_1(Y; \mathbb{Z}_2) = 0$, and let A be the closure of the selfintersection set $A(f) = \{x \in X : f^{-1}f(x) \neq x\}$. Then*

$$\beta_0(Y \backslash f(X)) \leqslant 2 + \beta_0(A) + 3T(f),$$

$$\beta_0(Y \backslash f(X)) \geqslant 2 + \beta_0(A) - \beta_0(f(A)) + T(f) + \tfrac{1}{2}C(f) - \beta_1(X),$$

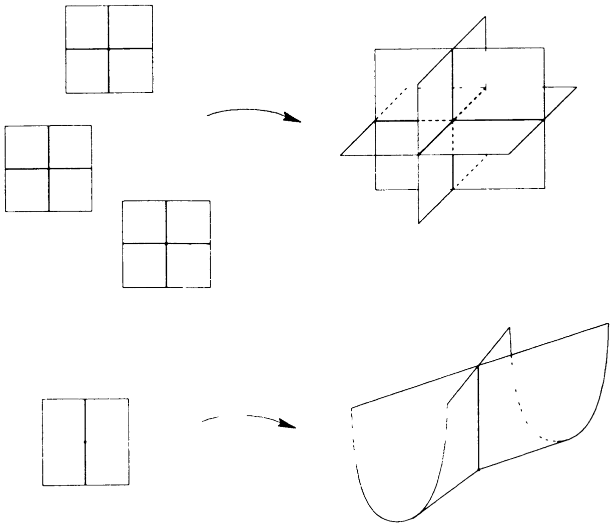

Fig. 2. Graph structure at a triple point and a cross cap.

where $T(f)$ is the number of triple points, $C(f)$ is the number of cross caps and $\beta_1(X)$ denotes the $\mathbb{Z}_2$-dimension of $H_1(X;\mathbb{Z}_2)$.

Proof. If we consider the map

$$(i_*, (f|_A)_*): H_1(A;\ \mathbb{Z}_2) \to H_1(X;\mathbb{Z}_2) \oplus H_1(f(A);\ \mathbb{Z}_2),$$

we have that

$$\begin{aligned}\dim_{\mathbb{Z}_2} \ker(i_*, (f|_A)_*) \geqslant \dim_{\mathbb{Z}_2} H_1(A;\ \mathbb{Z}_2) - \dim_{\mathbb{Z}_2} H_1(X;\ \mathbb{Z}_2)\\ - \dim_{\mathbb{Z}_2} H_1(f(A);\ \mathbb{Z}_2)\\ = \beta_1(A) - \beta_1(f(A)) - \beta_1(X),\end{aligned}$$

and

$$\dim_{\mathbb{Z}_2} \ker(i_*, (f|_A)_*) \leqslant \beta_1(A).$$

But the Euler characteristic of the graph A, $\chi(A) = \beta_0(A) - \beta_1(A)$, is also computed by the formula

$$\begin{aligned}\chi(A) &= \#\text{vertices} - \#\text{edges}\\ &= 3T(f) + C(f) - \tfrac{1}{2}(12T(f) + 2C(f)) = -3T(f),\end{aligned}$$

which gives $\beta_1(A) = \beta_0(A) + 3T(f)$.

Analogous computations in $f(A)$ give that

$$\beta_1(f(A)) = \beta_0(f(A)) + 2T(f) - \tfrac{1}{2}C(f),$$

and this concludes the proof.

$\square$

In [6], Izumiya and Marar get better upper bounds in a more restrictive case. In particular, they prove that if $C(f) = T(f) = 0$, then

$$\beta_0(Y \backslash f(X)) \leqslant 2 + \beta_0(f(A)),$$

and the equality is true provided that A is homologous to zero in X. We generalize this to higher dimensions in the next case.

C. *The case* $n \geqslant 3$. In general, it would be very ambitious to obtain some nice result if we consider a topologically stable map $f: X^n \to Y^{n+1}$, with X compact and $H_1(Y; \mathbb{Z}_2) = 0$, because we do not have a complete description of the local behavior of the map. Even in the case of an immersion with normal crossings, we have not got at the moment any interesting result. We just consider the case of an immersion with normal crossings having only double points, which is the generalization of the situation in the Izumiya-Marar formula. The following result has been obtained independently by Biasi, Motta and Saeki in [2].

THEOREM 4.3. *Let $f: X^n \to Y^{n+1}$ be an immersion with normal crossings with X compact and $H_1(Y; \mathbb{Z}_2) = 0$. Suppose that f has only simple or double points and let $A \subset X$ be the subset of double points. Then*

$$\beta_0(Y \backslash f(X)) \leqslant 2 + \beta_0(f(A)),$$

and the equality holds if $H_1(X; \mathbb{Z}_2) = 0$.

Proof. It is enough to prove that the kernel of $(f|_A)_*: H_{n-1}(A; \mathbb{Z}_2) \to H_{n-1}(f(A); \mathbb{Z}_2)$ has $\mathbb{Z}_2$-dimension $\beta_0(f(A))$. But this map is the direct sum of the maps $(f|_{f^{-1}(C)})_*: H_{n-1}(f^{-1}(C); \mathbb{Z}_2) \to H_{n-1}(C; \mathbb{Z}_2)$, where C are the connected components of $f(A)$. We prove that the kernel of each of these maps has $\mathbb{Z}_2$-dimension 1.

Note that $f^{-1}(C)$, C are compact $(n-1)$-manifolds and the map $f|_{f^{-1}(C)}: f^{-1}(C) \to C$ is a double covering. By the path lifting property of the covering maps, $f^{-1}(C)$ only can have one or two connected components. If it has two connected components, D_1, D_2, the composition

$$H_{n-1}(D_i; \mathbb{Z}_2) \to H_{n-1}(f^{-1}(C); \mathbb{Z}_2) \to H_{n-1}(C; \mathbb{Z}_2)$$

is an isomorphism, which implies that $(f|_{f^{-1}(C)})_*$ is an epimorphism and the kernel is 1-dimensional.

On the other hand, if $f^{-1}(C)$ is connected, we prove that the map $(f|_{f^{-1}(C)})_*$ is zero and the result is also true. Let $D = f^{-1}(C)$ and consider $x \in C$ and $f^{-1}(x) = \{y_1, y_2\} \subset D$. Then we have a commutative diagram

$$\begin{array}{ccc} H_{n-1}(D;\ \mathbb{Z}_2) & \xrightarrow{\alpha} & H_{n-1}(D,\ D\backslash\{y_1;\ y_2\};\ \mathbb{Z}_2) \\ {\scriptstyle (f|_D)_*}\downarrow & & \downarrow{\scriptstyle \beta} \\ H_{n-1}(C;\mathbb{Z}_2) & \xrightarrow{\cong} & H_{n-1}(C,\ C\backslash\{x\};\mathbb{Z}_2), \end{array}$$

where the map in the bottom is an isomorphism, for C is $\mathbb{Z}_2$-orientable. By using the excision property, it is easy to see that the composition $\beta \circ \alpha$ is zero and thus $(f|_D)_*$ must be zero. □

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