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ALEXANDRU DIMCA

MORIIHIKO SAITO

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On the cohomology of a general fiber of a polynomial map

ALEXANDRU DIMCA¹ and MORIHIKO SAITO²

¹*School of Mathematics, The University of Sydney, Sydney NSW 2006, Australia;* ²*RIMS Kyoto University, Kyoto 606 Japan*

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Introduction

Let $X = \mathbf{C}^n$, $S = \mathbf{C}$, and $f: X \rightarrow S$ a map defined by a polynomial which is also denoted by f . Then f induces a topological fibration over a Zariski-open subset of S . It is interesting whether we can compute algebraically the cohomology of a generic fiber $F = f^{-1}(t)$ using the polynomial ring $\mathcal{O} := \mathbf{C}[x_1, \dots, x_n]$ and the polynomial f . Of course, we can calculate the cohomology using the de Rham cohomology of the (scheme theoretic) generic fiber of f , but it is not quite computable.

In the weighted homogeneous case, the answer was given by [3]. Let Ω denote the complex of global algebraic differential forms on X (i.e., Ω^p is a free \mathcal{O} -module with a basis $dx_{i_1} \wedge \dots \wedge dx_{i_p}$ ($i_1 < \dots < i_p$)). Define a differential D_f on Ω by

$$D_f(\omega) = d\omega - df \wedge \omega \quad \text{for } \omega \in \Omega^p. \quad (0.1)$$

Then we have an isomorphism

$$H^{k+1}(\Omega, D_f) \cong \tilde{H}^k(F, \mathbf{C}) \quad \text{for any } k, \quad (0.2)$$

if f is weighted homogeneous, where \tilde{H} denotes reduced cohomology. (In [loc. cit.], D_f was denoted by \bar{D}_f . See also (2.10) below.) In this paper, we prove

(0.3) **THEOREM.** *The isomorphism (0.2) holds for any polynomial f .*

The proof uses the theory of algebraic Gauss–Manin system which is a generalization of [4] (see, for example, [1]), and also the theory of monodromic algebraic \mathcal{D} -modules. Let $\int_f \mathcal{O}_X$ denote the algebraic Gauss–Manin system, which is defined by the direct image of \mathcal{O}_X by f as algebraic \mathcal{D} -module. Let t be the coordinate of S , and $\partial_t = \partial/\partial t$. Then we have a natural quasi-isomorphism

(see (2.7)):

$$R\Gamma\left(S, \text{Cone}\left(\partial_t - \text{id}: \int_f \mathcal{O}_X \rightarrow \int_f \mathcal{O}_X\right)\right) \xrightarrow{\sim} (\Omega^*, D_f)[n]. \tag{0.4}$$

Here $\partial_t - \text{id}$ is *analytically* equivalent to ∂_t (because $\partial_t - 1 = e^t \partial_t e^{-t}$ in $\mathcal{D}_{S^{\text{an}}}$). Let $\int_f^p \mathcal{O}_X$ denote the p th cohomology of $\int_f \mathcal{O}_X$. Its restriction to a Zariski open subset of S is a vector bundle (i.e., a locally free sheaf) whose fiber is isomorphic to the cohomology of the fiber of f (see (2.3)). We take the direct image of $\int_f^p \mathcal{O}_X$ by the compactification $S \rightarrow \mathbf{P}^1$, and compute its *analytic* local cohomology at infinity (see (2.8)). Then we get the assertion using the theory of monodromical \mathcal{D} -modules (see (2.9)).

It should be noted that Theorem (0.3) is essentially *of algebraic nature*, and the local analytic version of (0.3) does not hold. For example, $(\Omega_{X^{\text{an}}}, D_f)[1]$ is not quasi-isomorphic to Deligne’s vanishing cycle sheaf, because D_f is *analytically* equivalent to the natural differential d using e^f .

1. Monodromical D -modules of one variable

In this section, we gather some elementary facts from the theory of monodromical algebraic \mathcal{D} -modules of one variable, which should be well known to specialists.

(1.1) Let S denote the affine line \mathbf{C} with coordinate t (i.e., $S = \text{Spec } \mathbf{C}[t]$). Let $S^* = S \setminus \{0\}$ with a natural inclusion $j: S^* \rightarrow S$. Let \mathcal{D}_S be the sheaf of algebraic differential operators on S [1], [5]. We denote by R the global sections of \mathcal{D}_S , which is the Weyl algebra $\mathbf{C}[t, \partial_t]$. Let $M_{\text{coh}}(\mathcal{D}_S)$ be the category of coherent \mathcal{D}_S -modules, and $M_{\text{fin}}(R)$ the category of finite R -modules. We have an equivalence of categories

$$M_{\text{coh}}(\mathcal{D}_S) = M_{\text{fin}}(R) \tag{1.1.1}$$

by the global section functor $\Gamma(S, *)$.

Let S^{an} denote the underlying complex analytic space of S . We have a functor

$$\text{An}: M_{\text{coh}}(\mathcal{D}_S) \rightarrow M_{\text{coh}}(\mathcal{D}_{S^{\text{an}}}) \tag{1.1.2}$$

by $M \rightarrow M^{\text{an}} := \mathcal{O}_{S^{\text{an}}} \otimes_{\mathcal{O}_S} M$, where the pull-back by the natural morphism $S^{\text{an}} \rightarrow S$ is omitted. Then the de Rham functor DR_S is given by

$$\text{DR}_S(M) = \text{Cone}(\partial_t: M^{\text{an}} \rightarrow M^{\text{an}}) \tag{1.1.3}$$

using the coordinate t to trivialize Ω_S^1 (see (2.1.2) below).

(1.2) For $M \in M_{\text{coh}}(\mathcal{D}_S)$, let $M(S) = \Gamma(S, M)$, and

$$M(S)^\alpha = \bigcup_{i \geq 0} \text{Ker}((t\partial_t - \alpha)^i: M(S) \rightarrow M(S)) \quad \text{for } \alpha \in \mathbf{C}. \tag{1.2.1}$$

Then

$$tM(S)^\alpha \subset M(S)^{\alpha+1}, \quad \partial_t M(S)^\alpha \subset M(S)^{\alpha-1}, \tag{1.2.2}$$

and we have isomorphisms

$$t: M(S)^{\alpha-1} \xrightarrow{\sim} M(S)^\alpha, \quad \partial_t: M(S)^\alpha \xrightarrow{\sim} M(S)^{\alpha-1} \quad \text{for } \alpha \neq 0. \tag{1.2.3}$$

In fact, $t\partial_t$ is bijective on $M(S)^\alpha$ for $\alpha \neq 0$, because $t\partial_t = \alpha$ on $\text{Gr}_i^K M(S)^\alpha$ with $K_i M(S)^\alpha = \text{Ker}(t\partial_t - \alpha)^{i+1}$ (similarly for $\partial_t t$).

(1.3) DEFINITION. We say that $M \in M_{\text{coh}}(\mathcal{D}_S)$ is *monodromical* if M is generated by $M(S)^\alpha$ ($\alpha \in \mathbf{C}$) over \mathcal{D}_S . Let $M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$ denote the full subcategory of $M_{\text{coh}}(\mathcal{D}_S)$ consisting of monodromical \mathcal{D}_S -modules. Then $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$ is called *meromorphic* (resp. *microlocal*) *type* if the action of t (resp. ∂_t) on $M(S)$ is bijective.

REMARK. The condition of monodromical \mathcal{D}_S -module is equivalent to that any element of $M(S)$ is annihilated by a polynomial of $t\partial_t$. So it is stable by extensions in $M_{\text{coh}}(\mathcal{D}_S)$.

(1.4) LEMMA. For $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$, we have a natural isomorphism

$$\bigoplus_{\alpha \in \mathbf{C}} M(S)^\alpha \xrightarrow{\sim} M(S), \tag{1.4.1}$$

and $M(S)^\alpha$ is finite dimensional over \mathbf{C} . In particular, the functor $M \rightarrow M(S)^\alpha$ is exact.

Proof. The injectivity of (1.4.1) is clear using the action of $t\partial_t$ on $M(S)$. Since the condition of monodromical \mathcal{D}_S -module is equivalent to the surjectivity of

$$\bigoplus_{\alpha \in \mathbf{C}} \mathcal{D}_S \otimes_{\mathbf{C}} M(S)^\alpha \rightarrow M, \tag{1.4.2}$$

the surjectivity of (1.4.1) follows from (1.2.3), taking the global section of (1.4.2). We have $\dim_{\mathbf{C}} M(S)^\alpha < \infty$, because $M(S)^\alpha$ is finitely generated over $\mathbf{C}[N]$ with $N = -(t\partial_t - \alpha)$.

REMARK. By (1.2.3) and (1.4.1), $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$ is meromorphic (resp.

microlocal) type if and only if

$$t: M(S)^{-1} \rightarrow M(S)^0 \text{ (resp. } \partial_t: M(S)^0 \rightarrow M(S)^{-1} \text{)} \tag{1.4.3}$$

is bijective.

(1.5) LEMMA. *Let $M \in M_{\text{coh}}(\mathcal{D}_S)$ such that $\text{supp } M \subset \{0\}$. Then M is monodromical, and $M(S)^\alpha = 0$ except for negative integers α .*

Proof. The assumption is equivalent to that any element of $M(S)$ is annihilated by a sufficiently high power of t . Then we can check the assertion using $\partial_t^i t^i = \prod_{0 < j \leq i} (t\partial_t + j)$.

REMARK. For M as above, M is a finite direct sum of \mathcal{B} in the proof of (1.8) by (1.2.3). This is a special case of Kashiwara’s equivalence (see [1]).

(1.6) LEMMA. *Let Λ be a subset of \mathbf{C} such that $0 \in \Lambda$ and the natural morphism $\Lambda \rightarrow \mathbf{C}/\mathbf{Z}$ is bijective. Let $\Lambda' = \Lambda \cup \{-1\}$. Let \mathcal{C} be the category whose object is a family of \mathbf{C} -vector spaces $V^\alpha (\alpha \in \Lambda')$ with morphisms $u: V^0 \rightarrow V^{-1}$, $v: V^{-1} \rightarrow V^0$, and $N: V^\alpha \rightarrow V^\alpha (\alpha \in \Lambda \setminus \{0\})$ such that $\bigoplus_{\alpha \in \Lambda'} V^\alpha$ is finite dimensional, and vu , uv and N are nilpotent. Then we have an equivalence of categories*

$$M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}} = \mathcal{C} \tag{1.6.1}$$

by associating $M(S)^\alpha$, ∂_t , t and $t\partial_t - \alpha$ to $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$.

Proof. This follows from (1.2.2–3) and (1.4.1).

(1.7) COROLLARY. *We have an equivalence of categories*

$$M_{\text{coh}}(\mathcal{D}_S)_{\text{mon},*} \text{ (resp. } M_{\text{coh}}(\mathcal{D}_S)_{\text{mon},!} \text{)} = V(\mathbf{C}, T), \tag{1.7.1}$$

where the left-hand side is the category of monodromical \mathcal{D}_S -modules of meromorphic (resp. microlocal) type, the right-hand side is the category of finite dimensional \mathbf{C} -vector spaces with an automorphism T , and the functor is defined by $M \rightarrow \bigoplus_{\alpha \in \Lambda} M(S)^\alpha$ with $T = \exp(-2\pi i t \partial_t)$.

REMARK. Using (1.6), we can show that the category $M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$ is equivalent to the category of regular holonomic $\mathcal{D}_{S^{\text{an}},0}$ -modules (for which an equivalence of categories similar to (1.6.1) holds). The terms ‘meromorphic’ and ‘microlocal’ are originally used in this case (see [8]).

(1.8) PROPOSITION. *Let $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$. Then M is regular holonomic [1].*

Proof. Since the action of $t\partial_t - \alpha$ on $M(S)^\alpha$ is nilpotent, we may assume $\sum_{\alpha \in \Lambda} \dim M(S)^\alpha = 1$ by (1.6), taking the graduation of a finite filtration on M (because regular holonomic \mathcal{D} -modules are stable by extensions [loc. cit.]). Then we can check that M is isomorphic to one of the following:

- (i) $\mathcal{O}_S = \mathcal{D}_S/\mathcal{D}_S\partial_t$,
- (ii) $\mathcal{B} := \mathcal{D}_S/\mathcal{D}_St$,
- (iii) $M(\alpha) := \mathcal{D}_S/\mathcal{D}_S(t\partial_t - \alpha)$ ($\alpha \in \Lambda \setminus \{0\}$),

depending on the α such that $M(S)^\alpha \neq 0$. So we get the assertion.

REMARK. We can show that a regular holonomic \mathcal{D}_S -module is monodromical, if and only if its restriction to S^* is finite over \mathcal{O}_{S^*} (i.e., a vector bundle with connection [2]). In fact, we may assume that $M|_{S^*}$ is a vector bundle by (1.10) below. Then the assertion is reduced to case where the action of t on M is bijective using the localization of M by t (because $M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$ is stable by extensions in $M_{\text{coh}}(\mathcal{D}_S)$, see Remark after (1.3)). Then the assertion follows [2] (see also (1.11) below).

(1.9) **PROPOSITION.** *For $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$, there exists uniquely $M' \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$ of meromorphic (resp. microlocal) type with a morphism $M \rightarrow M'$ (resp. $M' \rightarrow M$) inducing an isomorphism on S^* .*

Proof. By (1.6) there exists uniquely $M' \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$ with a morphism $M \rightarrow M'$ (resp. $M' \rightarrow M$), such that

$$\begin{aligned} M'(S)^\alpha &= M(S)^\alpha \quad \text{for } -\alpha \notin \mathbf{N} \setminus \{0\}, \\ \partial_t^i: M(S)^0 &\xrightarrow{\sim} M(S)^{-i} \quad (\text{resp. } t^i: M(S)^{-i} \xrightarrow{\sim} M(S)^0) \end{aligned} \tag{1.9.1}$$

for $i > 0$. Then the morphism induces an isomorphism on S^* by (1.5).

REMARK. In the standard notation (see [1]), M' is denoted by j_*j^*M (resp. $j_!j^*M$). Here j_* is really the direct image as Zariski sheaf (because M' is the localization of M by t), but $j_!$ is not. In fact, $j_!$ is defined by $\mathbf{D}j_*\mathbf{D}$ with \mathbf{D} the dual functor (see [loc. cit.]). We have

$$\mathbf{D}R_S(j_*j^*M) = Rj_*j^*\mathbf{D}R_S(M) \quad (\text{cf. [2]}), \tag{1.9.2}$$

$$\mathbf{D}R_S(j_!j^*M) = j_!j^*\mathbf{D}R_S(M) \quad (\text{cf. [1]}). \tag{1.9.3}$$

See also (1.12) below for (1.9.3).

(1.10) **COROLLARY.** *For $M \in M_{\text{coh}}(\mathcal{D}_S)_{\text{mon}}$, the restriction of M to S^* is a free \mathcal{O}_{S^*} -module of rank $\sum_{\alpha \in \Lambda} \dim M(S)^\alpha$. In particular, $M^{\text{an}}|_{S^*}$ is a vector bundle with connection [2], and $\mathbf{D}R_S(M)[-1]|_{S^*}$ is a local system.*

Proof. It is enough to show the first assertion. We may assume M meromorphic type by (1.9). Then $M(S)$ is a free $\mathbf{C}[t, t^{-1}]$ -module of rank $\sum_{\alpha \in \Lambda} \dim M(S)^\alpha$ by (1.2.3) and (1.4.3), and the assertion follows.

(1.11) **PROPOSITION.** *Let L be a local system on S^* with complex coefficients, L_∞ the group of multivalued sections of L with the monodromy T , and L_∞^α the*

$\exp(-2\pi i\alpha)$ -eigenspace of L_∞ with respect to T_s , where $T = T_s T_u$ is the Jordan decomposition. Then there exists uniquely $M \in M_{\text{coh}(\mathcal{D}_S)_{\text{mon}}}$ of meromorphic (resp. microlocal) type with an isomorphism

$$L = \text{DR}_S(M)[-1]_{|_{S^*}}, \tag{1.11.1}$$

where DR_S is as in (1.1.3). Furthermore, we have a canonical isomorphism

$$M(S)^\alpha = L_\infty^\alpha \tag{1.11.2}$$

for $\alpha \in \Lambda$, such that $-(t\partial_t - \alpha)$ corresponds to $N := (\log T_u)/2\pi i$.

Proof. By (1.9) it is enough to show the assertion for M meromorphic type. By (1.7), there exists uniquely $M \in M_{\text{coh}(\mathcal{D}_S)_{\text{mon}}}$ of meromorphic type with the isomorphism (1.11.2). Then M^{an} is identified with a $\mathcal{O}_S[t^{-1}]$ -submodule of $j_{\star}(\mathcal{O}_{S^*} \otimes_{\mathbb{C}} L)$ generated by

$$t^\alpha \exp(-(\log t)N)u \tag{1.11.3}$$

for $u \in L_\infty^\alpha$ with $\alpha \in \Lambda$ (see [2]), because (1.11.3) satisfies the same relation as the element of $M(S)^\alpha$ corresponding to $u \in L_\infty^\alpha$ by definition of M (and $M^{\text{an}}|_{S^*}$ and $\mathcal{O}_{S^*} \otimes_{\mathbb{C}} L$ have the same rank). In particular, $M^{\text{an}}|_{S^*} = \mathcal{O}_{S^*} \otimes_{\mathbb{C}} L$, and the assertion follows.

REMARK. This proposition shows that we have an equivalence of categories between the category of monodromical \mathcal{D}_S -modules of meromorphic type and the category of local systems on S^* , in a compatible way with (1.7.1). Note that the isomorphism (1.11.2) depends on the choice of the branch of $\log t$ (i.e., the choice of a lift of 1 to a universal covering of S^*).

(1.12) **PROPOSITION.** *If $M \in M_{\text{coh}(\mathcal{D}_S)_{\text{mon}}}$ is microlocal type, we have*

$$(\text{DR}_S(M))_0 = 0, \tag{1.12.1}$$

$$R\Gamma(S^{\text{an}}, \text{DR}_S(M)) = 0. \tag{1.12.2}$$

Proof. By (1.10), $\text{DR}_S(M)[-1]_{|_{S^*}}$ is a local system, and it is enough to show (1.12.1). Using a filtration defined in the category of monodromical \mathcal{D}_S -modules of microlocal type, we may assume $\sum_{\alpha \in \Lambda} \dim M(S)^\alpha = 1$. Then it is isomorphic to $M(\alpha) = \mathcal{D}_S/\mathcal{D}_S(t\partial_t - \alpha)$ if $M(S)^\alpha = \mathbb{C}$ for $\alpha \in \Lambda \setminus \{0\}$ (see the proof of (1.8)). In the other case, we can check that M is isomorphic to $\mathcal{D}_S/\mathcal{D}_S(t\partial_t)$. Then we can check the assertion (see also [8]).

2. Algebraic Gauss-Manin system

(2.1) Let $f: X \rightarrow Y$ be a morphism of smooth complex algebraic varieties. The direct image $\int_f M$ of a \mathcal{D}_X -module M is defined by

$$\int_f M = Rf_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^L M) \quad \text{with} \quad \mathcal{D}_{Y \leftarrow X} = \omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} (\omega_Y)^\vee),$$

(2.1.1)

where ω_X is the dualizing sheaf, and \vee denotes the dual line bundle. See [1], [6], [10], etc. Note that, if f is an open embedding, $\int_f M$ is defined by the sheaf theoretic direct image. In the case Y is the affine line S , the direct image $\int_f M$ will be more explicitly expressed later (see (2.6)).

We denote the cohomological direct image $\mathcal{H}^p \int_f M$ by $\int_f^p M$. For $M = \mathcal{O}_X$, the direct image $\int_f \mathcal{O}_X$ (or $\int_f^p \mathcal{O}_X$) is called the *Gauss-Manin system* of f [7].

For a \mathcal{D}_X -module M , let

$$DR_X(M) = (\Omega_X^*(M))^{\text{an}}[\dim X],$$

(2.1.2)

where $\Omega_X^*(M)$ denotes the de Rham complex as in [1], [2], and an is defined as in (1.1.2). By [1] we have:

(2.2) **PROPOSITION.** *If M is regular holonomic, the cohomological direct images $\int_f^p M$ are regular holonomic, and we have a natural isomorphism*

$$DR_Y \left(\int_f M \right) = Rf_*(DR_X(M)).$$

(2.2.1)

(2.3) **COROLLARY.** *Assume f smooth with relative dimension r , and $\int_f^p \mathcal{O}_X$ is a locally free \mathcal{O}_Y -module of finite rank (i.e., a vector bundle with integrable connection [2]). Let L^p be the local system defined by the horizontal sections of $(\int_f^p \mathcal{O}_X)^{\text{an}}$. Then we have natural isomorphisms*

$$DR_Y \left(\int_f^p \mathcal{O}_X \right) = L^p[\dim Y],$$

(2.3.1)

$$L^p = R^{p+r} f_* \mathbf{C}_{X^{\text{an}}}.$$

(2.3.2)

Proof. The first assertion follows from the Poincaré lemma. Then the second follows from (2.2.1) using $DR_X(\mathcal{O}_X) = \mathbf{C}_{X^{\text{an}}}[\dim X]$, because \mathcal{O}_X is regular holonomic by definition [1].

REMARK. If we assume that $R^{p+r}f_*\mathbf{C}_{X^{\text{an}}}$ is a local system, the corollary follows also from the relative version of [4] using a desingularization of the divisor at infinity of a compactification of f , because we may replace Y by its Zariski-open subset.

(2.4) LEMMA. *Let $f: X \rightarrow Y$ be as in (2.1), and assume Y is the affine line S with coordinate t as in Section 1 so that f is identified with a function on X . Let $\theta = \partial_t - \text{id}$, and*

$$K_f = \text{Cone} \left(\theta: \int_f \mathcal{O}_X \rightarrow \int_f \mathcal{O}_X \right). \tag{2.4.1}$$

Then we have a natural isomorphism

$$(K_f)^{\text{an}} = \text{DR}_S \left(\int_f \mathcal{O}_X \right), \tag{2.4.2}$$

where an is defined as in (1.1.2).

Proof. This follows from $\partial_t - 1 = e^t \partial_t e^{-t}$ in $\mathcal{D}_S^{\text{an}}$.

(2.5) PROPOSITION. *For $f: X \rightarrow S$ as above, assume X^{an} contractible and purely n -dimensional. Then we have a natural isomorphism*

$$R\Gamma(S^{\text{an}}, (K_f)^{\text{an}}) = \mathbf{C}[n]. \tag{2.5.1}$$

Proof. This follows from (2.2) and (2.4).

REMARK. We can apply (2.2) also to the direct image of \mathcal{O}_X by $X \rightarrow pt$ and the direct image of $\int_f^p \mathcal{O}_X$ by $S \rightarrow pt$. In this case, (2.2) means the commutativity of the direct image with the functor An in (1.1.2), and follows also from [4] and [2] (see also (1.9.2) above) respectively.

(2.6) Let $f: X \rightarrow S$ be as in (2.4), and M a \mathcal{D}_X -module. We define a structure of \mathcal{D}_X -module on $M \otimes_{\mathbf{C}} \mathbf{C}[\partial_t]$ by

$$g(u \otimes \partial_t^i) = gu \otimes \partial_t^i, \quad \zeta(u \otimes \partial_t^i) = \zeta u \otimes \partial_t^i - (\zeta f)u \otimes \partial_t^{i+1} \tag{2.6.1}$$

for $g \in \mathcal{O}_X$, $\zeta \in \Theta_X$ and $u \in M$. It has also the action of $R = \mathbf{C}[t, \partial_t]$ (see (1.1)) by

$$\partial_t(u \otimes \partial_t^i) = u \otimes \partial_t^{i+1}, \quad t(u \otimes \partial_t^i) = fu \otimes \partial_t^i - iu \otimes \partial_t^{i-1}, \tag{2.6.2}$$

which commutes with the action of \mathcal{D}_X . Then $M \otimes_{\mathbf{C}} \mathbf{C}[\partial_t]$ is identified with the direct image of M by the embedding i_f by the graph of f , and $u \otimes \partial_t^i$ is identified

with $\partial_t^i \delta(t - f) \otimes u$. Here $\delta(t - f)$ is the delta function with support $\{f = t\}$, and satisfies the relation

$$t\delta(t - f) = f\delta(t - f), \quad \zeta\delta(t - f) = -(\zeta f)\partial_t \delta(t - f), \tag{2.6.3}$$

which gives (2.6.1–2).

Since the direct image of a \mathcal{D} -module by a smooth projection with fiber X is given by the sheaf theoretic direct image of the relative de Rham complex shifted by $\dim X$ (see for example [1]), the direct image $\int_f M$ is expressed as

$$\int_f M = Rf_* (\Omega_X(M \otimes_{\mathbb{C}} \mathbb{C}[\partial_t])[\dim X]), \tag{2.6.4}$$

factorizing f into the closed embedding i_f and the projection. This can be also obtained by using induced \mathcal{D} -modules [9]. Note that, if f is an affine morphism, the derived direct image Rf_* can be replaced by f_* .

(2.7) PROPOSITION. For $f: X \rightarrow S$ as above, assume $X = \mathbb{C}^n$. Then we have a natural quasi-isomorphism (0.4) in the introduction.

Proof. By (2.6.4), $\int_f \mathcal{O}_X$ is expressed by $f_*(\Omega_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t])[n]$, where the differential of $\Omega_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$ is given by

$$\omega \otimes \partial_t^i \rightarrow d\omega \otimes \partial_t^i - df \wedge \omega \otimes \partial_t^{i+1}. \tag{2.7.1}$$

See (2.6.1). Then we have a short exact sequence of complexes

$$0 \rightarrow \Omega_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t] \xrightarrow{d} \Omega_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t] \rightarrow \Omega_X \rightarrow 0, \tag{2.7.2}$$

where the differential of Ω_X is D_f . Then we get the assertion taking the exact functors f_* and $\Gamma(S, *)$.

(2.8) PROPOSITION. Let $\bar{S} = \mathbb{P}^1$ with a natural inclusion $j: S \rightarrow \bar{S}$. Let M be a regular holonomic \mathcal{D}_S -module, and $K = \text{Cone}(\theta: M \rightarrow M)$ for θ as above. Let U be a Zariski-open subset of S on which M is locally free over \mathcal{O}_U , and denote the local system $\text{DR}_S(M)[-1]|_{U^{\text{an}}}$ by L . Then we have a canonical isomorphism

$$H_{\{t\}}^0((j_* K)^{\text{an}}) = L_t \tag{2.8.1}$$

for $t \in U^{\text{an}} (\subset \mathbb{C})$ such that $\text{Im } t = 0$ and $\text{Re } t \gg 0$. Furthermore,

$$H_{\{t\}}^i((j_* K)^{\text{an}}) = 0 \quad \text{for } i \neq 0. \tag{2.8.2}$$

Proof. Let $V = \{t \in \mathbb{C} : |t| > R\} \subset U^{\text{an}}$ for R sufficiently large, and $V' =$

$V \cup \{\infty\}$. Then $(j'_*M)^{\text{an}}|_V$ is an extension of $M^{\text{an}}|_V$ as regular holonomic \mathcal{D}_V -module such that the action of the local coordinate $s (= t^{-1})$ is bijective, and such an extension is unique by [2]. So $H^i_{\{\infty\}}((j'_*K)^{\text{an}})$ is uniquely determined by $M^{\text{an}}|_V$, or equivalently, by $L|_V$ (see [loc. cit.]). Since V is homotopy equivalent to $(S^*)^{\text{an}}$, we may assume $U = S^*$, and M is a monodromical \mathcal{D}_S -module of microlocal type by (1.11). Then

$$H^i(S^{\text{an}}, K^{\text{an}}) = H^i(S^{\text{an}}, \text{DR}_S(M)) \tag{2.8.3}$$

by the same argument as the proof of (2.4). So it is zero for any i by (1.12.2), and we may replace $H^i_{\{\infty\}}((j'_*K)^{\text{an}})$ in (2.8.1–2) by $H^i(\bar{S}^{\text{an}}, (j'_*K)^{\text{an}})$ using the long exact sequence:

$$\rightarrow H^i_{\{\infty\}}((j'_*K)^{\text{an}}) \rightarrow H^i(\bar{S}^{\text{an}}, (j'_*K)^{\text{an}}) \rightarrow H^i(S^{\text{an}}, K^{\text{an}}) \rightarrow \tag{2.8.4}$$

By GAGA, we have

$$H^i(\bar{S}^{\text{an}}, (j'_*K)^{\text{an}}) = H^i(\bar{S}, j'_*K) = H^i(S, K), \tag{2.8.5}$$

where the last isomorphism follows from the exactness of j'_* . Moreover,

$$R\Gamma(S, K) = \text{Cone}(\theta: M(S) \rightarrow M(S)) = \bigoplus_{\alpha \in \Lambda} M(S)^\alpha \tag{2.8.6}$$

using (1.2.3) and (1.4.3). So the assertion follows from the isomorphism (1.11.2). Here we identify L_t with L_∞ by taking a lift of t to a universal covering of S^* , at which $\log t$ is real valued.

(2.9) *Proof of Theorem (0.3).* Let K_f be as in (2.4), and $j': S \rightarrow \bar{S}$ as above. By GAGA and (2.7), we have

$$H^i(\bar{S}^{\text{an}}, (j'_*K_f)^{\text{an}}) = H^i(\bar{S}, j'_*K_f) = H^i(S, K_f) = H^{i+n}(\Omega, D_f). \tag{2.9.1}$$

We have the long exact sequence (2.8.4) with K replaced by K_f . Let $F = f^{-1}(t)$ for t as in (2.8.1). Then it is enough to show a canonical isomorphism

$$H^i_{\{\infty\}}((j'_*K_f)^{\text{an}}) = H^{i+n-1}(F, \mathbb{C}) \tag{2.9.2}$$

by (2.5.1), because we can check $H^0(\Omega, D_f) = 0$ so that the morphism

$$\mathbb{C} = H^{-n}(S^{\text{an}}, (K_f)^{\text{an}}) \rightarrow H^1_{\{\infty\}}((j'_*K_f)^{\text{an}}) \tag{2.9.3}$$

is injective. Then, applying (2.8) to $M = \int_f^p \mathcal{O}_X$, the assertion (2.9.2) follows from (2.3).

(2.10) REMARKS. (i) If f is weighted homogeneous, $\int_f^p \mathcal{O}_X$ ($p \neq 1-n$) and $\int_f^{1-n} \mathcal{O}_X/\mathcal{O}_S$ are monodromical \mathcal{D}_S -modules of microlocal type, and the decomposition (1.4.1) by the action of $t\partial_t$ is induced by the grading of Ω' compatible with f as in [3]. This implies that the isomorphism (0.2) is compatible with the action of monodromy as in [loc. cit.].

(ii) In Theorem A of [3], δ does not induce an isomorphism for $k=0$. The definition of D_f should be replaced by (0.1) in this paper, which is denoted by \bar{D}_f in [loc. cit.].

(iii) In the proof of (1.8) of [3], it is better to use Coim Δ instead of $\bar{\Omega} = \text{Ker } \Delta$.

(2.11) EXAMPLE: $f = x^2y + x$. This is a generalized weighted homogeneous polynomial admitting *negative* weights, and the spectral sequence as in [3] does not converge. In fact, the E_0 -complex is isomorphic to the Koszul complex $(\Omega', df \wedge)$, and is acyclic, but the general fiber $F = f^{-1}(t)$ is isomorphic to \mathbf{C}^* so that $\tilde{H}^0(F, \mathbf{C}) = 0$, $\tilde{H}^1(F, \mathbf{C}) = \mathbf{C}$. We can check $H^2(\Omega', D_f) = \mathbf{C}$ as follows.

We have $f_x = 2xy + 1$, $f_y = x^2$, and

$$D_f(-x^i y^j dx) = jx^i y^{j-1} dx dy - x^{i+2} y^j dx dy$$

$$D_f(x^i y^j dy) = ix^{i-1} y^j dx dy - 2x^{i+1} y^{j+1} dx dy - x^i y^j dx dy.$$

Let $\phi: \Omega^2 \rightarrow \mathbf{C}$ be a map defined by $\phi(x^i y^j dx dy) = (-1)^i j! / (2j - i + 1)!$ for $2j + 1 \geq i$, and 0 otherwise. Then ϕ induces the isomorphism $H^2(\Omega', D_f) = \mathbf{C}$.

Added in the proof. We are informed that a similar result is obtained by B. Malgrange and P. Deligne independently using the theory of Fourier transformation.

References

- [1] Borel, A.: *Algebraic D-Modules*, Academic Press, Boston, 1987.
- [2] Deligne, P.: Equation différentielle à points singuliers réguliers, *Lecture Notes in Math.* 163, Springer-Verlag, Berlin, 1970.
- [3] Dimca, A.: On the Milnor fibration of weighted homogeneous polynomials, *Compositio Math.* 76 (1990), 19–47.
- [4] Grothendieck, A.: On the de Rham cohomology of algebraic varieties, *Publ. Math. IHES* 29 (1966), 95–103.
- [5] Grothendieck, A. and Dieudonné, J.: *Eléments de géométrie algébrique IV*, *Publ. Math. IHES* 32 (1967).
- [6] Kashiwara, M.: B -function and holonomic systems, *Inv. Math.* 38 (1976), 33–53.
- [7] Pham, F.: Singularité des systèmes différentiels de Gauss-Manin, *Progress in Math.* 2., Birkhäuser, 1979.
- [8] Saito, M.: On the structure of Brieskorn lattice, *Ann. Institut Fourier* 39 (1989), 27–72.
- [9] Saito, M.: Induced \mathcal{D} -modules and differential complexes, *Bull. Soc. Math. France* 117 (1989), 361–387.
- [10] Sato, M., Kawai, T. and Kashiwara, M.: Microfunctions and pseudodifferential equations, in *Lecture Notes in Math.* vol. 287, Springer, Berlin (1973), 264–529.