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Consider an integral on degenerate variety $X \subset \mathbf{P}^N$. The projective geometry of X is very much reflected in the geometry of its conormal variety $C(X) \subset \mathbf{P}^N \times \mathbf{P}^{N*}$; $C(X)$ is the closure of the pairs (x, H) with $x \in X_{\text{reg}}$ and $T_x X \subseteq H$ (i.e. such that H is tangent to X at x). For instance if the dual variety X^* (i.e. the projection of $C(X)$ into $\mathbf{P}^{n*}$) is not a hypersurface then the general hyperplane tangent to X is tangent to X at infinitely many points. If the algebraically closed base field $\mathbf{F}$ has char($\mathbf{F}$) $= 0$, then it is known that in such a case the general hyperplane tangent to X is tangent to X exactly along a linear space. Without any assumption on X^*, when char($\mathbf{F}$) $= 0$, many nice things are true: biduality (i.e. $(X^*)^* = X$ under the identification of $(\mathbf{P}^{N*})^*$ with $\mathbf{P}^N$), the fact that in the enumerative formulas the solutions comes with multiplicity ones etc. In general such good things are not true when char($\mathbf{F}$) $= p > 0$. Wallace ([15]) was the first to note this fact and to show that there is a class of varieties (the reflexive varieties) for which such facts are true (in particular the biduality of X). A variety X is called reflexive if $C(X) = C(X^*)$ (up to the exchange of the factors in $\mathbf{P}^N \times \mathbf{P}^{N*}$ and the identification of $(\mathbf{P}^{N*})^*$ with $\mathbf{P}^N$). A variety $X \subset \mathbf{P}^N$ is called ordinary if it is reflexive and X^* is a hypersurface. When char($\mathbf{F}$) $= 0$, every variety is reflexive. When char($\mathbf{F}$) > 0 this is not true, but the class of reflexive varieties has remarkable stability properties and it is very nice: in the small universe of reflexive varieties everything is nice (see [2], [3], [4]). But there is a disturbing feature: char($\mathbf{F}$) $= 2$. When char($\mathbf{F}$) $= 2$ there is no ordinary variety of odd dimension. In particular the general hyperplane section Y of an ordinary variety cannot be ordinary. But it has many nice features: semi-reflexivity (see [3]) and Y^* is a hypersurface ([3], 5.9 and 5.12), i.e. the general hyperplane tangent to y is tangent only at finitely many points of Y. It was asked in [3], (5.11)(iii), if for such X the following property ($) holds:

($): the general hyperplane tangent to a general hyperplane section Y of an ordinary variety X is tangent exactly at one point of Y.

A proof of ($) for all ordinary X would make authomatically possible to drop the

unpleasant assumption "char($\mathbf{F}$) $\neq$ 2" in a few statements ([3], 4.10(ii); for the non-existence of bitangencies in the enumeration of the contact formula in [2], §2, part (c) of the main theorem, p. 162, see the discussion in [2], p. 169). Unfortunately, the answer is NO. In Section 1 we construct a very nice family of examples; they are hypersurfaces and we have explicit equations; they are related to the so-called null-correlations (and to the null-correlation bundles) (a very classical topic: see e.g. [1]). But the next best thing happens. These examples are the only ones. This is proved in Section 2. Thus in this paper we prove the following result.

THEOREM 0.1. *Fix an integral non degenerate variety* X *in* $\mathbf{P}^N$, $\dim(X) = n$. *Then a general hyperplane tangent to a general hyperplane section* Y *of* X *is tangent at more than one point of* Y *if and only if* $N = n + 1$, $n = 2k + 2$ *is even, and, up to a change of the homogeneous coordinates* $y_1, \ldots, y_k, z_1, \ldots, z_k$ *of* $\mathbf{P}^N$, *the equation* f *of* X *has the form*:

$$f = \left(\sum_{i=1}^{k} y_i z_i \right) h^2 + b^2 \tag{1}$$

with h *and* b *homogeneous polynomials.*

Every integral non degenerate hypersurface X satisfying (1) has two geometrical properties (see §1 for their proof): TX is the restriction to X of a null-correlation bundle of $\mathbf{P}^{n+1}$ and X is canonically isomorphic to its dual variety. Now we can check that such a variety X is not normal (proof: X is singular on the codimension one subset $\{h = b = 0\}$). Thus we have the following corollary.

COROLLARY 0.2 *Fix an integral non degenerate variety* $X \subset \mathbf{P}^N$. X *has property* ($\$$) *if it is normal or if it is not a hypersurface.*

1. The examples

In this paper we work always over an algebraically closed base field $\mathbf{F}$ with char($\mathbf{F}$) = 2. Fix an odd integer $m > 1$ and set $\mathbf{P} := \mathbf{P}^m$. Let O and Ω be structural sheaf and the cotangent bundle on $\mathbf{P}$. It is well-known that $\Omega(2)$ is spanned by its

global sections. Since m is odd, $c_m(\Omega(2)) = 0$; hence a general section of $\Omega(2)$ never vanishes. Choose a nowhere vanishing section s of $\Omega(2)$. The choice of s is equivalent to the choice of a surjection $t: T\mathbf{P} \to O(2)$. Set $N := \mathrm{Ker}(t)$. N is a rank-$(m - 1)$ vector bundle on $\mathbf{P}$. It is called a null-correlation bundle and classically considered (see e.g. [1]).

Thus s induces an exact sequence:

$$0 \to N \to T\mathbf{P} - O(2) \to 0. \tag{2}$$

Choose homogeneous coordinates $x_0, \ldots, x_m$ on $\mathbf{P}$; we will set $m = 2k - 1$, $y_i := x_{2i-2}, 1 \leqslant i \leqslant m, z_i := x_{2i-1}, 1 \leqslant i \leqslant m$. Consider the Euler sequence on $\mathbf{P}$:

$$0 \to O \to (m + 1)O(1) \to T\mathbf{P} \to 0. \tag{3}$$

The map $O \to (m + 1)O(1)$ in (3) send $c \in O$ into $(cx_0; \ldots; cx_m)$. By (3) the choice of t is equivalent to the choice of $m + 1$ linear forms $l_0, \ldots, l_m$ on $\mathbf{P}$ such that $x_0 l_0 + \cdots + x_m l_m = 0$; t is surjective exactly when the forms $l_0, \ldots, l_m$ have no common zero. Furthermore by the dual of (3) $H^0(\mathbf{P}, \Omega(2))$ can be identified to the set of antisymmetric $(m + 1) \times (m + 1)$ matrices over $\mathbf{F}$ (even if $\mathrm{char}(\mathbf{F}) = 2$); the matrix given by $l_0, \ldots, l_m$ has as elements in the ith row the coefficients of l_i. The nowhere vanishing of s is equivalent to the fact that the corresponding antisymmetric matrix L is invertible; hence up to a change of coordinates s is uniquely determined. We will choose the coordinates $x_0, \ldots, x_m$ in such a way that $l_{2i-2} = z_i$ and $l_{2i-1} = -y_i, 1 \leqslant i \leqslant m$.

Let X be a reduced hypersurface of $\mathbf{P}$ with homogeneous equation f, $\deg(f) = d > 2$. Assume that $(\partial/\partial x_i)f = l_i g$ for all i and for some form g of degree $d - 2$. Since X is reduced, $g \neq 0$. Then the map from $T\mathbf{P}|X$ to the normal bundle $O_X(d)$ to X factors through $O_X(2)$; since the l_i are nowhere vanishing, we see that the exact sequence

$$0 \to TX \to T\mathbf{P}|X \to O_X(d)$$

induces the exact sequence

$$0 \to TX \to T\mathbf{P}|X \to O_X(2) \to 0 \tag{4}$$

which is the restriction to X of (2). In particular TX is locally free and $TX = N|X$.

Claim: X is reflexive.

Proof of the claim. The alternating matrix L corresponding to s gives a null-correlation, i.e. an isomorphism $\mathbf{L}$ between $\mathbf{P}$ and its dual $\mathbf{P}^*$ such that $x \in \mathbf{L}(x)$ for all $x \in \mathbf{P}$. Since the matrix L is alternating, we check easily that if

$y \in \mathbf{L}(x)$ then $x \in \mathbf{L}(y)$. Let $J \subset \mathbf{P} \times \mathbf{P}^*$ be the incidence correspondence, $J = \{(x, H): x \in H\}$. $J = \mathbf{P}(TP(-1))$ (as space over $\mathbf{P}$) and t gives a section of $J \to \mathbf{P}$ with image the graph graph(**L**) of **L**. Furthermore, since $TX = N|X, J|(X \times \mathbf{P}^*)$ is the conormal variety $C(X)$ of X. Since **L** is an isomorphism, the projection graph(**L**) $\to \mathbf{P}^*$ is an isomorphism. Hence so is its restriction $C(X) \to X^*$. Thus X is reflexive (see [2] or [3] or [4]), proving the claim.

Since (4) is the restriction of (2), for any $x, y \in (X)_{\text{reg}}$, $y \in T_x X$ if and only if $x \in T_y X$. Furthermore if H is a general hyperplane, say $H = \mathbf{L}(a)$ with $a \in \mathbf{P}$ and a general, for all $x \in H \cap (X_{\text{reg}})$, we have $a \in T_x X$ (i.e. by definition $Y := X \cap H$ is strange with strange point a). In particular any line D in H with $a \in D$ is tangent to X at all points of $Y \cap D$. First assume that for a general line D in H with $a \in D$, card$(Y \cap D) \geqslant 2$. Fix any such D and $c, b \in Y \cap D$ with $c \neq b$. To show that X does not satisfy ($\$$), it is sufficient to check that $(T_c Y) \cap Y = (T_b Y) \cap Y$. Fix a general $e \in (T_c Y) \cap Y$. Then $c \in T_e Y$. Since $a \in T_e Y$, we get $b \in T_e Y$, hence $e \in T_b Y$, hence the thesis. We have to prove the assumption, i.e. that for general $\mathbf{L}(a)$ there is such a line D. First assume $m = 3$, i.e. X a surface. Since X is reflexive, for general $x \in X T_x X$ has order of contact 2 with x at X, hence for general $\mathbf{L}(a)$ a general line D tangent to $X \cap \mathbf{L}(a)$ (i.e. a general line in H through a) has contact order 2 with Y at each point of $Y \cap D$; since $d > 2$, the assumption follows. Now assume $m > 3$. By the case $m = 3$, we see (cutting X with a general $\mathbf{P}^3$) that for a general $x \in X$ and a general tangent line D to X at x, card$(D \cap X) > 1$; $\mathbf{L}(x)$ is the tangent space to X at x, hence contains D. Of course, $\mathbf{L}(x)$ is not a general hyperplane. But we may find a family of points $a(t), t \in T, a(t) \in \mathbf{P}$ for all t, with T smooth, irreducible affine curve, and a family of lines $D(t), t \in T$, with $a(t) \in D(t) \subset \mathbf{L}(a(t))$ for every t (hence $D(t)$ tangent to X at each point of $D(t) \cap X$ by the strangeness of $X \cap \mathbf{L}(a(t))$), with $a(0) = x$ for some point $0 \in T$, $a(t)$ general for general t. For general $t \in T$, $\mathbf{L}(a(t))$ and $D(t)$ will work.

Now we have to show that (assuming char(**F**) $= 2$) such varieties X exist, write their equations and find all possible X. First note that from $(\partial/\partial z_i)f = y_i g$ (resp. $(\partial/\partial y_i)f = z_i g$), $i = 1, \ldots, m$, we get that g contains every variables z_i (resp. y_i) at an even order. Thus $g = h^2$ for some polynomial h of degree $d/2$; in particular d is even. Furthermore, taking derivatives, we see that $f - (\sum_i y_i z_i)h^2$ is a square, say b^2. Viceversa, for any even d and any choice of polynomials h, b, with deg$(h) = -1 + (d/2)$, deg$(b) = d/2$, define f by the equation (1). The variety $X := \{f = 0\}$ will work if it is reduced and irreducible. Even the reducibility is not a big problem; if it is reducible, any of its reducible components of degree > 2 would work (if there are such components). The reader can easily construct examples of f with X integral. Here we show only the existence of at least one example, taking $m = 3, h = x_1, b = (x_0 + x_2 + x_3)^2$. Note that the restriction of this surface X to the plane $A := \{x_2 = 0\}$ is a curve C with exactly a singular point $(x_0 = x_3 \neq 0, x_1 = 0)$ while the line $\{x_0 = 0\}$ intersects C only at one point with $x_3 = 0$. Hence C cannot be the product of (maybe reducible) conics.

2. End of proof

In this section we will prove 0.1 and 0.2 showing that the examples given in section 1 are the only examples of reflexive varieties without property ($). We will fix an integral non degenerate variety $X \subset \mathbf{P}^N$, $\dim(X) = n$, X not with the property ($). The proof of 0.1 is divided into 2 steps; essentially, in the first step we will handle the case "$N = n + 1$", while in the second step we will handle the case "$N > n + 1$".

Step 1. By [5], Lemma 3 p. 334, the general projection Y of X into $\mathbf{P}^{n+1}$ is reflexive; of course, Y is ordinary and cannot have the property ($). In this step we will work with Y i.e. we will assume $N = n + 1$ and X a hypersurface. By the definition of ordinary variety and [4], 2.4(v), there is a proper closed subset A of X, A containing the singular set $\mathrm{Sing}(X)$ of X, such that for every $x \in X$, $x \notin A$, the tangent space $T_x X$ is tangent to the smooth locus of X only at x. Consider the following assertion (£):

(£): for a general $x \in X$ and a general hyperplane H through x, the proper morphism from the conormal variety of $X \cap H$ in H to the dual of $X \cap H$ has x as its fiber through x (set theoretically).

Note that, fixing H and taking $z \in H$ with z general, it is easy to check that (£) implies ($). Thus if we prove (£) we get a contradiction. For every $y \in X \backslash A$, set $X(y) := X \cap T_y X$, $X(y)' := X(y) \cap (X_{\mathrm{reg}})$. Since $N = n + 1$, by the definition of A for every $y \in X(x)'$, $y \neq x$, $T(x, y) := T_y X \cap T_x X$ is a hyperplane of $T_x X$. Varying y we get a family of hyperplanes of $T_x X$ of dimension at most $n - 1$.

Assume that for general x and general $y \in T(x)'$, $x \notin T(x, y)$.

Then a general hyperplane H through x does not contain any $T(x, y)$. Since $T_x(X \cap H) = (T_x X) \cap H$, (x, H) will prove (£).

Thus we may assume that for general $x \in X$ and every $y \in T(x)'$, $x \in T(x, y)$. If the family $\{T(x, y)\}$ (x fixed, y variable in $X(x)'$) has not dimension $n - 1$, again the general hyperplane H through x does not contain any $T(x, y)$, again giving (£). Thus till the end of this step we will assume that for general x, a general hyperplane of $T_x X$ containing x is of the form $T(x, y)$. Counting dimensions we see that this implies that for general $x \in X$ and general $y \in (X \cap T_x X)$, we have $x \in T_y X$. This means that for general $x \in X$, the image of $(X_{\mathrm{reg}} \cap T_x X)$ under the Gauss map $\mathbf{g} : X_{\mathrm{reg}} \to \mathbf{P}^*$ is contained in the hyperplane H_x of $\mathbf{P}^*$ corresponding to x. Thus for general $x \in X$, $\mathbf{g}(X_{\mathrm{reg}} \cap T_x X)$ has $X^* \cap H_x$ as closure in $\mathbf{P}^*$. Thus $\mathbf{g}^*(O_{\mathbf{P}^*}(1)) = O_X(1)|X_{\mathrm{reg}}$. We claim that this gives that the induced map in cohomology $\mathbf{g}'^* : H^0(\mathbf{P}^*, O_{\mathbf{P}^*}(1)) \to H^0(X_{\mathrm{reg}}, O_X(1))$ has image contained in the image of $H^0(\mathbf{P}, O_{\mathbf{P}}(1))$; indeed since X is non degenerate, $H^0(\mathbf{P}^*, O_{\mathbf{P}^*}(1))$ is generated by the sections corresponding to the hyperplanes H_x, with $x \in X$. The

map g'^* is injective because X^* is not contained in a hyperplane as X is ordinary. Note that (for dimensional reasons and the claim and the injectivity just proven) g'^* induces an isomorphism j of $\mathbf{P}^*$ onto $\mathbf{P}$. To show that if ($) does not hold X must be of the form considered in Section 1, it is sufficient to check that j^{-1} is a null-correlation, i.e. that $j(c) \in c$ for every $c \in \mathbf{P}^*$. Let M be the matrix associated to j^{-1}. M is antisymmetric if and only if j^{-1} is a null-correlation. Assume that M is not antisymmetric, i.e. that $Q := \{z \in \mathbf{P} : {}^t z M z = 0\}$ is a quadric. Since by construction $x \in j^{-1}(x)$ for all $x \in X$ (note that $x \in T_x X$), this means that X is a quadric (which trivially satisfy ($)). Thus we have described every ordinary hypersurface X without ($).

Step 2. (case $N > n + 1$). Now we assume that X is not a hypersurface, and that ($) fails for X; by [5] as at the beginning of step 1 we may assume that $N = n + 2$ and that a general projection of X into $\mathbf{P}^{n+1}$ does not have ($) (hence by step 1 is of the type considered in Section 1). Fix a point $a \in X_{\text{reg}}$. Fix a general point $P \in \mathbf{P}^{n+2}$ and let j (or j_P) be the projection of X into $\mathbf{P}^{n+1}$ from P; set $Z := j(X)$. Let $H := (P, T_a X)$ be the span of P and $T_a X$. For general P we have $H \cap X \neq H \cap X \cap T_a X$. Fix a general $b \in (H \cap X)$. Since $j(b) \in T_{j(a)} Z$, by the classification in the case of hypersurfaces we have $j(a) \in T_{j(b)} Z$, i.e. a is in the span H' of P and $T_b X$. Varying P in H, we see that this is equivalent to $H' = H$. Hence $\dim(T_a X \cap T_b X) = n - 1$. Moving P, we see that this is true for all $a, b \in X_{\text{reg}}$. Fix $a, b, c \in X_{\text{reg}}$. Then either $T_a X \cap T_b X \cap T_c X = T_a X \cap T_b X$ or $T_c X$ is contained in the span H'' of the union of $T_a X$ and $T_b X$. Since $\dim(T_a X \cap T_b X) = n - 1$, we have $\dim(H'') = n + 1$, hence $H'' \neq \mathbf{P}^{n+2}$. Move c. In the latter case we get $X_{\text{reg}} \subset H''$, contradicting the non degeneracy of X. In the former case all $T_c X$, $c \in X_{\text{reg}}$, contain $T_a X \cap T_b X$, hence X cannot be ordinary. This completes the proof of 0.1 and 0.2.

REMARK 2.1. By the classification given it follows that, except for the examples of Section 1, [3], 4.10(ii), holds true even in characteristic 2. For the examples of Section 1 it fails badly since for a general $H \in X^*$, the hyperplane tangent to X^* at H is the dual of $X \cap H$.

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