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A CHARACTERIZATION OF QUASI-HOMOGENEOUS GORENSTEIN SURFACE SINGULARITIES

Jonathan M. Wahl

Abstract

Following Steenbrink, we introduce 3 new resolution invariants of a complex normal surface singularity. For a complete intersection we give a formula for $\tau = \dim T^1$ in terms of these and the usual resolution invariants. As a corollary, we obtain Looijenga's result $\mu \geqslant \tau + b$, where μ is the Milnor number, and b is the number of loops in the resolution dual graph. We also prove that if $\mu = \tau$, then the singularity is quasi-homogeneous; for a hypersurface, this is a special case of a well-known theorem of K. Saito. As a corollary of the method, we show every quasi-homogeneous Gorenstein surface singularity, not a rational double point, admits a one-parameter equisingular deformation.

Introduction

Let $f \in \mathbb{C}\{z_0, \dots, z_n\} = P$ define a hypersurface with an isolated singularity at the origin. Define

$$\mu = \dim P\Big/\left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)$$

$$\tau = \dim P\Big/\left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}, f\right).$$

Clearly, $\mu \geqslant \tau$, with equality if $f = \Sigma a_i \partial f/\partial z_i$, i.e., if there is a derivation $D = \Sigma a_i \partial/\partial z_i$ with $Df = f$. f is called a q.-h. polynomial (quasi-homogeneous, or weighted homogenous) if for some positive integers $w_0, \dots, w_n$, d,

$$f(t^{w_0} z_0, \dots, t^{w_n} z_n) = t^d f(z_0, \dots, z_n).$$

If f is q.-h., then $D = (1/d)\Sigma w_i z_i (\partial/\partial z_i)$ has $Df = f$, whence $\mu = \tau$. Conversely a theorem of K. Saito [11] says that if $\mu = \tau$, then after a holomorphic change of coordinates, f is q.-h. (cf. also Zariski's theorem [21] on irreducible plane curves).

Now suppose $(X, 0) \subset (\mathbb{C}^N, 0)$ is the germ of an n-dimensional isolated complete intersection singularity, with $n \geqslant 1$. Compatibly with the hypersurface case, define

$$\mu = rk\, H_n(F)$$

$$\tau = \dim T^1_{X,0};$$

here, F is the Milnor fibre of a smoothing ([7], [17]), and τ is the dimension of the base space of a semi-universal deformation. From the defining equations of $(X, 0)$, one can give formulae for μ and τ as dimensions of certain finite length modules; but it is no longer clear what is the relation between these invariants. This problem was first considered by G.-M. Greuel [3], who conjectures $\mu \geqslant \tau$, and proves the inequality in case $n = 1$ or if the link of X is a rational homotopy sphere. Greuel also proves that (in every dimension > 0) $\mu = \tau$ if $(X, 0)$ is q.-h. E. Looijenga has recently proved [6] that for $n = 2$, $\mu \geqslant \tau + b$ where $b =$ number of loops in the resolution dual graph of $(X, 0)$. (More recently, Looijenga and J. Steenbrink [25] have generalized this result for all $n \geqslant 2$, yielding in particular that $\mu \geqslant \tau$). We shall prove the analogue of Saito's theorem for $n = 2$, yielding (with numbers referring to location in the text):

THEOREM 3.3: *Let $(X, 0)$ be a two-dimensional isolated complete intersection. Then $\mu \geqslant \tau$, and $\mu = \tau$ iff $(X, 0)$ is* q.-h.

We actually prove a sharper result; namely, $\mu \geqslant \tau +$ b, and $\mu = \tau +$ b iff $(X, 0)$ is q.-h. $(b = 0)$ or $(X, 0)$ is cusp $(b = 1)$.

Our proof involves first writing $\mu - \tau$ as a positive integral combination of b and some resolution invariants α, β, and γ introduced by J. Steenbrink (Theorem 2.7). Then, the hard work is to show that the vanishing of this expression implies the existence of an interesting global vector field on a resolution of $(X, 0)$, hence a derivation on $\mathcal{O}_{X,0}$. (Only in the hypersurface case does the condition $\mu = \tau$ immediately produce a derivation). Finally, there is a key theorem of G. Scheja and H. Wiebe [12] saying that in dimension 2, the existence of a non-nilpotent derivation implies quasi-homogeneity.

In fact, we prove much more.

THEOREM 3.2: *Let $(X, 0)$ be a two-dimensional Gorenstein (see 2.1) surface singularity. Then $\alpha = \beta = \gamma = 0$ iff either $(X, 0)$ is quasi-homogeneous (so $b = 0$), or $(X, 0)$ is a cusp (so $b = 1$).*

Another corollary of the expression for $\mu - \tau$ is that one gets a topological lower bound for the analytic invariant τ, viz. $\tau \geqslant \mu_- + \mu_0$ (Corollary 2.9). This should be compared with the problem of finding the minimum value of τ in μ-constant families ([1], [22], and Examples 4.6 and 4.7 below).

If a Gorenstein $(X, 0)$ is smoothable, we can define $\mu =$ Milnor number of the smoothing $= rk\ H_n(F)$, and $\tau =$ dimension of the corresponding smoothing component in the semi-universal deformation. In dimension 2, $\mu - \tau$ is expressed via the same formula as in the complete intersection case (this depends on recent work of Greuel and Looijenga [23]); so, also in this case, $\mu \geqslant \tau$, with = iff $(X, 0)$ is q.-h. For a 1-dimensional Gorenstein singularity, Greuel has proved [4] $\mu \geqslant \tau$ with $\mu = \tau$ iff $(X, 0)$ is q.-h. (See also the recent paper of Greuel-Martin-Pfister [5]).

The invariants α, β, and γ for a normal surface singularity are introduced in §1, a chapter due almost entirely to Steenbrink. We give his proof of the basic Theorem 1.9, which yields an expression (in terms of these and more usual resolution invariants) for the irregularity q of $(X, 0)$. q, studied by Stephen Yau [20], is the dimension of the space of holomorphic 1-forms on X-$\{0\}$ modulo those which extend holomorphically on a resolution. From Theorem 1.9 one can compute q in the q.-h. case, and also prove a theorem of Pinkham and Wahl [10]: for a rational surface singularity, $q = 0$. In §4, we compute α, β, γ, and q for some hypersurface singularities; these invariants are not constant in equisingular families. q turns out to be a rather subtle invariant, and we pose two questions:

Question 1 (See 5.7): Does every hypersurface singularity admit an equisingular deformation to one whose general fibre has $q = 0$?

Question 2 (See 5.10): Is q a semi-continuous invariant?

To examine Question 1 in special cases (as in §4), one considers the problem of "general moduli" for a given equisingularity type, as is done for irreducible plane curves by Zariski in [22]. In fact, we use some calculations and ideas of Zariski to answer Question 1 affirmatively in several cases. While Yau conjectures in [20] that essentially all Gorenstein singularities have $q > 0$, examples indicate otherwise, and Question 1 even asks if the opposite should be true.

One surprising result is that q is the dimension of the tangent space of a functor of deformations of $(X, 0)$.

THEOREM 5.3: *Let $(X, 0)$ be a Gorenstein surface singularity. Then there is a naturally defined smooth q-dimensional subspace of the base space of the semi-universal deformation of $(X, 0)$, corresponding to a certain class of equisingular deformations.*

COROLLARY 5.4: *Let $(X, 0)$ be a q.-h. Gorenstein surface singularity, not a rational double point. Then there exists a non-trivial one-parameter equisingular deformation of $(X, 0)$.*

We are apply to thank Josef Steenbrink, whose two pages of handwritten notes (in response to some questions we posed) form the basis of Chapter 1. We also thank E. Looijenga for discussions, and the University of Nijmegen and the National Science Foundation for support during part of the preparation of this paper.

NOTATION AND TERMINOLOGY: A singularity shall mean a Stein germ $(X, 0)$ of an analytic space with an isolated singularity at 0. $(X, 0)$ is called (by abuse of language) q.-h., or said to have a good $\mathbb{C}^*$-action, if the complete local ring of X at 0 is the completion of a positively graded ring; equivalently, the analytic isomorphism type of $(X, 0)$ is that of a variety defined by weighted homogeneous polynomials. h^i means dim H^i; RDP means rational double point. A simple elliptic singularity is one whose resolution consists of one smooth elliptic curve; and a cusp is one whose resolution consists of one rational curve with a node, or a cycle of smooth rational curves. The invariants α and β in this paper have nothing to do with those of the same name in [17]. A good resolution of a normal surface singularity is one for which all exceptional components are smooth, and all intersections are transversal; there is a unique minimal one.

§1. Steenbrink's invariants

(1.1) Let $(X, 0)$ be a normal surface singularity, $\tilde{X} \to X$ a good resolution, and $E \subset \tilde{X}$ the (reduced) exceptional fibre. E is a union of smooth curves E_i, $i = 1, \dots, k$; let $g_i =$ genus of E_i, $g = \Sigma g_i$, and denote by $\tilde{E}$ the disjoint union of the E_i. Also, define $b =$ first betti number of the dual graph of E ($=$ number of loops). Then $h^1(\mathcal{O}_E) = g + b$, dim $H^1(E; \mathbb{C}) = 2g + b$. We also define the geometric genus $p_g = h^1(\mathcal{O}_{\tilde{X}})$.

(1.2) The sheaf of germs of logarithmic 1-forms $\Omega^1_{\tilde{X}}(\log E)$ is defined by the kernel of the restriction map:

$$0 \to \Omega^1_{\tilde{X}}(\log E)(-E) \to \Omega^1_{\tilde{X}} \to \Omega^1_{\tilde{E}} \to 0. \tag{1.2.1}$$

It follows that $\Lambda^2 \Omega^1_{\tilde{X}}(\log E) = \Omega^2_{\tilde{X}}(E)$, and there is an exact sequence

$$0 \to \Omega^1_{\tilde{X}} \to \Omega^1_{\tilde{X}}(\log E) \to \mathcal{O}_{\tilde{E}} \to 0; \tag{1.2.2}$$

here, the map on the right is the residue map.

LEMMA 1.3:

a) *The composition*

$$H^0(\mathcal{O}_{\tilde{E}}) \to H^1(\Omega^1_{\tilde{X}}) \to H^1(\Omega^1_{\tilde{E}})$$

is an isomorphism.

b) $H^0(\Omega^1_{\tilde{X}}) \tilde{\to} H^0(\Omega^1_{\tilde{X}}(\log E))$.

PROOF: b) follows from a) and (1.2.2). As for a), the image of $1 \in H^0(\mathcal{O}_{E_i})$ in $H^1(\Omega^1_{\tilde{X}})$ is the class of the line bundle $\mathcal{O}_{\tilde{X}}(E_i)$; projecting then to $H^1(\Omega^1_{E_j})$ gives the class of the line bundle $\mathcal{O}_{E_j}(E_i)$ on the curve E_j. Thus, the composition in a) is essentially the intersection pairing on E; by negative-definiteness, this is an isomorphism.

(1.4) Since $H^0_E(\mathcal{O}_{\tilde{E}}) \tilde{\to} H^0(\mathcal{O}_{\tilde{E}})$, the map $H^0(\mathcal{O}_{\tilde{E}}) \to H^1(\Omega^1_{\tilde{X}})$ factors via $H^1_E(\Omega^1_{\tilde{X}})$. Therefore, by (1.3.a), we may define an integer $\gamma \geq 0$ by

$$k + \gamma = rk\left(H^1_E\left(\Omega^1_{\tilde{X}}\right) \to H^1\left(\Omega^1_{\tilde{X}}\right)\right). \tag{1.4.1}$$

(Recall k = number of components of E).

(1.5) Exterior differentiation is seen, via a straightforward local argument, to give rise to an exact sequence

$$0 \to j_!\mathbb{C}_{\tilde{X}-E} \to \mathcal{O}_{\tilde{X}}(-E) \overset{d}{\to} \Omega^1_{\tilde{X}}(\log E)(-E) \overset{d}{\to} \Omega^2_{\tilde{X}} \to 0. \tag{1.5.1}$$

Here, j: $\tilde{X} - E \to \tilde{X}$ is the inclusion, and $j_!\mathbb{C}_{\tilde{X}-E}$ is the sheaf of locally constant functions which vanish on E, defined via the map

$$0 \to j_!\mathbb{C}_{\tilde{X}-E} \to \mathbb{C}_{\tilde{X}} \to \mathbb{C}_E \to 0.$$

If X is contractible, then $\tilde{X}$ retracts topologically onto E, so $H^i(\mathbb{C}_{\tilde{X}}) \tilde{\to} H^i(\mathbb{C}_E)$, all i; so, all cohomology of $j_!\mathbb{C}_{\tilde{X}-E}$ is 0. In particular,

$$H^i\left(\mathcal{O}_{\tilde{X}}(-E)\right) \tilde{\to} H^i\left(d\mathcal{O}_{\tilde{X}}(-E)\right), \quad \text{all } i. \tag{1.5.2}$$

(1.6) As in (1.5), an easy local argument gives, for every exceptional cycle $Y \geq 0$ with support in E, an exact sequence

$$0 \to j_!\mathbb{C}_{\tilde{X}-E} \to \mathcal{O}_{\tilde{X}}(-Y-E) \overset{d}{\to} \Omega^1_{\tilde{X}}(\log E)(-Y-E) \overset{d}{\to} \Omega^2_{\tilde{X}}(-Y) \to 0. \tag{1.6.1}$$

With the natural inclusion of this sequence into (1.5.1), one deduces

PROPOSITION 1.7: *For every exceptional cycle $Y \geq 0$, there is an exact $\mathbb{C}$-linear sequence of locally free sheaves on Y:*

$$0 \to \mathcal{O}_Y(-E) \to \Omega^1_{\tilde{X}}(\log E) \otimes \mathcal{O}_Y(-E) \to \Omega^2_{\tilde{X}} \otimes \mathcal{O}_Y \to 0.$$

(1.8) Besides γ (see (1.4)), Steenbrink introduces two other invariants:

$$\alpha = \dim H^0(\Omega^2_{\tilde{X}})/dH^0(\Omega^1_{\tilde{X}}(\log E)(-E))$$

$$\beta = \dim H^0(\Omega^1_{\tilde{E}})/\operatorname{Im} H^0(\Omega^1_{\tilde{X}}).$$

THEOREM 1.9 (*Steenbrink*): *Let $E \subset \tilde{X} \to X$ be a good resolution of a normal surface singularity, with p_g, g, and b as usual, and α, β, $\gamma \geqslant 0$ as in* (1.4) *and* (1.8). *Then the irregularity $q = \dim H^0(\Omega^1_{\tilde{X}-E})/H^0(\Omega^1_{\tilde{X}})$ is given by*

$$q = p_g - g - b - \alpha - \beta - \gamma.$$

PROOF: From

$$0 \to \mathcal{O}_{\tilde{X}}(-E) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_E \to 0,$$

we see $h^1(\mathcal{O}_{\tilde{X}}(-E)) = p_g - g - b$. Next, consider

$$0 \to d\mathcal{O}_{\tilde{X}}(-E) \to \Omega^1_{\tilde{X}}(\log E)(-E) \to \Omega^2_{\tilde{X}} \to 0.$$

But $h^1(\mathcal{O}_{\tilde{X}}(-E)) = h^1(d\mathcal{O}_{\tilde{X}}(-E))$ (1.5.2) and $h^1(\Omega^2_{\tilde{X}}) = 0$ (Grauert-Riemenschneider), so

$$h^1(\Omega^1_{\tilde{X}}(\log E)(-E)) = p_g - g - b - \alpha. \tag{1.9.1}$$

From (1.2.1) we deduce

$$h^1(\Omega^1_{\tilde{X}}) = k + p_g - g - b - \alpha - \beta. \tag{1.9.2}$$

By Serre duality,

$$h^1_E(\Omega^1_{\tilde{X}}) = h^1(\Omega^1_{\tilde{X}}). \tag{1.9.3}$$

Now consider the exact sequence

$$H^0(\Omega^1_{\tilde{X}}) \to H^0(\Omega^1_{\tilde{X}-E}) \to H^1_E(\Omega^1_{\tilde{X}}) \to H^1(\Omega^1_{\tilde{X}}).$$

As the rank of the last map is $k + \gamma$, the cokernel of the first map has dimension

$$q = h^1_E(\Omega^1_{\tilde{X}}) - (k + \gamma).$$

Use (1.9.2) and (1.9.3) now to get the result.

COROLLARY 1.10: *The invariants* α, β, $\gamma \geqslant 0$ *satisfy*

a) $\alpha+\beta+\gamma \leqslant p_g - g - b$

b) $\beta \leqslant g$

c) $p_g - g - b - \alpha \geqslant \Sigma h^1(\mathcal{O}_{E_i}(-E))$.

PROOF: a) follows from the theorem; b) is because $g = h^0(\Omega^1_{\tilde{E}})$; c) is a consequence of (1.9.1) and the surjection from (1.2.2):

$$\Omega^1_{\tilde{X}}(\log E)(-E) \to \oplus \mathcal{O}_{E_i}(-E) \to 0.$$

COROLLARY 1.11: *Suppose* $(X, 0)$ *is quasi-homogeneous. Then* $q = p_g - g$, *and* $\alpha = \beta = \gamma = 0$.

PROOF: The resolution graph is star-shaped, so $b = 0$; and one could use the $\mathbb{C}^*$-action to show $\alpha = \beta = \gamma = 0$. However, it is easiest to use Stephen Yau's theorem [20] that in the q.-h. case, $q \geqslant p_g - g$. Combining with Theorem 1.9 yields $q = p_g - g$, hence $\alpha = \beta = \gamma = 0$.

COROLLARY 1.12 (*Pinkham-Wahl* [10], *p.* 178): *On the resolution of a rational singularity, all* 1*-forms on* $\tilde{X} - E$ *extend holomorphically across* E; *that is*, $q = 0$.

PROOF: Immediate from Theorem 1.9, as rational means $p_g = 0$.

REMARKS: (1.13.1) α, β, and γ are independent of the good resolution chosen. For instance, if π: $\tilde{X}_1 \to \tilde{X}_2$ is a blowing-up at an exceptional point of $\tilde{X}_2$, we have $\Omega^1_{\tilde{X}_2} \xrightarrow{\sim} \pi_*\Omega^1_{\tilde{X}_1}$ and $R^1\pi_*\Omega^1_{\tilde{X}_1}$ has length 1; thus,

$$H^0\left(\Omega^1_{\tilde{X}_2}\right) \xrightarrow{\sim} H^0\left(\Omega^1_{\tilde{X}_1}\right)$$

$$h^1\left(\Omega^1_{\tilde{X}_2}\right) = h^1\left(\Omega^1_{\tilde{X}_1}\right) + 1.$$

Therefore, β and q are the same on $\tilde{X}_1$ and $\tilde{X}_2$, as well as (see (1.9.2))

$$p_g - g - b - \alpha - \beta.$$

As p_g, g, and b are the same, this proves the assertion.

(1.13.2) As will be seen below, α, β, and γ are not constant in equisingular (i.e., simultaneous resolution) families (see especially 5.1); this contrasts with p_g, g, and b. However, standard semi-continuity theorems plus (1.9.1) and (1.9.2) imply that in such a family, α and $\alpha+\beta$ cannot go down under deformation. It would be interesting to relate α, β, and γ to other invariants which distinguish fibres in an equisingular family, such as those involving the "b-function" [19].

(1.13.3) Lemma (1.3.a) may fail in characteristic p (as Corollary 1.12 may), since the determinant of the intersection matrix may be divisible by p (cf. the characteristic 0 vanishing theorems of [14]).

(1.13.4) In lieu of α and β, one could define two other invariants δ and ϵ, arising from the Hodge spectral sequence on $\tilde{X}$:

$$\delta = \dim H^0(\Omega^2_{\tilde{X}})/dH^0(\Omega^1_{\tilde{X}})$$

$$g + b + \epsilon = rk\left(H^1(\tilde{X}; \mathbb{C}) \to H^1(\mathcal{O}^1_{\tilde{X}})\right).$$

$\epsilon \geqslant 0$ because $H^1(\tilde{X}; \mathbb{C}) \xrightarrow{\sim} H^1(E; \mathbb{C})$, and $H^1(E; \mathbb{C}) \to H^1(\mathcal{O}_E)$ is a surjection onto a $(g+b)$-dimensional space. One shows easily that $\delta \leqslant \alpha$, $\beta \leqslant \epsilon$, and $\alpha + \beta = \delta + \epsilon$. It also follows from [6] that $g - \epsilon = \dim H^1{}_{\{0\}}(d\mathcal{O}_X)$.

§2. Gorenstein singularities and smoothing components

(2.1) In this section we assume $(X, 0)$ Gorenstein, i.e., there exists a nowhere-0 holomorphic 2-form on X-$\{0\}$; recall that a complete intersection is Gorenstein. On the MGR (minimal good resolution) $\tilde{X} \to X$, this 2-form has a polar divisor $Z \geqslant 0$; and in fact $Z \geqslant E$ unless X is a RDP (see 3.6 below). Thus

$$K_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}(-Z),$$

and we will write $K \cdot K = Z \cdot Z$. Recall the MGR is equivariant, i.e. $H^0(\Theta_{\tilde{X}}) = H^0(U, \Theta_{\tilde{X}}) = \Theta_X$ (where $U = \tilde{X} - E$).

Proposition 2.2: *Let $\tilde{X} \to X$ be the* MGR *of a Gorenstein surface singularity. Then*

$$h^1(\Theta_{\tilde{X}}) = 2(p_g - g - b - \alpha - \beta) - \gamma + k - K \cdot K$$

$$= 2p_g + \chi_{\mathrm{top}}(E) - K \cdot K - 1 - (b + 2\alpha + 2\beta + \gamma).$$

Proof: The result being true for an RDP (both sides equal k), we assume $Z > 0$. A rank 2 vector bundle F satisfies

$$F \simeq F^* \otimes \Lambda^2 F$$

("duality"), so

$$\Theta_{\tilde{X}} \simeq \Omega^1_{\tilde{X}}(Z).$$

By equivariance,

$$H^0(\Omega^1_{\tilde X}(Z) = H^0(U, \Omega^1_{\tilde X}(Z)) = H^0(U, \Omega^1_{\tilde X}).$$

From

$$0 \to \Omega^1_{\tilde X} \to \Omega^1_{\tilde X}(Z) \to \Omega^1_{\tilde X} \otimes \mathcal{O}_Z(Z) \to 0,$$

we deduce

$$q - X(\Omega^1_{\tilde X} \otimes \mathcal{O}_Z(Z)) + h^1(\Omega^1_{\tilde X}) - h^1(\Omega^1_{\tilde X}(Z)) = 0. \tag{2.2.1}$$

One computes q from Theorem 1.9 and $h^1(\Omega^1_{\tilde X})$ from (1.9.2). The Euler characteristic on Z is computed as in [16], (A.7.3), since $\Lambda^2\Omega^1_{\tilde X}(Z) \otimes \mathcal{O}_Z = \mathcal{O}_Z(Z)$:

$$\chi(\Omega^1_{\tilde X} \otimes \mathcal{O}_Z(Z)) = Z \cdot (K + 2Z) - Z \cdot (Z + K) = K \cdot K.$$

Since $h^1(\Theta_{\tilde X}) = h^1(\Omega^1_{\tilde X}(Z))$, the result follows from (2.2.1).

(2.3) Recall the following:

THEOREM 2.4. ([17],[23],[24]). *The dimension of any smoothing component of the semi-universal deformation of a Gorenstein surface singularity* $(X, 0)$ *is*

$$\tau = h^1(\Theta_X) + 10p_g + 2K \cdot K. \tag{2.4.1}$$

(2.5) The preceding result was conjectured in [17], and proved there for complete intersections. It was also shown there how a general proof would follow from two now-verified results. First, there is a general deformation theoretic assertion that the dimension of a smoothing component is the "number" of derivations of $(X, 0)$ that do not lift during the smoothing; this was subsequently proved by Greuel-Looijenga [23]. Second, the formula for this number of derivations can be found if one can globalize the smoothing; Looijenga proved in [24] that this can always be done.

(2.6) Putting together (2.4.1) with Proposition 2.2 gives the dimension for a smoothing component of

$$12p_g + \chi_{\text{top}}(E) + K \cdot K - 1 - (b + 2\alpha + 2\beta + \gamma). \tag{2.6.1}$$

THEOREM 2.7: *Let* $(X, 0)$ *be a Gorenstein surface singularity. For a given smoothing, let* $\mu =$ *Milnor number and* $\tau =$ *dimension of the corresponding smoothing component* (*so,* $\tau = \dim T^1_X$ *if* X *is unobstructed, e.g., a com-*

plete intersection). Also, let b = *number of loops in the dual graph of a resolution, and* α, β, γ *the invariants of* §1. *Then*

$$\mu = 12p_g + \chi_{\mathrm{top}}(E) + K \cdot K - 1$$

$$\mu - \tau = b + 2(\alpha + \beta) + \gamma.$$

PROOF: The formula for μ is due to Laufer, Wahl [17], and Steenbrink [13]. Put this together with Theorem 2.4 and (2.6.1).

COROLLARY 2.8 (*Looijenga* [6]): *For a two-dimensional complete intersection,* $\mu \geqslant \tau + b$; *in particular,* $\mu \geqslant \tau$.

COROLLARY 2.9: *For a two-dimensional smoothable Gorenstein singularity, write* $\mu = \mu_0 + \mu_+ + \mu_-$, *from diagonalizing the intersection pairing on* $H_2(F; \mathbb{R})$. *Then*

$$\tau \geqslant \mu_0 + \mu_- = \mu - (2p_g - 2g - b).$$

PROOF: From Theorems 2.7 and 1.9, we deduce

$$\tau = \mu - (2p_g - 2g - b) + \gamma + 2q.$$

But $\mu_+ = 2p_g - 2g - b$ (e.g., [17], (1.5.1) and (3.13.d)).

REMARKS: (2.10.1) Corollary 2.9 is new even for hypersurfaces, and gives a lower bound for the minimum value of τ in a μ-constant family (see §4).

(2.10.2) We will show below that a Gorenstein surface singularity is quasi-homogeneous iff $b = \alpha = \beta = \gamma = 0$; if $(X, 0)$ is smoothable, Theorem 2.7 says this is equivalent to the condition μ = dimension of a smoothing component.

(2.10.3) In §4, we use Theorem 2.7 to compute α, β, γ, and the irregularity q in many cases.

§3. The main theorem

(3.1) Our goal in this section is the converse to Corollary 1.11 in the Gorenstein case.

THEOREM 3.2: *Let* $(X, 0)$ *be a Gorenstein surface singularity. Then* $\alpha = \beta = \gamma = 0$ *iff* $(X, 0)$ *is* q.-h. $(b = 0)$ *or* $(X, 0)$ *is a cusp* $(b = 1)$.

COROLLARY 3.3: *Let* $(X, 0)$ *be a complete intersection surface singularity. Then* $\mu \geqslant \tau$, *and* $\mu = \tau$ *iff* $(X, 0)$ *admits a good* $\mathbb{C}^*$*-action.*

(3.4) Corollary 3.3 follows from Theorem 2.7 and 3.2; in fact, $\mu > \tau + \mathrm{b}$ excepting the q.-h. and cusp cases. That $\mu = \tau$ if $(X, 0)$ is q.-h. was proved for complete intersections in any dimension by Greuel [3]; in dimension 2, it follows from Theorem 2.7 and Corollary 1.11.

(3.5) We shall always work on the MGR $\tilde{X} \to X$. Denote by $S = S_{\tilde{X}}$ the sheaf of derivations of $\tilde{X}$, logarithmic along E. Thus,

$$S = \left(\Omega^1_{\tilde{X}}(\log E)\right)^*, \tag{3.5.1}$$

and $\Lambda^2 S \simeq \mathcal{O}_{\tilde{X}}(Z - E)$, hence ("duality")

$$S \simeq \Omega^1_{\tilde{X}}(\log E)(Z - E). \tag{3.5.2}$$

S is also defined by

$$0 \to S \to \Theta_{\tilde{X}} \to \oplus \mathcal{O}_{E_i}(E_i) \to 0,$$

so $H^0(S) = H^0(\theta_{\tilde{X}})$. By equivariance of the MGR, $H^0(S) = H^0(U, S)$; combining with (3.5.2) gives

$$H^0\left(\Omega^1_{\tilde{X}}(\log E)(Z - E)\right) = H^0\left(U, \Omega^1_{\tilde{X}}\right). \tag{3.5.3}$$

(This gives a bound on the polar order along E of 1-forms on U). If C is one of the smooth E_i's, there is a natural exact sequence (cf. [14], 1.10.2)

$$0 \to \mathcal{O}_C \to S \otimes \mathcal{O}_C \to \Theta_C(-(E - C)) \to 0. \tag{3.5.4}$$

Our first task will be to locate a C so that the image of $H^0(S) \to H^0(S \otimes \mathcal{O}_C)$ contains the global section of (3.5.4). We start with a generally known

LEMMA 3.6: *On the minimal good resolution of a Gorenstein singularity, let* $Z = \Sigma r_i E_i$. *Excluding the* RDP *case, all* $r_i \geqslant 1$, *and* $r_i = 1$ *implies either*

a) E_i *is rational, with at most* 2 *intersection points with other curves.*

b) $Z = E$ *is one non-singular elliptic curve, and* $(X, 0)$ *is simple elliptic.*

PROOF: See e.g. ([2], 4.6), for a proof due to M. Reid of the fact that $r_i \geqslant 1$. If some $r_i = 1$, then

$$-K \cdot E_i = Z \cdot E_i = 2 - 2g_i + E_i \cdot E_i = E_i \cdot E_i + \Sigma r_j,$$

the last sum being over the intersection points with neighbors of E_ι. The result follows.

LEMMA 3.7: *Consider the MGR of a Gorenstein singularity which is not an RDP, simple elliptic, or a cusp singularity. Then there is a smooth E_ι with $Z \geqslant E + E_\iota$, for which $g_\iota > 0$, or $g_\iota = 0$ and E_ι has $\geqslant 3$ intersections with other curves.*

PROOF: This is a simple consequence of lemma 3.6. For, the only graphs in which all curves are of type 3.6. a) are chains of rational curves (from cyclic quotient singularities) and cycles of rational curves (from cusps). The only Gorenstein quotient singularities, however, are the RDP's.

(3.8) For a cusp singularity, $p_g = b = 1$, $g = 0$, and there is no $\mathbb{C}^*$-action. A simple elliptic singularity always admits a $\mathbb{C}^*$-action. Let us exclude these cases, and assume

$$C \text{ is an exceptional curve with } Z \geqslant E + C, \text{ and } g(C) > 0 \text{ or}$$
$$g(C) = 0 \text{ and has at least 3 intersection points withother curves.} \tag{3.8.1}$$

(3.9) If Y is any effective exceptional divisor, then the dualizing sheaf on Y is an invertible sheaf given by

$$\omega_Y = K_{\tilde{X}} \otimes \mathcal{O}_Y(Y) = \mathcal{O}_Y(Y - Z). \tag{3.9.1}$$

Riemann-Roch says that for any locally free F on Y,

$$h^0(F) = h^1(\omega_Y \otimes F^*). \tag{3.9.2}$$

LEMMA 3.10: *Suppose C is a smooth exceptional curve with $Z \geqslant C + E$. Then*

a) *$Z = C + E$ implies $p_g = g + b + 1$*
b) *If $Z > C + E$, then $h^1(\mathcal{O}_{Z-C-E}(-E)) = p_g - g - b - 1$.*

PROOF: By Grauert-Riemenscheider, $h^1(\mathcal{O}_{\tilde{X}}(-Z)) = 0$, so $p_g = h^1(\mathcal{O}_Z)$. By (3.9.1), $\omega_Z \simeq \mathcal{O}_Z$, $\omega_{Z-C} \simeq \mathcal{O}_{Z-C}(-C)$. Thus (3.9.2) $p_g = h^0(\omega_Z) = h^1(\mathcal{O}_Z)$. From

$$0 \to \mathcal{O}_{Z-C}(-C) \to \mathcal{O}_Z \to \mathcal{O}_C \to 0$$

we get $p_g = 1 + h^0(\mathcal{O}_{Z-C}(-C)) = 1 + h^0(\omega_{Z-C}) = 1 + h^1(\mathcal{O}_{Z-C})$. If $Z = C + E$, then $h^1(\mathcal{O}_{Z-C}) = h^1(\mathcal{O}_E) = g + b$, whence a). Otherwise, consider

$$0 \to \mathcal{O}_{Z-C-E}(-E) \to \mathcal{O}_{Z-C} \to \mathcal{O}_E \to 0.$$

Since $H^0(\mathcal{O}_{Z-C})$ maps onto the constant functions $H^0(\mathcal{O}_E) = \mathbb{C}$, we have

$$h^1(\mathcal{O}_{Z-C-E}(-E)) = h^1(\mathcal{O}_{Z-C}) - h^1(\mathcal{O}_E)$$
$$= p_g - 1 - g - b,$$

as desired.

LEMMA 3.11: *Suppose* C *is a curve satisfying* (3.8.1). *Then* $\alpha = \beta = \gamma = 0$ *implies*

$$H^0(S) \twoheadrightarrow H^0(S \otimes \mathcal{O}_C) \simeq \mathbb{C}$$

is surjective.

PROOF: The hypothesis on C implies $H^0(\Theta_C(-(E-C))) = 0$, so (3.5.4) $h^0(S \otimes \mathcal{O}_C) = 1$. We must show

$$H^0(S(-C)) \subset H^0(S)$$

has cokernel of dimension $u \geqslant 1$. By (3.5.2) and (3.5.3), this inclusion is

$$H^0\big(\Omega^1_{\tilde{X}}(\log E)\big)(Z-C-E)\big) \subset H^0\big(U, \Omega^1_{\tilde{X}}\big). \qquad (3.11.1)$$

So, if we let v be the dimension of the cokernel of

$$H^0\big(\Omega^1_{\tilde{X}}(\log E)\big) \subset H^0\big(\Omega^1_{\tilde{X}}(\log E)(Z-C-E)\big), \qquad (3.11.2)$$

we see that $u + v = q$ (recall (1.3.b) and the definition of q). By Theorem 1.9 and the hypotheses, $q = p_g - g - b$; we want $u \geqslant 1$, so we must show $v \leqslant p_g - g - b - 1$.

If $Z = C + E$, then $v = 0$. Lemma 3.10 a) yields $p_g = g + b + 1$, so the desired inequality is true.

Assume $Z > C + E$. By (3.11.2),

$$v \leqslant h^0\big(\Omega^1_{\tilde{X}}(\log E)(Z-C-E) \otimes \mathcal{O}_{Z-C-E}\big); \qquad (3.11.3)$$

by (3.9.1), (3.9.2), and (3.5.2), this last dimension is

$$h^1\big(\Omega^1_{\tilde{X}}(\log E) \otimes \mathcal{O}_{Z-C-E}(-E)\big).$$

Now consider the exact sequence (1.7), with $Y = Z - C - E$:

$$0 \to \mathcal{O}_{Z-C-E}(-E) \to \Omega^1_{\tilde{X}}(\log E) \otimes \mathcal{O}_{Z-C-E}(-E)$$
$$\to \Omega^2_{\tilde{X}} \otimes \mathcal{O}_{Z-C-E} \to 0.$$

As $H^1(\Omega^2_{\tilde{X}}) = 0$, so is H^1 of the last term; combining with (3.11.3) yields

$$v \leqslant h^1(\mathcal{O}_{Z-C-E}(-E)).$$

Applying lemma 3.10 b) gives $v \leqslant p_g - g - b - 1$, as desired.

(3.12) Continuing with C as above let $D \in H^0(S)$ map onto the natural section of $S \otimes \mathcal{O}_C$. Choose analytic local coordinates (x, y) at a point of C, so that in this patch, $C = E$ is given by $x = 0$. Let A be the ring of local holomorphic functions. D, being logarithmic along E, is given locally by

$$D = fx\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}, \quad f, g \in A.$$

In terms of (3.5.4), the image of D in $H^0(\Theta_C(-(E-C))) = 0$ is $g(0, y)\partial/\partial y$; thus $g = xh$, some $h \in A$. As D induces a non-0 section of $S \otimes \mathcal{O}_C$, we have $f(0, y)$ is a unit, so $f \in A$ is a unit.

LEMMA 3.13: *$D \in H^0(S)$ as above induces an injection of $(x^n)/(x^{n+1})$ into itself, for all integers $n \geqslant 1$.*

PROOF: Clearly, D: $A \to A$ sends the ideal (x^n) into itself. Suppose $D(x^n p) \in (x^{n+1})$, some $p \in A$. As $D = fx\partial/\partial x + hx\partial/\partial y$, we deduce $nfx^n p \in (x^{n+1})$. Since f is a unit, p must be in (x), as needed to complete the proof.

(3.14) Since $D \in H^0(S) = H^0(\Theta_{\tilde{X}}) = \Theta_X$, D gives a derivation (still denoted D) of the complete local ring R of X. Let $I_n \subset R$ be the ideal of all functions on R vanishing to order $\geqslant n$ on C (when the function is considered on $\tilde{X}$). As vanishing order can be determined in any patch, we have

$$I_n \subset (x^n)$$

$$I_n \cap (x^{n+1}) = I_{n+1}$$

so that $I_n/I_{n+1} \subset (x^n)/(x^{n+1})$. Certainly $D(I_n) \subset I_n$, since D is logarithmic along E; so, by (3.13), D induces an injection on I_n/I_{n+1}.

LEMMA 3.15: *$D \in \Theta_R$ is a non-nilpotent derivation, i.e. $\overline{D}$: $m/m^2 \to m/m^2$ (m = maximal ideal of R) is a non-nilpotent linear map.*

PROOF: Choose the largest r so that $m = I_r$, and pick $g \in I_r - I_{r+1}$. (Note $m^2 \subset I_{r+1}$). Then $D^k g \notin I_{r+1}$, all k, by (3.14). Thus, if $\bar{g} \in m/m^2$ is the image of g, then $\overline{D}^k \bar{g} \neq 0$, all k.

(3.16) The proof of Theorem 3.2 is now a consequence of Lemma 3.15 and the Theorem of G. Scheja and H. Wiebe [12] that a complete normal domain of dimension 2 admits a good $\mathbb{C}^*$-action if there exists a non-nilpotent derivation.

§4. Calculating α, β, γ, and q

(4.1) In this chapter, we shall calculate α, β, γ, and q for some hypersurface singularities which are fibres in a positive weight deformation of a quasi-homogeneous singularity. We can thus compute μ, p_g, g, and $b = 0$ at the beginning, and use different values for τ in the family to compute (with the aid of Theorem 2.7 and the inequalities of (1.10)) the other invariants.

(4.2) Suppose $R = \mathbb{C}[x, y, z]/f$ is q.-h., with $wt\ x = w_1$, $wt\ y = w_2$, $wt\ z = w_3$, and $wt\ f = d$; these positive integers are to have no common factors. Let us define

$$l = d - \Sigma w_i.$$

It is known that $l < 0$ iff R is an RDP, $l = 0$ iff R is simple elliptic. The Jacobian algebra

$$J = \mathbb{C}[x, y, z]/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

is graded with $J = \oplus_0^N J_i$, where $N = 2l + d$. J_N has dimension 1, and is spanned by the Hessian of f. Multiplication followed by projection to the Hessian component gives, by local duality, a perfect pairing

$$J \times J \to J_N \simeq \mathbb{C}.$$

Lemma 4.3: *Considering the singularity $X =$ Spec R, we have*

$$p_g = \dim \bigoplus_{i \leqslant l} J_i$$

$$g = \dim J_l$$

$$\mu = \prod_i \left(\frac{d}{w_i} - 1\right).$$

Proof: See [18], or use [8]: for, if $i \leqslant l$, then $\dim J_i = \dim R_i$ ($R = \oplus_{i=0}^{\infty} R_i$), and R_i can be expressed as the global sections of some line bundle L_i on the curve $C = \text{Proj } R$. Then $L_l = K_C$, $L_i \otimes L_{l-i} = K_C$, $0 \leqslant i \leqslant l$; use Serre duality on C and Theorem 5.7 of [8] to complete the calculation of p_g (and g). The calculation of μ is well-known.

EXAMPLE 4.4: Consider the (bimodular) family

$$X_{s,t}\colon z^2+y^3+x^{10}+sx^7y+tx^8y=0.$$

Calculating with $s=t=0$, we find weights (3, 10, 15; 30), so $l=2$, $p_g=1$, $g=0$, $\mu=18$. Calculating $\tau=\dim J/(f)$,

$$\tau_{s,t}=\begin{cases}18 & (s,t)=(0,0)\\ 17 & s=0,\ t\neq 0\\ 16 & s\neq 0\end{cases}$$

As $g=0$, $\beta=0$; so (2.7)

$$18-\tau=2\alpha+\gamma.$$

By (1.9), $q=1-\alpha-\gamma\geqslant 0$. We find in the respective 3 cases above: $(\alpha,\gamma)=(0,0)$, (0, 1), and (1, 0). Note that $q=0$ except in the first (q.-h.) case.

(4.5) One can imitate the last example, by using the stratification by $\tau=\dim T^1$ on the deformations of positive weight on a q.-h. R. In particular, one is interested in finding the minimum value of τ, i.e., τ for a generic positive weight deformation. This is a problem of Zariski in the case of curves ([22], see also Teissier's appendix).

EXAMPLE 4.6: Consider R defined by $z^2+x^{2a+1}+y^{2a+2}=0$. Using (4.3), one calculates $p_g=a(a-1)/2$, $g=0$, $\mu=2a(2a+1)$. Thus, for any positive weight deformation, $\beta=0$ and $q=a(a-1)/2-\alpha-\gamma\geqslant 0$. The minimum value of τ is the same as for the positive weight deformations of the curve $x^{2a+1}+y^{2a+2}=0$; according to [22], p. 126, this value is $\tau_{\min}=3\mathrm{a}(\mathrm{a}+1)$. [This is also $\mu-2p_g$, as in (2.9)]. So, for these singularities,

$$2\alpha+\gamma=\mu-\tau=a(a-1).$$

Therefore, $\gamma=0$ and $\alpha=a(a-1)/2$, whence these singularities have $q=0$.

EXAMPLE 4.7: Consider R defined by $x^n+y^n+z^n=0$, $n\geqslant 4$. Then $p_g=\binom{n}{3}$, $g=\binom{n-1}{2}$, $\mu=(n-1)^3$, and $p_g-g=\binom{n-1}{3}$. So,

$$q=\binom{n-1}{3}-\alpha-\beta-\gamma\geqslant 0.$$

A long calculation similar to [22], p. 114–127 yields the minimum τ for deformations of positive weight:

$$\tau_{\min}=(2n-3)(n+1)(n-1)/3.$$

So,

$$\mu - \tau_{\min} = 2\binom{n-1}{3} = 2\alpha + 2\beta + \gamma.$$

Therefore, for these singularities with minimum τ, $\gamma = 0$ and $\alpha + \beta = \binom{n-1}{3}$; in particular, $q = 0$. Using $\beta \leqslant g$ and Corollary 1.10c), we see

$$\binom{n-2}{2}\beta \leqslant \binom{n-1}{2}.$$

We do not know if α and β are constant on the $\tau_{\min}$-stratum. Note that again $\tau_{\min} = \mu - 2(p_g - g)$.

§5. The irregularity q and deformation theory

(5.1) Let $(X, 0)$ be a normal surface singularity, $\tilde{X} \to X$ the MGR. Denote by Def the functor (on artin rings) of deformations of $(X, 0)$; it is (weakly) represented by the semi-universal deformation space, denoted $\overline{\text{Def}}$. $\overline{\text{Def}}$ may be thought of as a local analytic space, or even (since $(X, 0)$ is algebraic) as the spectrum of an algebraic local ring. In [15], we introduced the equisingular functor ES of deformations of X to which all E_i lift, and which blow down to deformations of $\tilde{X}$. ES has a good deformation theory, and a key theorem [14] is that ES $\to$ Def is an injection. Thus, there is a closed subspace $\overline{\text{ES}}$ of $\overline{\text{Def}}$ which represents ES (on the category of artin rings - we make no convergence assertions). If $(X, 0)$ is q.-h., it is proved in [18] that ES is the functor of deformations of $(X, 0)$ to which the weight filtration lifts; one may think of $\overline{\text{ES}}$ as that part of $\overline{\text{Def}}$ of weight $\geqslant 0$ (see also Pinkham [9]).

(5.2) Now suppose $(X, 0)$ is Gorenstein, with $K_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}(-Z)$. As $H^1(\mathcal{O}_{\tilde{X}}(-Z)) = 0$, there is a subfunctor $TR_Z \subset$ ES, introduced in [16]. TR_Z consists of deformations of $\tilde{X}$ to which all E_i lift, in such a way that the induced deformation of the divisor Z (as a scheme) is trivial, i.e., a product. (Thus, some infinitesimal neighborhood of E is held fixed). TR_Z is represented by a closed subscheme of $\overline{\text{ES}}$.

THEOREM 5.3: *Let $(X, 0)$ be a Gorenstein surface singularity. Then there exists a smooth q-dimensional equisingular family in* $\overline{\text{Def}}$; *specifically*, TR_Z *is a smooth functor*, *of dimension q.*

PROOF: Theorem 4.10 of [16] implies TR_Z is smooth, of dimension $h^1(S) - h^1(\Theta_Z)$. By the exact sequence

$$0 \to \Theta_{\tilde{X}}(-Z) \to S \to \Theta_Z \to 0,$$

this dimension is that of

$$\operatorname{Im}\left(H^1(\Theta_{\tilde{X}}(-Z)) \to H^1(S)\right).$$

By the main theorem of [14], $H^1_E(S) = 0$, so

$$H^1(S) \subset H^1(U, S) = H^1(U, \Theta_{\tilde{X}}(-Z)).$$

Thus, the desired dimension is that of

$$\mathrm{Im}\big(H^1(\Theta_{\tilde{X}}(-Z)) \to H^1(U, \Theta_{\tilde{X}}(-Z)\big).$$

Now, $\Theta_{\tilde{X}}(-Z) = \Omega^1_{\tilde{X}}$, and the dimension of

$$\mathrm{Im}\big(H^1(\Omega^1_{\tilde{X}}) \to H^1(U, \Omega^1_{\tilde{X}})\big)$$

is equal to $h^1(\Omega^1_{\tilde{X}})$-dim(kernel). By (1.4.1) and (1.9.2), this quantity is

$$k + p_g - g - b - \alpha - \beta - (k + \gamma),$$

which equals the irregularity q by Theorem 1.9.

COROLLARY 5.4: *Let* $(X, 0)$ *be a* q.-h. *Gorenstein surface singularity, not an* RDP. *Then* $(X, 0)$ *admits a non-trivial one-parameter equisingular deformation.*

PROOF: It follows from §3 that unless $(X, 0)$ is simple elliptic, $p_g \geqslant g + 1$, whence (1.11) $q > 0$; the theorem implies the result. If $(X, 0)$ is simple elliptic, the result is well-known.

REMARK (5.5): It can be shown [18] that if R is Gorenstein and q.-h., then the tangent space of TR_Z is

$$\bigoplus_{i > l} T^1(i),$$

where l was defined in (4.2) for a hypersurface, and as in the proof of lemma 4.3 in general. In fact,

$$p_g = \bigoplus_{i \geqslant l} T^1(i),$$

and this space is the tangent space of another equisingular functor.

(5.6) In discussing the irregularity q of a Gorenstein surface singularity, S.S.-T. Yau conjectured [20] that $q > 0$, except in a few special cases (e.g., $p_g \leqslant 1$). In the q.-h. case, then $q > 0$, except for an RDP or simple elliptic singularity, as already observed by Yau. But, as the examples of §4 indicate, Yau's conjecture is incorrect; instead we ask practically the opposite:

QUESTION 5.7: Let R be a Gorenstein surface singularity. Is there an equisingular deformation (as in 5.1) to one with irregularity 0?

THEOREM 5.8: *The answer to* (5.7) *is affirmative if* $p_g \leqslant g+1$.

PROOF: As $q = p_g - g - b - \alpha - \beta - \gamma$, $q > 0$ would imply $q = 1$, $p_g = g + 1$, and $b = \alpha = \beta = \gamma = 0$. By Theorem 3.2, R is q.-h. By Theorem 5.3, Spec R has a one-parameter equisingular deformation, obtained from blowing down an equisingular deformation of $\tilde{X}$ for which the induced deformation of Z is trivial. We will show the general fibre is not a q.-h. singularity, so $\alpha + \beta + \gamma > 0$ there, so $q = 0$ there.

The general fibre in this family has the same resolution graph as R, and the same central curve C (up to analytic isomorphism) and the same interesection points on C with the other exceptional curves (because E has been deformed trivially). Further, since $Z \geqslant 2C$ (3.6), the isomorphism class of the normal bundle of C is the same, for special and general fibre. But, e.g., by [8], these data determine a q.-h. singularity up to isomorphism. Since R cannot appear as a singularity in the general fibre of a deformation in $\overline{\text{Def}}$, the general fibre of our family cannot be q.-h.

REMARK (5.9): Theorem 5.8 applies to all minimally elliptic singularities,as well as to Gorenstein singularities whose resolution graph is a smooth curve C of genus $\geqslant 2$, with $K_C =$ conormal bundle. Yau's Example 2.8 of [20] (p. 836) is incorrect; the error is that (in that notation) a derivation in $H^0(S(-A))$ will by definition have vanishing order at least 2 along A (and not 1, as claimed).

QUESTION 5.10: Is the irregularity q a semi-continuous invariant?

(5.11) The adjacencies of a singularity are usually described in terms of resolution diagrams (or equisingularity types), without distinguishing analytic type. If (5.10) is affirmative, this would give a helpful way of specifying analytic types.

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Note added in proof: According to a recent preprint of Steenbrink and van Straten, the answer to Question (5.10) is negative.