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## CONVEX POLYTOPES AS MATRIX INVARIANTS

Gerard Sierksma and Klaas de Vos

### Abstract

For a convex polytope  $P$  which is the convex hull of a finite number of points, the set  $\pi(P)$  consists of all real square matrices  $A$  such that  $AP \subset P$ , i.e. that leave  $P$  invariant. In this paper the extremals of  $\pi(P)$  are characterized for  $P$  being a convex simplex, and the number of its extremes is determined.

### 1. Introduction

In Berman and Plemmons [2] the first chapter deals with matrices that leave a cone invariant, i.e.  $\pi(K) = \{A \in \mathbb{R}^{d \times d} \mid AK \subset K\}$  with  $K$  a cone in  $\mathbb{R}^d$ . An extensive bibliography on properties of  $\pi(K)$  can be found in this book. In e.g. Tam [8] it is shown that  $\pi(K)$  is a polyhedral cone if  $K$  is a polyhedral cone. One of the main problems is to characterize the extremals of such a polyhedral cone  $\pi(K)$ ; see e.g. Adin [1]. Instead of taking a cone as matrix invariant we consider in this paper convex polytopes, with a convex polytope being the convex hull of a finite nonempty set of points in  $\mathbb{R}^d$ ; see e.g. Eggleston [4] and Sierksma [6]. In Valentine [9] the term convex polyhedron is used. In a recent paper by Elsner [3] is also deviated from the idea of using cones; here so-called nontrivial convex sets are used as matrix invariants. In this paper we restrict ourselves mainly to convex simplices  $S_0$  with one vertex at the origin. By a convex simplex  $P$  in  $\mathbb{R}^d$  we mean the convex hull of  $d+1$  points in  $\mathbb{R}^d$  with nonempty interior. We shall characterize the extremes of  $\pi(S_0)$  and calculate its number. In general, we define for  $X \subset \mathbb{R}^d$  the set of matrices

$$\pi(X) = \{A \in \mathbb{R}^{d \times d} \mid AX \subset X\}.$$

Note that if  $X = \{0\}$ , then  $\pi(X) = \mathbb{R}^{d \times d}$ . If  $X = \{x\}$  with  $x \neq 0$ , then  $\pi(X)$  consists of all  $(d, d)$ -matrices with eigenvector  $x$  and eigenvalue 1. Before restricting ourselves to convex polytopes we give the following result for arbitrary sets. Note that if  $X$  is convex then so is  $\pi(X)$ . By cone  $X$  we mean the convex cone generated by  $X$ , i.e. all nonnegative linear combinations of  $X$ . The set cone  $X$  is also denoted by  $X^G$ ; see [2].

**THEOREM 1:** *Let  $X \subset \mathbb{R}^d$ . Then the following holds*

- (a)  $\text{cone } \pi(X) \subset \pi(\text{cone } X)$ ;  
 (b)  $\text{cone } \pi(X) = \pi(\text{cone } X)$  if  $X$  is compact, convex and contains 0.

**PROOF:** (a) Take any  $A \in \text{cone } \pi(X)$ . Then there are matrices  $A_1, \dots, A_n \in \pi(X)$  such that  $A = \sum_{i=1}^n \lambda_i A_i$  with  $\lambda_i \geq 0$  for all  $i$ . Furthermore let  $x = \sum_{i=1}^k \mu_i x_i \in \text{cone } X$  with  $\mu_i \geq 0$  and  $x_i \in X$  for all  $i$ . Then it follows that  $Ax = A(\sum_{i=1}^k \mu_i x_i) = \sum_{i=1}^k \mu_i A x_i = \sum_{i=1}^k \mu_i (\sum_{j=1}^n \lambda_j A_j x_i) = \sum_{i,j} \lambda_j \mu_i A_j x_i \in \text{cone } X$ . Hence,  $A \in \pi(\text{cone } X)$ .

(b) Take any  $A \in \pi(\text{cone } X)$  and let  $X \neq \{0\}$ . Hence  $A(\text{cone } X) \subset \text{cone } X$ . As  $0 \in X$  and  $X$  convex it follows that for each  $x \in X$  there are  $\lambda, \mu > 0$  and  $y \in X$  such that

$$A(\lambda x) = \mu y,$$

or that  $(\lambda/\mu)Ax \in X$ . Let  $\lambda^*$  be the infimum of all  $\lambda/\mu$  over  $x$ . Then  $\lambda^* \neq 0$ , and  $(\lambda^* A)X \subset X$ . This means that  $\lambda^* A \in \pi(X)$ . As  $\pi(X)$  is convex and contains 0, it follows that  $A \in \text{cone } \pi(X)$ .

In Theorem 1(b) we have a sufficient condition in order to obtain equality in (a). Note that we also have equality if  $X = \text{cone } X$ , because in that case both  $X$  and  $\pi(X)$  are convex cones; see e.g. Berman and Plemmons [2]. The following example shows that equality does not hold in general in Theorem 1. Take  $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 0, x_1 + x_2 = 1\}$ . Then  $\text{cone } X = \mathbb{R}_+^2$ , and  $\pi(\text{cone } X)$  consists of all nonnegative (2,2)-matrices. On the other hand  $\pi(X)$  consists of all matrices

$$\begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}$$

with  $0 \leq a, b \leq 1$ , so  $\text{cone } \pi(X)$  consists of all nonnegative multiples of these matrices. Hence  $\pi(\text{cone } X) \neq \text{cone } \pi(X)$ .

In the following chapters we replace ‘‘cone’’ by ‘‘conv’’ and ‘‘extr’’, so we consider  $\pi(\text{conv } X)$ ,  $\text{conv } \pi(X)$  and  $\pi(\text{extr } X)$ ,  $\text{extr } \pi(X)$ .

## 2. Polytopes and simplices as matrix invariants

The main purpose of this chapter is to study the commutativity of  $\pi$  and  $\text{conv}$ , i.e.  $\pi(\text{conv } X) = \text{conv } \pi(X)$  for  $X$  a polytope. Clearly, if  $X$  is convex so is  $\pi(X)$ , and in that case we have  $\text{conv } \pi(X) = \pi(\text{conv } X)$ .

**THEOREM 2:** *For each  $X$  in  $\mathbb{R}^d$  the following holds*

$$\text{conv } \pi(X) \subset \pi(\text{conv } X).$$

PROOF: Take any  $X \subset \mathbb{R}^d$ . Clearly  $\pi(\text{conv } X)$  is convex in  $\mathbb{R}^{d \times d}$ . So we only have to show that  $\pi(X) \subset \pi(\text{conv } X)$ . Take any  $A \in \pi(X)$ . Then  $AX \subset X$ . Now let  $x \in A(\text{conv } X)$ . Then there are  $x_1, \dots, x_s \in X$  and  $\lambda_1, \dots, \lambda_s \geq 0$  with  $\lambda_1 + \dots + \lambda_s = 1$ , such that  $x = A(\sum_{i=1}^s \lambda_i x_i) = \sum_{i=1}^s \lambda_i Ax_i$ . As  $Ax_i \in X$  for each  $i$  we have  $x \in \text{conv } X$ , and it follows that  $A(\text{conv } X) \subset \text{conv } X$ . Hence,  $A \in \pi(\text{conv } X)$ .

Equality does not hold in general in the above theorem as is shown by the following example. Take  $X = \{(1,0), (0,1), (\frac{1}{2},0), (1,1), (0,0)\}$ . Then

$$\pi(X) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then  $\text{conv } X = \{(x_1, x_2) \mid 0 \leq x_1, x_2 \leq 1\}$ , and  $\pi(\text{conv } X)$  consists of all non-negative matrices with row sums  $\leq 1$ . On the other hand the (2,1)-th element of each matrix in  $\text{conv } \pi(X)$  is zero. Equality also does not hold, in general, in case  $X$  consists of the extremals of a convex cone  $X$ , i.e.  $X = \text{extr } K$ .

It is well-known that  $\text{conv } X = \text{conv}(\text{extr } K) = K$  (the Krein-Milman theorem). However, in general,  $\pi(\text{conv } X) = \pi(K) \neq \text{conv } \pi(\text{extr } K)$ . In order for  $\pi(K)$  being equal to  $\text{conv } \pi(\text{extr } K)$  it is therefore necessary that  $\text{extr } \pi(K) \subset \pi(\text{extr } K)$  which is a conjecture of Loewy and Schneider [5].

In the following we shall write  $0 \cup S$  instead of  $\{0\} \cup S$ . In the next theorem  $S_0 = \text{conv}(0 \cup S)$  is a convex simplex i.e. a convex simplex with one vertex at the origin and  $|S| = d$ . The theorem shows the commutativity of  $\pi$  and  $\text{conv}$  for  $0 \cup S$ . The proof of it is after Theorem 6.

**THEOREM 3:** *Let  $0 \cup S$  be the vertices of a convex simplex in  $\mathbb{R}^d$ . Then the following holds*

$$\pi(\text{conv}(0 \cup S)) = \text{conv } \pi(0 \cup S).$$

In order to prove this theorem we need a nonsingular transformation  $T: \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that

$$T(e_1, \dots, e_d) = S$$

with  $S$  as in the above theorem. We denote  $E_d = \{e_1, \dots, e_d\}$ , i.e. the set of unit vectors in  $\mathbb{R}^d$ .

**THEOREM 4:** *The following assertions are equivalent:*

- (i)  $A \in \pi(\text{conv}(0 \cup E_d))$ ;
- (ii)  $Ae_1, \dots, Ae_d \in \text{conv}(0 \cup E_d)$ ;
- (iii)  $A \geq 0$ , column sums of  $A \leq 1$ ;

PROOF: We shall show that (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i).

(i)  $\Rightarrow$  (ii): Let  $A \in \pi(\text{conv}(0 \cup E_d))$ , so  $A(\text{conv}(0 \cup E_d)) \subset \text{conv}(0 \cup E_d)$ . As  $e_i \in \text{conv}(0 \cup E_d)$ , it follows that  $Ae_i \in \text{conv}(0 \cup E_d)$  for each  $i = 1, \dots, d$ .

(ii)  $\Rightarrow$  (iii): As  $Ae_j \in \text{conv}(0 \cup E_d)$  for each  $j$ , it follows that the  $j$ -th column of  $A$  is equal to  $\sum_{i=1}^d \lambda_{ij} e_i$  with  $\lambda_{1j}, \dots, \lambda_{dj} \geq 0$  and  $\lambda_{1j} + \dots + \lambda_{dj} \leq 1$ , so the  $j$ -th column sum of  $A$  is equal to  $\sum_{i=1}^d \lambda_{ij}$  and is  $\leq 1$ . The matrix  $A$  is nonnegative, because all  $\lambda_{ij}$ 's are nonnegative.

(iii)  $\Rightarrow$  (i): Take any  $x \in \text{conv}(0 \cup E_d)$ . Then there are scalars  $\alpha_1, \dots, \alpha_d \geq 0$  with  $\alpha_1 + \dots + \alpha_d \leq 1$  such that  $x = \sum_{i=1}^d \alpha_i e_i$ . Hence,  $Ax = A(\sum_{i=1}^d \alpha_i e_i) = \sum_{i=1}^d \alpha_i Ae_i$ . As the column sums of  $A$  are  $\leq 1$ , it follows directly that  $Ae_i \in \text{conv}(0 \cup E_d)$  for each  $i = 1, \dots, d$ , and therefore we have  $Ax \in \text{conv}(0 \cup E_d)$ , and hence  $A \in \pi(\text{conv}(0 \cup E_d))$ .

Theorem 4 implies that all matrices in  $\pi(0 \cup E_d)$  have *Perron-Frobenius eigen-value*  $\leq 1$ ; this is the well-known Minkowski-theorem, see e.g. Sierksma [7].

LEMMA 5: Let  $T: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a nonsingular transformation and let  $X \subset \mathbb{R}^d$ . Then the following assertions hold:

- (a)  $\text{conv}(TX) = T(\text{conv } X)$ ,
- (b)  $\pi(TX) = T\pi(X)T^{-1}$ ;
- (c)  $\text{conv } \pi(TX) = T[\text{conv } \pi(X)]T^{-1}$ .

PROOF: (a) is left to the reader. In order to prove (b), take any  $A \in \pi(TX)$ . Hence,  $A(TX) \subset TX$ , and this implies that  $(T^{-1}AT)(X) \subset X$ , so that  $T^{-1}AT \in \pi(X)$ , or  $A \in T[\pi(X)]T^{-1}$ . The converse inclusion is shown similarly. To prove (c), take any  $A \in \text{conv } \pi(TX) = \text{conv}[T\pi(X)T^{-1}]$ . Then there are matrices  $B_1, \dots, B_k \in \pi(X)$ , and scalars  $\lambda_1, \dots, \lambda_k \geq 0$  with  $\lambda_1 + \dots + \lambda_k = 1$  such that  $A = \sum_{i=1}^k \lambda_i (TB_i T^{-1}) = T(\sum_{i=1}^k \lambda_i B_i)T^{-1}$ . As  $\sum_{i=1}^k \lambda_i B_i \in \text{conv } \pi(X)$ , it follows that  $A \in T[\text{conv } \pi(X)]T^{-1}$ . The other conclusion is also shown similarly.

The following theorem gives the commutativity of  $\pi$  and  $\text{conv}$  for  $0 \cup E_d$ .

THEOREM 6:  $\pi(\text{conv}(0 \cup E_d)) = \text{conv } \pi(0 \cup E_d)$ .

PROOF: According to Theorem 2, we only have to show that  $\pi(\text{conv}(0 \cup E_d)) \subset \text{conv } \pi(0 \cup E_d)$ . Take any  $A \in \pi(\text{conv}(0 \cup E_d))$ . Then  $A(\text{conv}(0 \cup E_d)) \subset \text{conv}(0 \cup E_d)$ . Theorem 4 then gives that  $A \geq 0$  and that all column sums of  $A$  are  $\leq 1$ . We must show now that  $A$  can be written as a convex combination of matrices from  $\pi(0 \cup E_d)$ . To show this, we first define  $A = A_1 = \{a_{ij}^{(1)}\}$ . Moreover, we define

$$\lambda_1 = \min_j \max_i a_{ij}^{(1)}.$$

and  $I_1$  is the matrix with precisely one 1 in the  $j$ -th column in the  $(i, j)$ -th position if  $a_{i,j}$  is the maximum in the  $j$ -th column of  $A_1$  (if there are more maxima in the  $j$ -th column choose one!) and zeroes otherwise ( $j = 1, \dots, d$ ). Then consider the matrix

$$A_2 = A_1 - \lambda_1 I_1,$$

with  $A_2 = \{a_{i,j}^{(2)}\}$  and define

$$\lambda_2 = \min \max a_{i,j}^{(2)}.$$

Also define  $A_3 = A_2 - \lambda_2 I_2 = A_1 - \lambda_1 I_1 - \lambda_2 I_2$ , where  $I_2$  is defined for the matrix  $A_2$  in the same way as  $I_1$  for  $A_1$ . Continuing this process we obtain, after at most  $d^2$  steps, the zero-matrix. So we obtain

$$0 = A_{d^2+1} = A - \lambda_1 I_1 - \lambda_2 I_2 - \dots - \lambda_{d^2} I_{d^2}.$$

Hence,  $A = \sum_{i=1}^{d^2} \lambda_i I_i$ . Clearly,  $A \geq 0$ , and each  $\lambda_i \geq 0$ . Note that a column of  $I_i$  becomes zero if the corresponding column of  $A_i$  is zero. If after  $d^2 - 1$  steps there still is some nonzero element in  $A_{d^2-1}$  we have, say in the  $j$ -th column,

$$a_{i,j} + \dots + a_{d,j} - (\lambda_1 + \dots + \lambda_{d^2}) = 0,$$

or  $\sum_{i=1}^{d^2} \lambda_i = \sum_{i=1}^d a_{ij} \leq 1$ . And this means that in fact  $A \in \text{conv } \pi(0 \cup E_d)$ .

The number of steps in the proof of the above theorem is, in general  $\leq d^2$ . Question: under what conditions is the number of steps equal to  $d^2$ ?

**PROOF OF THEOREM 3:** First note that  $0 \cup S$  is the set of vertices of a simplex, so  $|S| = d$ . We must show that

$$\pi(\text{conv}(0 \cup S)) \subset \text{conv } \pi(0 \cup S).$$

Clearly, there is a nonsingular transformation  $T: \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that  $S = TE_d = T(e_1, \dots, e_d)$ . According to Lemma 5 and Theorem 6 we have  $\pi(\text{conv}(0 \cup S)) = \pi(\text{conv}(0 \cup TE_d)) = \pi(\text{conv } T(0 \cup E_d)) = \pi(T\text{conv}(0 \cup E_d)) = T(\pi(\text{conv}(0 \cup E_d)))T^{-1} = T(\text{conv } \pi(0 \cup E_d))T^{-1} = \text{conv } \pi(T(0 \cup E_d)) = \text{conv } \pi(0 \cup TE_d) = \text{conv } \pi(0 \cup S)$ .

### 3. The extremes of $\pi(S_0)$

In this chapter we shall characterize the extreme vertices of  $\pi(S_0)$  and determine their number. If  $P$  is a convex polytope then, in general,  $\pi(P)$  is not a polytope. For instance if we take the two points  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

then  $P = \text{conv} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$  is convex but  $\pi(P)$  is not: in  $\pi(P)$  there are matrices  $A = (a_{i,j})$  with  $a_{11} = -a_{12}$ , so  $a_{11}$  can be as large as possible which means that  $\pi(P)$  is not bounded, so certainly is not a convex polytope. The following theorem gives a sufficient condition in order to save the boundedness of  $\pi(P)$ .

**THEOREM 7:** *If  $X$  is a bounded set in  $\mathbb{R}^d$  with  $\text{intconv}(0 \cup X) \neq \emptyset$  then  $\pi(X)$  is bounded.*

**PROOF:** Suppose, to the contrary, that  $\pi(X)$  is not bounded. Then there is a sequence of matrices  $A_k$  in  $\pi(X)$  such that one of the elements, say the  $(i, j)$ -th element, of the  $A_k$ 's goes to infinity. As the interior of  $\text{conv}(0 \cup X)$  is nonempty, there is an element  $y \in \text{intconv}(0 \cup X)$  with  $y_i \neq 0$ . Then  $y = \sum_{i=1}^s \lambda_i x_i$  with  $x_i \in X$ ,  $\lambda_i \geq 0$ , and  $\lambda_1 + \dots + \lambda_s \leq 1$ . Then  $(A_k y)_j \rightarrow \infty$  for  $k \rightarrow \infty$ . As  $A_k y = \sum_{i=1}^s \lambda_i A_k x_i$  is a finite sum, we have  $A_k x_i \rightarrow \infty$  for  $k \rightarrow \infty$  and for some  $i$ . As  $A_k x_i \in X$ , it follows that  $X$  is not bounded which is a contradiction. Therefore we have in fact that  $\pi(X)$  is bounded.

It is open question whether  $\pi(X)$  is a convex polytope in case  $X$  is a convex polytope in  $\mathbb{R}^d$  with  $\text{intconv}(0 \cup X) \neq \emptyset$ . Question: Is the number of extreme vertices of  $\pi(X)$  less then or equal to  $(d+1)^d$  (see the following theorem)?

**THEOREM 8:** *Let  $0 \cup S$  be the vertices of a convex simplex. Then the following holds:*

$$|\pi(0 \cup S)| = |\pi(0 \cup E_d)| = (d+1)^d.$$

**PROOF:** There is a nonsingular transformation  $T: \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that  $0 \cup S = T(0 \cup E_d)$ , so we have directly that  $|\pi(0 \cup S)| = |\pi(0 \cup E_d)|$ . We only have to show that

$$|\pi(0 \cup E_d)| = (d+1)^d.$$

First note that  $A0 = 0$  for each  $A \in \pi(0 \cup E_d)$ . The problem of determining the number of matrices in  $\pi(0 \cup E_d)$  is therefore equivalent to the problem of finding the number of bipartite graphs  $(G, H)$  on  $2(d+1)$  vertices with  $|G| = |H| = d+1$ , with one edge fixed, and such that the degree of each vertex in  $G$  is 1. Let there be a fixed edge between  $a \in G$  and  $b \in H$ . Then there are for each vertex  $\neq a$  in  $G$  precisely  $d+1$  possibilities in  $H$ . This holds for all of the vertices in  $G$  that are  $\neq a$ . So the number of such bipartite graphs is equal to

$$\frac{(d+1) \times \dots \times (d+1)}{d \text{ times}}.$$

Hence,  $|\pi(0 \cup E_d)| = (d + 1)^d$ .

To illustrate the above theorem we consider the following example. Let  $d = 2$  and  $S_0 = \text{conv}\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \text{conv}(0 \cup E_2)$ . Then we have

$$\pi(0 \cup S) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}, \text{ and}$$

$$\pi(S_0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \geq 0, a + c \leq 1, b + d \leq 1 \right\}.$$

Note that  $|\pi(0 \cup S)| = 3^2 = 9$ . In order to characterize the extreme vertices of  $S_0$  we need the following two lemmas.

LEMMA 9: *Let  $X \subset \mathbb{R}^{d \times d}$  be convex and  $T: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be nonsingular. Then the following holds:*

$$\text{extr}(TXT^{-1}) = T(\text{extr } X)T^{-1}$$

PROOF: Take any  $A \in \text{extr}(TXT^{-1})$ . Hence,  $A \notin \text{conv}(TXT^{-1}) \setminus \{A\}$ . Suppose, to the contrary, that  $A \notin T(\text{extr } X)T^{-1}$ . We shall show first that  $T^{-1}AT \in \text{extr } X$ , or that  $T^{-1}AT \notin \text{conv } X \setminus \{T^{-1}AT\}$ . Taking  $T^{-1}AT \in \text{conv } X \setminus \{T^{-1}AT\}$ , there should exist matrices  $B_1, \dots, B_k \in X$ , all  $\neq T^{-1}AT$ , and scalars  $\lambda_1, \dots, \lambda_k \geq 0$  with  $\lambda_1 + \dots + \lambda_k = 1$ , such that  $T^{-1}AT = \sum_{i=1}^k \lambda_i B_i$ , or  $A = \sum_{i=1}^k \lambda_i TB_i T^{-1}$ , and  $B_i \neq T^{-1}AT$  for all  $i$ . Because  $TB_i T^{-1} \in \text{conv}(TXT^{-1})$  for all  $i$ , we have  $A \in \text{conv}(TXT^{-1}) \setminus \{A\}$ , hence  $A \notin \text{extr}(TXT^{-1})$ , and this is a contradiction. Therefore we have,  $T^{-1}AT \in \text{extr } X$ , and this means that  $A \in T(\text{extr } X)T^{-1}$ . The converse can be shown similarly.

LEMMA 10:  $\text{extrconv } \pi(0 \cup E_d) = \pi(0 \cup E_d)$ .

PROOF: As all columns of the matrices in  $\pi(0 \cup E_d)$  consists of zero or unit vectors, no such a matrix can be written as a convex combination of the other ones in  $\pi(0 \cup E_d)$ .

The next theorem characterizes the extreme vertices of  $\pi(S_0) = \pi(\text{conv}(0 \cup S))$ ; they are precisely the matrices that leave the vertices invariant.

THEOREM 11:  $\text{extr } \pi(S_0) = \text{extr } \pi(\text{conv}(0 \cup S)) = \pi(0 \cup S)$ .

PROOF: Let  $T: \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a nonsingular transformation such that

$0 \cup S = T(0 \cup E_d)$ . Then we have

$$\begin{aligned}
 \text{extr } \pi(S_0) &= \text{extr } \pi(\text{conv}(0 \cup S)) = \text{extr } \pi(\text{conv}(T(0 \cup E_d))) \\
 &= \text{extrconv } \pi(T(0 \cup E_d)) = \text{extrconv}[T\pi(0 \cup E_d)T^{-1}] \\
 &= \text{extr}(T[\text{conv } \pi(0 \cup E_d)]T^{-1}) \\
 &= T[\text{extrconv } \pi(0 \cup E_d)]T^{-1} = T[\pi(0 \cup E_d)]T^{-1} \\
 &= \pi(T(0 \cup E_d)) = \pi(0 \cup S).
 \end{aligned}$$

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