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Ziv Ran

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THE STRUCTURE OF GAUSS-LIKE MAPS

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Holomorphic Gauss maps have classically been attached to subvarieties of projective spaces and complex tori. If $X \subset Y$ is such a subvariety, the Gauss map $\Gamma = \Gamma_X$ assigns to $x \in X$ the "embedded tangent space" to X at x, suitably interpreted. More generally, a Gauss map can be attached to any immersion into a homogeneous space, and still more general cases are discussed below. It is natural to inquire as to the structure of Γ, in particular the dimension of its fibres. As for the generic fibre, the situation is well-understood, at least for subvarieties of projective spaces and complex tori. For surfaces in $\mathbb{P}^3$ it was apparently known classically that Γ fails to be generically finite only for cones and developable surfaces, and Griffiths and Harris [2] have extended this result to arbitrary subvarieties of $\mathbb{P}^n$, showing in particular that the only smooth subvarieties $X \subset \mathbb{P}^n$ whose Gauss map is not generically finite are the linear spaces. For $X \subset Y$, Y a complex torus, it is well-known (see [2] or [3]), that the generic fibre of Γ_X essentially coincides with the largest subtours $Y_0 \subseteq Y$ with respect to which X is invariant.

The "finer" structure of Γ, however, seems to be less well-known. It was only recently proved by Zak (see [1]) that the Gauss map of any smooth nonlinear subvariety of $\mathbb{P}^n$ is in fact *finite*. His proof uses the Fulton-Hansen connectedness theorem. For smooths subvarieties X of a complex torus Y, not invariant under any subtorus (i.e. such that Γ_X is generically finite), Ueno [3] has made a conjecture which is equivalent to the finiteness of Γ_X.

The purpose of this note is to observe that a certain class of maps which includes Gauss maps enjoys very strong structural properties. In particular, when proper, not only are they, up to a Stein factorization, smooth and flat, but they are in fact (analytically) locally trivial fibrations with a homogeneous space as fibre (see Proposition below). As applications we obtain generalizations of Zak's Theorem and Ueno's conjecture (Cor. 1, 2), which imply the ampleness of certain line bundles, as well as bounds on the dimensions of "linear"subvarieties of X. Other applications include a bound on the size of the poles of meromorphic

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1-forms (Cor. 3), and some information about morphisms defined in a neighborhood of a subvariety of a homogeneous space (Cor. 4).

We shall be working in the complex-analytic category, which is more general. But the proof goes through – in fact becomes somewhat simpler – in the algebraic category (over an algebraically closed field of characteristic 0). The complex spaces we consider will be reduced and irreducible.

DEFINITION: Let Y be a complex manifold. A set of *Gauss data* on Y consists of the following:

(i) An injection of vector bundles λ: $TY \to E$, where TY denotes the tangent bundle.

(ii) A surjection μ: $V \otimes L \to E$, where V is a finite-dimensional vector space, and L is a line bundle on Y generated by finitely many of its global sections. (1)

Given a set of Gauss data and an immersion Φ: $X \to Y$, a *Gauss map* Γ_Φ (sometimes (abusively) denoted Γ_X) is defined as follows: Let $d\Phi$: $TX \to \Phi^*TY$ denote the differential of Φ and let the following diagram be cartesian:

$$\begin{array}{ccccc} F & \longrightarrow & & \longrightarrow & V \times \Phi^*L \\ \downarrow & & & & \downarrow {\scriptstyle \Phi^*\mu} \\ TX & \xrightarrow{d\Phi} & \Phi^*TY & \xrightarrow{\sigma^*\lambda} & \Phi^*E \end{array}$$

Γ_Φ: $X \to G$ is the classifying map associated to the injection $F \otimes \Phi^*L^{-1} \to V \otimes \mathcal{O}_Y$. Here G is the Grassmannian $G(\dim X + \dim V - rkE, V)$.

EXAMPLES:

(1) Suppose TY is such that TY is generated by finitely many global sections. Then we have a surjection $V \otimes \mathcal{O}_Y \to TY$ with $\dim V < \infty$, whence, taking $E = TY$, a set of Gauss data. The corresponding Gauss maps are said to be *untwisted*.

(2) In particular, when Y is a complex torus, we obtain in this manner the usual Gauss map associated to an immersion Φ: $X \to Y$. In fact, if B is the tautological subbundle on the grassmannian, then $\Gamma^*B = TX$.

(3) When $Y = \mathbb{P}^n$, however, the procedure of Example 1 does not yields the usual Gauss map. To obtain the latter, take $E = TY$ and the natural map $V \otimes \mathcal{O}_Y(1) \to TY$ where $Y = \mathbb{P}(V)$. When Φ: $X \to Y$

(1) Note that if we work, as we can, in the algebraic category, then a vector bundle generated by global sections already implies it is generated by finitely many such. This is because of the noetherian nature of the Zariski topology.

is an immersion, we have the exact diagram

$$\begin{array}{ccccccc} 0 \to \mathcal{O}_X & \to & V \otimes \mathcal{O}_X(1) & \to & \Phi^* TY & \to & 0 \\ \| & & \uparrow & & \uparrow & & \\ 0 \to \mathcal{O}_X & \to & F & \to & TX & \to & 0 \end{array}$$

where $\mathcal{O}_X(1) = \Phi^*\mathcal{O}_Y(1)$ and the top row is the "Euler sequence". This shows that our $\Gamma = \Gamma_\Phi$ indeed coincides with the usual Gauss map. Moreover note that if B is the tautological subbundle on G, then $\det B = \mathcal{O}_G(-1)$, the dual of the line bundle defining the Plucker embedding. Since $\Gamma^* \det B = \det(F \otimes \mathcal{O}_X(-1)) = \det(F) \otimes \mathcal{O}_X(-m-1) = \det(TX) \otimes \mathcal{O}_X(-m-1)$ by (1), where $m = \det X$, we obtain the well known formula

$$\Gamma^*\mathcal{O}_G(1) = K_X \otimes \mathcal{O}_X(m+1) \tag{2}$$

where $K_X = \det(TX)^{-1}$ is the Canonical bundle.

For the purpose of determining their structure, the essential property of Gauss maps is given by the following:

OBSERVATION: *Let Γ: $X \to G$ be a gauss map, $Z \subset X$ a fibre of Γ. Then in some neighborhood of z, TX is generated by finitely many sections.*

To see this let $\Gamma = \Gamma_\Phi$, notations being as in the Definition. The tautological subbundle B is trivial in some neighborhood U of $\Gamma(Z)$, hence so is $F \otimes L^{-1}$ in $\Gamma^{-1}(U)$. Since L is generated by finitely many global sections, it follows that so is F, hence TX, in $\Gamma^{-1}(U)$.

Now our general result can be stated as follows:

PROPOSITION: *Let p: $X \to U$ be a morphism of a complex manifold to a complex space. Assume the tangent bundle TX is generated by finitely many global sections. Then*

(*i*) *If $Z \subset p^{-1}(u)$ is any compact subvariety of a fibre, then there exists a subvariety Z' of $p^{-1}(u)$, containing Z and smooth there, such that* $\dim Z' \leqslant \dim X - \dim p(X)$ [1] *and that the untwisted Gauss map Γ_Z(cf. Example 1) is constant (where defined). In particular* $\dim Z \leqslant \dim X - \det p(X)$ *and if $Z = p^{-1}(u)$ is compact then Γ_Z is constant.*

(*ii*) *If moreover p is proper and $p = g \circ p'$ is a Stein factorization, then p' is a locally trivial fibration with fibre a homogeneous space (homogeneous – space form, for short.)*

This will follow from:

LEMMA: *With the assumptions of the Proposition, part (i), if p is not constant than there exists a principal divisor $D = (f)$ defined in a neighbor-*

[1] The dimension of an arbitrary subset of V is defined to be that of its Zariski closure.

hood of Z, *containing* Z *and smooth there, such that* $\dim p(D) \geqslant \dim p(X) - 1$ *and that* Γ_D *is constant on* Z.

Let's see that the Lemma implies the Proposition. For part (i), letting D be as in the Lemma, note that the Lemma applies again with D, $p|_D$ in place of X, p. Continuing in this manner, we obtain a chain $D = D_1 \supset D_2 \supset \ldots$ with codim $D_k = k$. The only way this chain could stop is if $p|_{D_k}$ is constant for some k, i.e., $D_k \in p^{-1}(u)$. Since $\dim p(D_k) \geqslant \dim U - k =: n - k$, this implies $k \geqslant n$, i.e. codim $D_k \geqslant n$. Now apply the Lemma again, only this time with X, p replaced by D_k, Γ_{D_k}. Repeating this step, we finally obtain a subvariety Z as required.

For part (ii), we may assume $p = p'$, i.e. $p_* \mathcal{O}_X = \mathcal{O}_U$. Note that this implies that any function f defined in a neighborhood of $Z = p^{-1}(u)$ has the form $f = g \circ p$ for some function g on a neighborhood of u. Hence, applying the Lemma repeatedly as above we obtain functions $g_1 \ldots g_n$ such that $Z = \{g_1 \circ p = \ldots = g_n \circ p = 0\}$ and is smooth and reduced. This implies that p is smooth. Moreover, Γ_Z is constant. Hence, we have a diagram:

$$\begin{array}{ccccccccc} 0 \to & V_1 \otimes \mathcal{O}_Z & \to & V \otimes \mathcal{O}_Z & \to & (V/V_1) \otimes \mathcal{O}_Z & \to 0 \\ & \downarrow \mu' & & \downarrow \mu & & \downarrow \mu'' & \\ 0 \to & TZ & \to & TX|Z & \to & N_Z|X & \to 0. \end{array}$$

As μ is surjective. TZ is generated by global sections, hence Z is homogenous. As μ'' is an isomorphism, it follows that $H^0(Z, TX|_Z) \to H^0(Z, N_{Z|X})$ is surjective, i.e. the Kodaira-Spencer mapping $T_uU \xrightarrow{\sim} H^0(Z, N_{Z|X}) \to H^1(Z, TZ)$ vanishes. As is well-known and easy to prove, this implies that p is locally trivial.

Turning to the proof of the Lemma, let us introduce some notation. For a space Y and a point $y \in Y$, let $\mathcal{M}_y \subset \mathcal{O}_{Y,y}$ denote the maximal ideal. For $f \in \mathcal{O}_{Y,y}$ put $\mu_y(f) = \max\{j: f \in \mathcal{M}_y^j\}$ and for a subset $S \subset Y$ put $\mu_S(f) = \min\{\mu_y(f): y \in S\}$. Likewise we can define $\mu_y(D)$, $\mu_S(D)$ where D is a divisor.

Now by shrinking X and changing U we may assume that U is Stein, $p(X)$ is Zariski-dense in U and, moreover, that there are only finitely many divisors $E \subset X$ such that $\dim p(E) \leqslant n - 2$ where $n = \dim V$, call them $E_1, \ldots, E_N$. By shrinking X we may assume $E_i \cap Z \neq \phi$ for all. Let B_i denote the closure of $p(E_i)$ and $E_i' = p^{-1}(B_i)$. Pick a point $z_0 \in Z$ and a curve $A \subset X$ containing z_0 and not contained in $\cup E_i' \cup p^{-1}(u)$. Let g be a function on V, vanishing identically on $p(A)$ but not on B_i for any i, and put $f_0 = g \circ p$. Then the principal divisor (f_0) can be decomposed as

$$(f_0) = \sum_{i=1}^{N} n_i E_i + P, \quad P \not\supset E_i, \quad i = 1, \ldots, N.$$

As $A \not\subset (f_0)$, P must be nontrivial. Note that $n_i = \mu_{E_i}(f_0)$ and put $k_0 = \mu_{z_0}(f_0)$, $k = \mu_z(f_0)$.

Now if $k = 1$ put $f = f_0$. Suppose $k > 1$. We identify the tangent bundle TX with the sheaf of derivations of $\mathcal{O}_X$. As TX is generated by global sections, it follows that for a sufficiently general global derivation ∂_1 of $\mathcal{O}_X$ and $f_1 = \partial_1 f_0$, we have $\mu_z(f) = k - 1$ and $\mu_{E_i}(f_1) = n_i - 1$ whenever $n_i > 0$. On the other hand note that as f_0 vanishes on Z and $Z \cap E_i \neq \phi$, f_0 cannot equal a nonzero constant on E_i. Hence, if $n_i = 0$ then we may assume $\mu_{E_i}(f_1) = 0$. Also note that $\mu_z(f_1) \geqslant \mu_z(f_0) - 1$ for any $z \in Z$.

Continuing to differentiate up to $k - 1$ times in this manner, we obtain at the j-th step a function f_j with

$$\mu_Z(f_j) = k - j,\ \mu_{E_i}(f_j) = \max(n_i - j, 0),\ \mu_z(f_j) \geqslant \mu_z(f_0) - j,$$

$$\forall z \in Z.$$

Put $f = f_{k-1}$, $D = (f)$. Thus D is smooth at a generic point $z' \in Z$. Let $V \times \mathcal{O}_X \to TX$ be a surjection and for $v \in V$ denote by ∂_v the corresponding derivation of $\mathcal{O}_x$. If Γ_D: $D - D_{\text{sing}} \to G(\dim V - 1, V) = \mathbb{P}(V^*)$ is the Gauss map then $\Gamma_D(x)$ may be identified as the hyperplane: $\{v \in V$: $\partial_v(f)(x) = 0\}$. Take $v \in V \setminus \Gamma_D(z')$. Then $\partial_v(f)(z') \neq 0$. As Z is compact, $\partial_v(f)$ is constant on Z hence never vanishes there. In particular, D is smooth along Z, so that Γ_D is everywhere defined there. If Γ_D were not constant, we would have $\bigcup_{z \in Z} \Gamma_D(z) = \mathbb{P}(V^*)$, hence there would be $z' \in Z$ such that $v \in \Gamma_D(z'')$, i.e. $\partial_v(f)(z'') = 0$, which is not the case.

Now as D has only one connected component containing Z we may, after shrinking X, assume D is irreducible. It remains to show that $\dim p(D) \geqslant n - 1$, i.e. that $D \neq E_i$ for any i. If $D = E_i$ then $E_i \supseteq Z$ and $n_i = k$. Now for any $z \in Z$ we have

$$1 = \mu_z(f) \geqslant \mu_z(f_0) - k + 1 \geqslant \mu_Z(f_0) - k + 1 = 1$$

hence $\mu_z(f_0) = k$. But recall that $z_0 \in Z \cap P \subseteq E_i \cap P$, and the divisor (f_0) contains $n_i E_i + P$. Hence $\mu_{z_0}(f_0) > \mu_{z_0}(n_i E_i) \geqslant n_i = k$ which is a contradiction, proving the Lemma.

We turn now to some corollaries. These are mostly based on the Observation, made above, that the conclusions of the Proposition apply to Gauss maps.

Corollary 1: *Let X be a compact manifold and Φ: $X \to \mathbb{P}^n$ an immersion with $\Phi(X)$ not a linear space, then*

(*i*) Γ_Φ *is finite*

(*ii*) $K_X \otimes \mathcal{O}_X(m+1)$ *is ample*, $m = \dim X$.

(*iii*) *If $L \subset \Phi(X)$ is a linear space of dimension k, then*

$$k \leqslant \frac{n-m}{n-m+1} m$$

PROOF: It is known that Γ_Φ is generically finite (cf. [2]; this can also be derived from the Proposition: Exercise.) By the Proposition, Γ_Φ is finite. In view of the discussion following Example 3, this implies (ii). Finally (iii) follows because, for all $x \in \Phi^{-1}(L)$, $\Gamma_X(x) \supseteq L$, hence $\Gamma_X(\Phi^{-1}(L))$ is contained in the set of all m-spaces containing L, which has dimension $(n-m)(m-k)$ and hence $k \leqslant (n-m)(m-k)$.

For instance, a smooth m-dimensional hypersurface cannot contain a linear space of dimension $m/2$. Part (i) was first proved, for embeddings, by Zak [1]. Finally we note that a suitably localized version of the Corollary holds: for instance, in part (iii) it suffices to assume that Φ is an immersion in a neighborhood of $\Phi^{-1}(L)$. Indeed, it suffices to show that Γ_Φ is generalically finite: but it is not hard to see that if it weren't, then the ramification locus of Φ must meet (the closure of) every fibre of Γ_Φ, which cannot happen in our case. For Φ an embedding, (iii) also follows from the Barth-Lefschetz theorem.

COROLLARY 2: *Let Φ: $X \to Y$ be an immersion where X is a compact manifold and Y is a complex torus.* (1) *Assume X is of general type (equivalently, $\Phi(X)$ is not invariant under any nontrivial subtours). Then:*

(*i*) *Γ_Φ is finite.*
(*ii*) *The canonical bundle K_X is ample (in particular, X is projective,* (2) *hence, so is Y if X generates Y).*
(*iii*) *If $A \subset \Phi(X)$ is an abelian subvariety of dimension k then*

$$k \leqslant \frac{n-m}{n-m+1} m.$$

PROOF: It is well-known that Γ_Φ is generically finite (cf. [2] or [3]). Hence we can argue as above.

For Φ an embedding, part (ii) was conjectured by Ueno [3]. Note that a localized version of this Corollary again holds.

COROLLARY 3: *Suppose X is a compact manifold, L a line bundle on X such that L and $\Omega_X \otimes L$ are generated by global sections. Then $K_X \otimes L^m$ is ample, $m = \dim X$, unless there is a nontrivial homogeneous-space form p: $X \to Y$ with fibre a complex torus and a line bundle L_0 on Y such that $L = p^* L_0$.*

PROOF: By assumption, there is a surjection $V \otimes \mathcal{O}_X \to \Omega_X \otimes L$ whence an injection λ: $TX \to V \otimes L$, which yields a set of Gauss data with $E = V \otimes L$

(1) As is well-known, the existence of such a map Φ for given X is equivalent to Ω_X being generated by global sections.
(2) This also follows from Moishezon's theorem that Moishezon + Kähler ⇒ Projective.

and a Gauss map $\Gamma = \Gamma_{\text{identity}}$. Calculating as in Example 3 above, we get $\Gamma^*\mathcal{O}_G(1) = K_X \otimes L^{\otimes m}$ which is ample if Γ is finite. If Γ is not finite, it Stein-factorizes through a homogeneous-space form $p\colon X \to Y$. As $\Gamma^*\mathcal{O}_G(1)$ is trivial on each fibre Z of p, while $L\,|\,Z$ and $K_{X|Z} = K_Z$ are both generated by global sections, both $L_{|Z}$ and K_Z must be trivial. It is well-known that the triviality of L on the fibres of p implies that $L_0 = p_* L$ is a line bundle and $p^*L_0 \cong L$. As for Z, we know TZ is generated by global sections, hence is of the form f^*Q where $f\colon Z \to G'$ is a map to a Grassmannian and Q is the universal quotient bundle. But $(K_Z)^{-1} = f^*\mathcal{O}_{G'}(1)$ is trivial, hence f is constant, TZ is trivial and hence Z is a complex torus.

This result may be interpreted as giving a lower bound on the size of the poles of meromorphic 1-forms on X. For instance, if X is a $K3$ surface, it says that of L is generated by global sections but is not ample, then $\Omega_X \otimes L$ cannot be generated by global sections.

Corollary 4: *Let $Y = G/H$ be a compact homogeneous space with H connected. Let $Z \subset Y$ be a closed subvariety and, X a neighborhood of Z and $p\colon X \to U$ a morphism constant on Z. Then Z is contained in a homogeneous subspace $K/H \subset Y$ on which p is constant; here K is a closed connected subgroup containing H. If moreover p is proper, then it factors through a canonical map $G/H \to G/K$.*

Proof: This follows directly from the Proposition upon noting that any subvariety $Z' \subset Y$ with $\Gamma_{Z'}$ constant is of the form K/H.

In particular, if Φ is a Grassmannian, then H is a maximal connected subgroup of G, and we recover the well-known result that any holomorphic function defined in a neighborhood of a subvariety of a Grassmannian must be constant.

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Ziv Ran
Department of Mathematics
University of Chicago
5734 University Avenue
Chicago, IL 60637
USA