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THE GEOMETRY OF THE PERIOD MAPPING ON CYCLIC COVERS OF $\mathbb{P}_1$

Gonzalo Riera

Introduction

The tangent space to the period space for Riemann surfaces of genus g at a curve C is naturally isomorphic to the second symmetric product

$$S^{(2)}H^0(C; \Omega_C^1)^* \tag{0}$$

of the dual of the vector space of holomorphic differentials on C. If C is Galois, them its group of automorphisms acts on the vector space (0), and the representation theory of this situation was analyzed classically by Chevalley and Weill. Our aim in this and a future paper is to analyze the relationship between subspaces of (0) described representation-theoretically and the geometric properties of deformations of C in directions lying in these subspaces. The present paper deals with the case in which C is cyclic over $\mathbb{P}_1$.

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Algebraic differential on $\mathbb{P}_1$

A rational differential form ω on $\mathbb{C}$-{0} is the pull-back of a form on $\mathbb{P}_1$ if is homogeneous of degree 0 and satisfies

$$\langle \omega, \theta \rangle = 0$$

where $\theta = x_0 \partial/\partial x_0 + x_1 \partial/\partial x_1$ (see [6]).

Thus an algebraic one-form on $\mathbb{P}_1$ can be expressed as

$$p(x_0, x_1)\Omega/q(x_0, x_1)$$

where $\Omega = \langle \theta, \mathrm{d}x_0 \wedge \mathrm{d}x_1 \rangle = x_0 \mathrm{d}x_1 - x_1 \mathrm{d}x_0$ and p, q are homogeneous polynomials such that

$$\deg p = \deg q - 2$$

Algebraic differentials on a cyclic cover of $\mathbb{P}_1$

Let n, m be integers $n \geqslant 2$, $m \geqslant 1$ and consider a divisor

$$a_1 + a_2 + \dots + a_{nm} \tag{1}$$

of distinct non-zero complex numbers in $\mathbb{C} \cup \{\infty\} = P_1$. Also, denote by M the line bundle associated to that divisor. Since $H^1(\mathbb{P}_1, \mathcal{O}^*) = \mathbb{Z}$ and the Chern class $c_1(M) = nm$ there exists a line bundle L such that the following diagram commutes

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \downarrow & \text{id} & \downarrow \\ \mathbb{P}_1 & \rightarrow & \mathbb{P}_1 \end{array}$$

where λ raises a local section to the power n.

If s is a global section of M given by a homogeneous polynomial

$$F(x_0, x_1) = \prod_{i=1}^{nm} (x_0 - a_i x_1)$$

then $C = \lambda^{-1}(s(\mathbb{P}_1))$ is a cyclic covering whose ramification divisor lies over the given divisor (1).

The function $y = F^{1/n}$ is a homogeneous form of degree m on C so that in affine coordinates for $\mathbb{P}_1$, the surface is the Riemann surface of the equation

$$y^n = (x - a_1)\dots(x - a_{nm}).$$

Let $\sigma: C \to C$ be a generator of the natural cyclic group of analytic automorphisms of the curve. The automorphism σ^* acts on differentials on C and thus any differential can be expressed as a linear combination of

$$\omega_k \qquad k = 0, 1, \dots, n-1$$

where

$$\sigma^*(\omega_k) = \zeta^{-k}\omega_k \qquad (\zeta = \exp(2\pi i/n)).$$

If we multiply this differential by the function

$$y^k/x_0^{mk}$$

we obtain a differential on $\mathbb{P}_1$:

$$y^k \omega_k / x_0^{mk} = p\Omega/q.$$

Thus redefining p and q we have that any algebraic differential on C is a linear sum of

$$p\Omega/qy^k \tag{2}$$

with p, q relatively prime homogeneous polynomials such that

$$\deg p + 2 = \deg q + km.$$

A differential (2) will have no-poles on C if and only if $q = 1$ and we can write

$$H^{1,0}(C, \mathbb{C}) = \bigoplus_{k=1}^{n-1} H_k^{1,0}$$

where

$$H_k^{1,0} = \{ p\Omega/y^k;\ \deg p = km - 2\} \tag{3}$$

for all $k = 1, 2 \ldots n-1$. Adding up the dimensions of these eigenspaces we obtain

$$\sum_{k=1}^{n-1} (km - 1) = m(n-1)/2 - (n-1) = g$$

where g is the genus of C.

Algebraic Hodge decomposition

Since we have just obtained an explicit expression for $H^{1,0}$ we have to characterize the quotient

$$H^1(C, \mathbb{C})/H^{1,0}(C, \mathbb{C}) = H^{0,1}(C, \mathbb{C}).$$

The answer is given in terms of meromorphic differentials on the curve that have poles on the branch points.

THEOREM: *The automorphism $\sigma : C \to C$ induces a decomposition in eigenspaces*

$$H^1/H^{1,0} = \bigoplus_{k=1}^{n-1} (H^1/H^{1,0})_k$$

where the k^{th} *summand can be naturally identified with*

$$\{\omega^k = p\Omega/y^{n+k};\ \deg p = m(n+k)-2\}/I_k \tag{4}$$

where I_k *consists of those forms* ω_k *that satisfy*

$$p \subset (\partial F/\partial x_0, \partial F/\partial x_1)I,$$

the homogeneous Jacobian ideal.

PROOF: Let B be the divisor on C which maps to the divisor (1) on P. The natural inclusion C–$B \to C$ induces in cohomology the exact sequence

$$0 \to H^1(C) \to H^1(C\text{–}B) \overset{R}{\to} \bigoplus_{p\in B} \mathbb{C}_p \overset{\Sigma}{\to} \mathbb{C} \to 0$$

where R applied to a differential form gives the residues at the branch points.

The cyclic group generated by σ acts on each term of this sequence and if we set

$$H_k^1(C) = \{w \in H^1(C);\ \sigma^*(w) = \zeta^{-k}w\}$$

($k = 0,\dots,n-1$) and similarly for $H_k^1(C-B)$ we obtain the exact sequences:

$$0 \to H_0^1(C) \to H_0^1(C-B) \to \bigoplus_{p\in B} \mathbb{C}_p \to \mathbb{C} \to 0$$

$$0 \to H_k^1(C) \to H_k^1(C-B) \to 0 \qquad k = 1,\dots,n-1.$$

Since $H_0^1(C)$ is the space of forms invariant under σ, it is the space of forms on $\mathbb{P}_1$ with no singularities, thus

$$H_0^1(C) = 0$$

and

$$H^1(C) = \bigoplus_{k=1}^{n-1} H_k^1(C) \cong \bigoplus_{k=1}^{n-1} H_k^1(C-B).$$

Moreover since σ is an analytic map, the decomposition into eigenspaces is compatible with the Hodge decomposition, that is

$$H_k^{1,0} \subset H_k \quad \text{and}$$

$$H^1/H^{1,0}(C) \cong \bigoplus_{k=1}^{n-1} H_k^1(C-B)/H_k^{1,0}(C-B).$$

We will compute these last terms using the "algebraic de Rham theorem" by Grothendieck (cf. [5]).

For an affine variety S, $H^1(S) \cong H^1(A^*)$, where A^* is the complex of algebraic differentials.

The decomposition into eigenspaces gives

$$H_k^1(C-B) \cong H^1(A_k^*, d)$$

where the complex $A_k^0 \overset{d}{\rightarrow} A_k^1$ has an increasing filtration

$$\begin{aligned} A_k^0(1) &= \{p/y^{nl+k}\} \qquad & l &= 0, 1, 2, \ldots \\ A_k^1(l) &= \{q\Omega/y^{n(l+1)+k}\} \qquad & l &= 1, 0, 1, \ldots \end{aligned} \tag{5}$$

for homogeneous polynomials p and q of appropriate degrees. We can now write a Koszul resolution as in Clemens (cf. [4]).

Recall that

$$\theta = x_0\partial/\partial x_0 + x_1\partial/\partial x_1$$

and set

$$v = \mathbb{C}\mathrm{d}x_0 \oplus \mathbb{C}\mathrm{d}x_1.$$

For l, $r \in \mathbb{Z}$ let $P_{k,l}^r$ be the vector space of homogeneous forms of degree $mn(l+r) + mk - 1$ in x_0 and x_1. We can then define natural epimorphisms

$$\alpha \colon P_{k,l}^0 v \rightarrow A_k^0(1)/A_k^0(l-1)$$

$$\omega \rightarrow \langle \theta, \omega \rangle / y^{nl+k}$$

and

$$\beta \colon P_{k,l}^1 \oplus \Lambda^2 v \rightarrow A_k^1(l)/A_k^1(l-1)$$

$$\omega \rightarrow \langle \theta, \omega \rangle / y^{n(l+1)+k}.$$

Next recall that

$$\langle \theta, \mathrm{d}F \wedge \omega \rangle = mnF\omega - \mathrm{d}F \wedge \langle \theta, \omega \rangle. \tag{6}$$

Now $\alpha(\omega) = 0$ if and only if F divides $\langle \theta, \omega \rangle$ that is

$$\langle \theta, \omega \rangle = F\gamma.$$

But then

$$\langle \theta, \omega - \gamma \mathrm{d}F/nm \rangle = 0$$

so that

$$\omega - \gamma \mathrm{d}F/nm = \langle \theta, q\mathrm{d}x_0 \wedge \mathrm{d}x_1 \rangle .$$

Thus

$$\ker \alpha = \{ p\mathrm{d}F + q\Omega \}.$$

Also

$$\ker \beta = \{ pF\mathrm{d}x_0 \wedge \mathrm{d}x_1 \}.$$

Using (6) to insure commutativity, we can write the following diagram

$$\begin{array}{ccccccccc}
0 \to P^{-1}_{k,l} \to & \ker \alpha \to & \ker \beta \to & P^1_{k,l}/I_{k,l} \to 0 \\
\downarrow & \downarrow & \downarrow & \\
0 \to P^{-1}_{k,l} \to & P^0_{k,l} \oplus V \to & P^1_{k,l} \oplus \Lambda^2 V \to & P^1_{k,l}/I_{k,l} \to 0 \\
\downarrow & \downarrow & \downarrow & \\
0 \to 0 \to & A^0_k(l)/A^0_k(l-1) \to & A^1_k(l)/A^1_k(l-1) \to 0 &
\end{array} \tag{7}$$

where $I_{k,l} = I \cap P^1_{k,l}$.

The middle row is exact, it is just a Koszul resolution. The top row is exact except at $P^1_{k,l}/I_{k,l}$. To see this, exactness at ker β is just the identity (6) that, for two-forms ω,

$$mnF\omega = \mathrm{d}F \wedge \langle \theta, \omega \rangle .$$

Exactness at ker α is just the fact that

$$\mathrm{d}F \wedge \Omega = mnF\mathrm{d}x_0 \wedge \mathrm{d}x_1 .$$

So (7) is a short exact sequence of complexes. Our remarks on each of the other two complexes then gives, via the long exact sequence in cohomology,

$$H^0\big(A^*_k(l)/A^*_k(l-1)\big) = 0$$

$$H^1\big(A^*_k(l)/A^*_k(l-1)\big) = P^1_{k,l}/I_{k,l}. \tag{8}$$

The complexes $A^*_k(l)$ filter the complex A^*_k whose cohomology is

$H_k^*(C-B)$. The spectral sequence associated to this filtration has

$$E_1^{l,q} = H^{l+q}\big(A_k^*(l)/A_k^*(l-1)\big)$$

so that

$$E_1^{l,q} = 0$$

unless $l+q=1$. So this spectral sequence degenerates at E_1 and we can compute the resulting filtration on $H_k^1(C-B)$ via (8), namely,

$$E_\infty^{-1,2} = H^1\big(A_k^*(-1)\big) = P_{mk-2}$$

$$E_\infty^{0,1} = H^1\big(A_k^*(0)/A_k^*(-1)\big)$$

$$= P_{m(n+k)-2}/I_{m(n+k)-2}$$

etc.

To finish the proof of the theorem, notice that, referring to (3),

$$E_\infty^{-1,2} = H_k^{1,0}$$

and, by the exactness of the middle row of (7), we have that if $l \geqslant 1$,

$$\dim E_\infty^{l,1-l}$$

$$= \big[m(l+1)n+k)-1\big] - 2\big[m(l+1)n+k)-mn\big]$$

$$+ \big[m(l+1)n+k)-2mn+1\big] = 0.$$

We will compute the dimension of $E_\infty^{0,1}$ in order to motivate our next result. Namely $\dim E_\infty^{0,1} = [m(n+k)-1] - 2[m(n+k)-mn] = -[m(n+k)-2mn+1] = mn-k+1 = \dim H_{n-k}^{1,0}$.
This is as it should be since, under the cup-product pairing, if $w \in H_r^{1,0}$, $y \in H_s^1$, then

$$w \wedge y \in H_{r+s}^2 = 0 \quad \text{unless} \quad r+s=n.$$

More precisely, we have the following result.

PROPOSITION: *Under the cup product $H^{1,0} \times H^{0,1} \to \mathbb{C}$ the space $H_k^{1,0}$ is orthogonal to $(H^1/H^{1,0})i$ for $i \neq n-k$.*

For differential forms

$$\varphi = p\Omega/y^k \in H_k^{1,0}$$

$$\psi = q\Omega/y^{2n-k} \in \big(H^1/H^{1,0}\big)_{n-k}$$

the following identity holds

$$(\varphi, \psi) = \int \varphi \wedge \psi$$

$$= (2\pi i/n-k)n^2 \sum_{j=1}^{nm} p(a_j)q(a_j)/$$

$$\times (a_j - a_1)^2 \ldots \wedge \ldots (a_j - a_{nm})^2. \tag{9}$$

PROOF: The first statement follows from the fact that, under the cup-product,

$$H_{k_1}^{1,0}(C) \otimes (H^1(C)/H^{1,0}(C))_{k_2}$$

goes into $H_{k_1+k_2}^{1,1}(C)$, which is zero for $k_1 + k_2 \neq n$.

For the second statement, we proceed as follows. Using affine coordinates for our curve C

$$y^n = F(x) = \prod_{j=1}^{nm} (x - a_j)$$

so that

$$ny^{n-1}\mathrm{d}y = F'(x)\mathrm{d}x.$$

Near $x = a_j$ we write

$$\varphi = p(x)\mathrm{d}x/y^k = p(x)y^{n-k-1}\mathrm{d}x/y^{n-1} = np(x)\mathrm{d}y/F'(x)y^{n-k-1}$$

$$= \beta_j(x)y^{n-k-1}\mathrm{d}y.$$

Similarly $\psi = nq(x)\mathrm{d}y/F'(x)y^{n-k+1} = \alpha_j(x)\mathrm{d}y/y^{n-k+1}$.

For each $j = 1, \ldots, nm$, let ρ_j be a C^∞ function on C supported in a neighborhood $|y| < \varepsilon$ of $x = a_j$ and identically equal to one the smaller neighborhood $|y| \leqslant \varepsilon/2$.

The form

$$\tilde{\psi} = d\left(\frac{1}{n-k} \sum_{j=1}^{nm} \rho_j \alpha_j / y^{n-k}\right) + \psi$$

is in the same cohomology class as ψ but is C^∞ in C, analytic outside the neighborhoods $|y| < \varepsilon$.

We the have

$$(\varphi, \psi) = (\varphi, \tilde{\psi}) = \sum_{j=1}^{nm} \int_{|y| \leqslant \varepsilon} \varphi \wedge \tilde{\psi}.$$

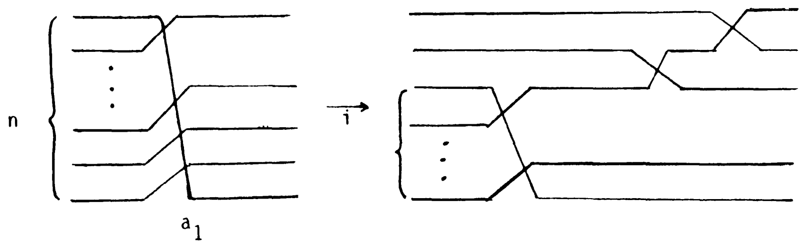

Figure 1

Since $\tilde{\psi}$ is analytic also in $|y| \leqslant \varepsilon/2$ we get

$$(\varphi, \psi) = \sum_{j=1}^{nm} \int_{\varepsilon/2 \leqslant |y| \leqslant \varepsilon} \varphi \wedge \partial/\partial\bar{y}\left(1/n - k\rho_j\alpha_j/y^{n-k}\right) \mathrm{d}\bar{y}$$

and by Stokes theorem

$$(\varphi, \psi) = \frac{1}{n-k} \sum_{j=1}^{nm} \int_{|y| = \varepsilon/2} \beta_j(x) y^{n-k-1} \alpha_j \mathrm{d}y/y^{n-k}$$

$$= \frac{2\pi i}{n-k} \sum_{j=1}^{nm} \beta_j(a_j)\alpha_j(a_j).$$

proving the identity.

Deformations of a cyclic cover

Let τ_n^1 be the period space of curves of a given genus that can be expressed as an n to 1 branced cover of $\mathbb{P}_1$. We will consider a neighborhood of the cyclic cover C given by an equation

$$y^n = (x - a_1) \dots (x - a_{nm}).$$

A deformation within τ_n^1 will split each branch point of order n into several branch points of lower order; a stratification of these deformations can be given in the following form:

At... a a branch point, a_1 say, there is associated a cyclic permutation $(1, 2 \dots n)$ of the sheets of the Riemann surface. For each $i = 2, \dots, n$ we will say that a deformation is of type i if it splits a_1 into a branch point with associated permutation $(1, 2 \dots i)$ and $(n - i)$ simple branch points (see Fig. 1).

A locally irreducible, analytic curve in the period space will determine a non-zero tangent vector (which generates its tangent cone at C) which we consider as an element in

$$Sym^{(2)}(H^{1,0})^*.$$

We denote by v_i the tangent cone to deformations of type greater than or equal to i. Thus $v_{i+1} \subset v_i$, v_n is the set of all tangent vectors to deformations by cyclic covers and v_2 is the tangent space to τ_n^1.

On the other hand, the cyclic group of order n generated by the automorphism $\sigma : C \to C$ acts on the space $Sym^{(2)}(H^{1,0})$. We say that a quadratic differential q_j is of type j, $1 \leqslant j \leqslant n$, if it is an eigenvector for the eigenvalue ζ^j, i.e.

$$\sigma^*(q_j) = \zeta^j q_j \qquad (\zeta = \exp(2\pi i/n)).$$

An element q in $Sym^{(2)}(H^{1,0})$ is a linear sum of these eigenvectors, that is

$$q = \sum_{j=1}^{n} q_j,$$

and we will call q_j the component of type j of the quadratic differential q.

There is a close relationship between eigenvalues and splitting of branch points given by the following.

THEOREM: *The tangent cone v_i is always a (possibly non reduced) linear space.*

A quadratic differential is orthogonal to v_i if and only if its components of type $j = 2, \dots, n - i + 1$, n vanish to orders

$$n, \dots, 2n - i - 1, 2n - 2 \text{ at branch points.}$$

In particular the cotangent space of cyclic deformations consists of those quadratic differentials whose invariant component vanishes at branch points to order $2n - 2$.

PROOF: We first fix the notation for a basis of $H^{1,0}(C)$.

$$\text{Let } e_\nu(\lambda) = x^\nu \mathrm{d}x / y^\lambda$$

where $0 \leqslant \nu \leqslant \lambda m - 2$, $\lambda = 1, \dots, n - 1$. For fixed λ these differentials form a basic of $H_\lambda^{1,0}(C)$; $\sigma^*(e_\nu(\lambda)) = \zeta^{-\lambda} e_\nu(\lambda)$.

In terms of a local parameter centered at a_1,

$$(x - a_1) = z^n$$

we can write

$$e_\nu(\lambda) = p(z)\mathrm{d}z, p(z) = C_{\lambda,\nu} z^{n-\lambda}(1 + \alpha z^n + \dots)$$

where $C_{\lambda,\nu} = na_1^{\nu}/[(a_1 - a_2)\ldots(a_1 - a_{nm})]^{\lambda/n}$ and $\alpha \in C$. (There will of course be analogous expressions for $a_2, \ldots, a_{nm}$.)

The vectors of the dual basis of $(H^{1,0})^*$ will be denoted by $e^{\nu}(\lambda)$.

To deform the curve C we will take a covering of C by coordinate charts whose changes of parameters depend on t, $|t| < \varepsilon$ (see [1]).

Here we take a local deformation of the parameter z centered at a_1 given by

$$\psi(z, t) = (x - a_1)$$

with $\psi(z, 0) = z^n$; the rest of the local parameters remain fixed. A differential of the first kind on C_t will be written in terms of z as

$$p(z, t)\mathrm{d}z = p(z, t)\mathrm{d}x/(\partial\psi/\partial z)$$

where $p(z, t)$ is an analytic function. Since

$$\frac{\partial\psi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial\psi}{\partial t} = 0$$

we compute the derivate of the differential at $t = 0$ to be

$$\left(\frac{\partial\psi}{\partial z}\right)^{-2}\left[p(z)\left(-\frac{\partial^2\psi}{\partial z\partial t}\frac{\partial\psi}{\partial z} + \frac{\partial^2\psi}{\partial z^2}\frac{\partial\psi}{\partial t}\right) - \frac{\partial p}{\partial z}\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial t}\right]\mathrm{d}z + \frac{\partial p}{\partial t}\mathrm{d}z. \tag{10}$$

This is in general a differential of the 2^{nd} kind with pole at a_1, that is, an element in

$$H^1/H^{1,0}.$$

The linear map $\varphi \in \mathrm{Hom}(H^{1,0}, H^1/H^{1,0}) \cong H^{1,0^*} \otimes H^{1,0^*}$ corresponding to the deformation associates to $p(z)\mathrm{d}z$ the differential (10) (where we can disregard the analytic part $\partial p/\partial t \mathrm{d}z$). In general, a deformation ψ depends on $n - 1$ parameters $t_2, \ldots, t_n$ given as

$$\psi(z, t_2, \ldots, t_n) = z^n + t_2 z^{n-2} + \ldots + t_n = (x - a_1).$$

(See [2] also.) We compute separately the linear mapping φ_i associated to

$$\psi_i(z, t_i) = z^n + t_i z^{n-i} \qquad i = 2, \ldots n.$$

In this case (10) gives

$$(nz^i)^{-1}((i-1)p(z) - z(p'(z))\mathrm{d}z$$

and replacing $p(z)$ by the corresponding expressions for the differentials $e_{\nu}(\lambda)$ we obtain

$$n^{-1}C_{\lambda,\nu} z^{n-(\lambda+i)-1}[(i + \lambda - n) + \beta z^n + \ldots]\mathrm{d}z$$

for some constant $\beta \in \mathbb{C}$. This differential has no poles for $i + \lambda \leqslant n$ and for $i + \lambda \geqslant n + 1$ we can compute the cup product

$$\left(e_\mu(\lambda'), \varphi_i(e_\nu(\lambda))\right)$$

$$= C_{\lambda,\nu}C_{\lambda',\mu}\int_{|z|=\varepsilon} z^{2n-(\lambda+\lambda'+i)-1}(1 + \gamma z^n + \dots)\mathrm{d}z.$$

This integral vanishes unless $\lambda + \lambda' + i = 2n$ and we obtain

$$\varphi_i = \sum_{\nu,\mu} d_{\nu,\mu} \sum_{\lambda \geqslant n-i+1} e^\nu(\lambda) \otimes e^\mu(2n - (i + \lambda))$$

where

$$d_{\nu,\mu} = (2\pi\sqrt{-1})a_1^{\nu+\mu}/[(a_1 - a_2)\dots(a_1 - a_{nm})]^{2-i/n}.$$

Thus $\sigma^*(\varphi_i) = \zeta^{-i}\varphi_i$ and it follows that a quadratic differential will be orthogonal to φ_i if and only if its component of type i vanishes at a_1 at the order $n + i - 2$. More precisely, let

$$q_i = \sum_{\nu,\mu} \alpha^{\nu,\mu}\left(\sum_{\lambda+\lambda'=n-i} e_\nu(\lambda) \otimes e_\mu(\lambda') + \sum_{\lambda+\lambda'=2n-i} e_\nu(\lambda) \otimes e_\mu(\lambda')\right)$$

be a differential of type i; the first sum in parenthesis vanishes always to order $n + i - 2$. q_i is orthogonal to φ_i if

$$(q_i, \varphi_i) = (cte)\sum_{\nu,\mu} \alpha^{\nu,\mu}a_1^{\nu+\mu} = 0.$$

But this means that in terms of the local parameter z the expansion of the second sum has a first coefficient equal to zero, and thus q_i vanishes to order $n + i - 2$.

It remains to characterize the tangent cones v_i in terms of the parameters $t_2, \dots, t_n$. We have drawn in Fig. 2 the situation for $n = 4$.

The mapping, from the z-disk to a neighborhood of a_1 given by

$$\psi(z, t_2, \dots, t_n) = x - a_1$$

will be branched at the zeroes of the derivative with respect to z. Two of these branch points will coincide for the values of $(t_2, \dots, t_n)$ that satisfy both equations

$$\partial\psi/\partial z = nz^{n-1} + (n-2)t_2z^{n-3} + \dots + t_{n-1} = 0$$

$$\partial^2\psi/\partial z^2 = n(n-1)z^{n-2} + \dots + 2t_{n-2} = 0.$$

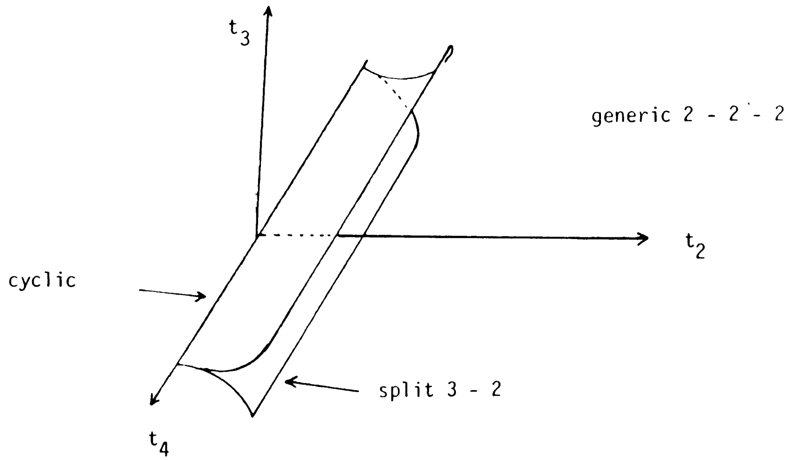

Figure 2

These equations in z will have a common root if their resultant

$$A = \begin{vmatrix} n & 0 & & \cdots & t_{n-1} & 0 & 0 \\ 0 & n & 0 & \cdots & t_{n-1} & 0 & 0 \\ & & & & & & 0 \\ & & & & & & t_{n-1} \\ n & (n-1) & 0 & \cdots & 2t_{n-2} & 0 & 0 \\ 0 & n & (n-1) & \cdots & 2t_{n-2} & 0 & 0 \\ & & & \ddots & \cdots & & \\ & & & & & \cdot & 0 \\ 0 & & & \ldots n(n-1)\ldots & & 2t_{n-2} & \end{vmatrix}$$

vanishes. This represents an hypersurface in $(t_2, \ldots, t_n)$ space corresponding to a triple branch point or more. More generally the rank of the matrix is smaller or equal to $2n - i - 1$ if and only if $\psi(z, t_2 \ldots t_n)$ has a branch point of order i at least.

The hypersurface $A = 0$ is singular at 0 and its tangent cone is given by

$$\sum_{\alpha_2 + \ldots + \alpha_n = n-2} \partial^{n-2} A / \partial t_2^{\alpha_2} \ldots \partial t_n^{\alpha_n} t_2^{\alpha_2} \ldots t_n^{\alpha_n} = 0$$

But from the last $n - 2$ columns in the matrix it is clear that all partial derivatives vanish at zero except

$$\partial^{n-2} A / \partial t_{n-1}^{n-2}(0)$$

and so the tangent cone is given by

$$t_{n-1}^{n-2} = 0, \quad \text{or} \quad t_{n-1} = 0.$$

This is then the tangent cone corresponding to v_3, the tangent cone to deformations of type greater or equal to 3.

For a deformation to be of type greater than or equal to 4 it is necessary and sufficient that the discriminant of $\partial^2\psi/\partial z^2$, $\partial^3\psi/\partial z^3$ vanishes also. Denoting this discriminant by B, the locus in $(t_2, \ldots, t_n)$ space is given by $A = 0$ and $B = 0$. (Furthermore, the singular locus of the surface $A = 0$ is given by this intersection.)

The tangent cone at 0 is then the intersection of the tangent cones to both surfaces, that is

$$\{t_{n-1}^{n-2} = 0, t_{n-2}^{n-3} = 0\} \quad \text{or} \ \{t_{n-1} = t_{n-2} = 0\},$$

and so on.

The tangent cone at 0 corresponding to v_i will be given by

$$\{t_{n-1} = \ldots = t_{n-i+2} = 0\}$$

in general, and then the tangent cone to v_i will be the linear space generated by

$$\{\varphi_n, \varphi_2, \ldots, \varphi_{n-i+1}\}.$$

A quadratic differential will be orthogonal to this subspace if and only if its components

$$\{q_n, q_2, \ldots, q_{n-i+1}\}$$

vanish at branch points to orders $2n-2$, n, $n+1, \ldots, 2n-i-1$ as required.

We observe also that in the generic case, v_2, we can obtain for these cyclic covers the result proved in general by Donagi-Green concerning the orthogonal space to deformations (see [1]).

Finally, it is interesting to consider the cases $n = 2$, 3 since it is possible to write global variations for the original equation.

For $n = 2$, the cyclic cover is given as

$$C: y^2 = (x - a_1) \ldots (x - a_{2m})$$

and a basic for $H^{1,0}(C)$ is

$$e_i = x^i/y\,dx \qquad 0 \leqslant i \leqslant m-2.$$

The only possible deformation is given by the families of curves

$$C_t\colon y^2 = (x-a_1)\dots(x-a_k+t)\dots(x-a_{2m})$$

with a basis for $H^{1,0}(C_t)$

$$e_i(t) = x^i \mathrm{d}x/((x-a_1)\dots(x-a_k+t)\dots(x-a_{2m}))^{1/2}.$$

The derivative at $t=0$ gives

$$(-1/2)x^i(x-a_1)\dots\hat{k}\dots(x-a_{2m})\mathrm{d}x/y^3$$

and in view of the proposition, its cup product with e_j is

$$\lambda_k^{i,j} = (2\pi\sqrt{-1})a_k^{i+j}/(a_k-a_1)\dots(a_k-a_{2m}).$$

The corresponding tangent vector is then

$$\sum_{i,j}\lambda_k^{i,j}e^i \otimes e^j$$

and a quadratic differential is orthogonal to all deformations if it vanishes at the branch points $a_1,\dots,a_{2m}$.

For $n=3$, the equation of C is

$$y^3 = (x-a_1)\dots(x-a_{3m})$$

with a basis of holomorphic differentials

$$e_i = x^i\mathrm{d}x/y, \qquad f_j = x^j\mathrm{d}x/y^2.$$

A cyclic variation is similar to the preceding case the tangent vector being

$$\sum \lambda_k^{i,j} e^i \otimes f^j.$$

However, to write a global equation in the case of splitting of a branch point the following considerations are necessary:

We have to find an equation

$$f(x,y) = y^3 + p(x)y + q(x) = 0$$

where p, q are polynomials in x whose coefficients depend on a parameter t. The polynomials should be chosen so that the equation represents a

Riemann surface ramified of order 3 over the points $a_2 \ldots a_{3m}$ and branched of order 2 over a_1, $a_1 - t$ and such that for $t = 0$ reducs itself to C.

The branch points are the common solutions of $f = 0$ and $\partial f/\partial y = 0$, and they are found among the solutions of the equation obtained by setting the discriminant equal to zero:

$$\Delta = -\left(4p^3 + 27q^2\right) = 0.$$

The branch points of order three have to satisfy also

$$\partial f/\partial y, \quad \partial^2 f/\partial y^2 = 0$$

and this implies that they are among the roots of

$$36p = 0.$$

We choose then

$$p(x) = t^{2/3}(x - a_2)\ldots(x - a_{3m})$$

where the particular power of t not only appears there for convenience but also reflects the fact that the parameter t is not natural. A natural parameter will be taken to be $s = t^{2/3}$. With this value for p we may compute again the discriminant

$$-\Delta = (x - a_2)^2 \ldots (x - a_{3m})^2 \Delta_2$$

for q has to vanish also at these $3m - 1$ points. Here we have written

$$\Delta_2 = 4t^2(x - a_2)\ldots(x - a_{3m}) + 27h^2$$

where $q(x) = (x - a_2)\ldots(x - a_{3m})h(x)$. The polynomial h must be chosen so as to satisfy the last condition concerning the branch points at a_1, $a_1 - t$. This forces the vanishing of Δ_2 there exactly to the first order, and we are led to an equation of the form

$$4t^2(x - a_2)\ldots(x - a_{3m}) + 27h^2(x) = g^2(x)(x - a_1)(x - a_1 + t).$$

This is a sort of Pell's equation for the polynomials $h(x)$, $g(x)$ and can be solved explicitly; to make a long story short, the solution appears as power series in t with polynomial coefficients, that is

$$h(x) = h_0(x - a_1) + h_1(x)t + \ldots$$

$$g(x) = g_0 + g_1(x)t + \ldots$$

for some constants h_0 and g_0 that can be computed directly from the equation.

Given now the formula for C_t, we have to write a basis for the analytic differentials:

$$\left(x^i y - l(x)\mathrm{d}x/f_y,\quad x^i g(x)\mathrm{d}x/f_y\right)$$

where l is some polynomial (see [3] for the details). We compute now the derivative with respect to $s = t^{2/3}$ and the answer does not depend on $l(x)$ or $g_1(x)$ but is of the form

$$x^i(x-a_2)\dots(x-a_{3m})/y^4\mathrm{d}x,$$

in the second case say, and the tangent vector then is obtained as

$$\sum \lambda_1^{i,j}\ f^i \otimes f^j$$

as in the proof of the theorem.

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