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**A BANACH SPACE WITH A SYMMETRIC  
BASIS WHICH CONTAINS NO  $\ell_p$  OR  $c_0$ , AND ALL  
ITS SYMMETRIC BASIC SEQUENCES ARE EQUIVALENT**

Z. Altshuler\*

**Abstract**

A Banach space having the properties described in the title of this paper is constructed.

In this paper we investigate the symmetric basic sequences in a Banach space with a symmetric basis. It is well known that the unit vector basis in the spaces  $c_0$  and  $\ell_p$  ( $1 \leq p < \infty$ ), is a symmetric basis, and every symmetric basic sequence in each of these spaces is equivalent to it. A natural question is whether there exists any other Banach space  $X$ , with a symmetric basis  $\{e_n\}_{n=1}^\infty$ , which has the same property. Let us recall that by [1], it turns out that if, in addition to the assumption that every symmetric basic sequence in  $X$  is equivalent to  $\{e_n\}_{n=1}^\infty$ , we know that the same holds in  $X^*$ , the dual of  $X$ , with respect to  $\{f_n\}_{n=1}^\infty$ , the sequence of the biorthogonal functionals associated to  $\{e_n\}_{n=1}^\infty$ , then  $\{e_n\}_{n=1}^\infty$  is equivalent to the unit vector basis of  $c_0$  or  $\ell_p$ , for some  $1 \leq p < \infty$ .

We answer the question raised above affirmatively by proving the following

**THEOREM:** *There exists a Banach space  $X$ , with a symmetric basis  $\{e_n\}_{n=1}^\infty$ , such that all symmetric basic sequences in  $X$  are equivalent to each other, and  $X$  is not isomorphic to  $c_0$  or  $\ell_p$ , for any  $1 \leq p < \infty$ .*

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Clearly a Banach space having the properties described in the theorem above contains no subspace isomorphic to  $c_0$  or  $\ell_p$  ( $1 \leq p < \infty$ ). A natural candidate for such an example is the space constructed by Figiel and Johnson [2], which has a symmetric basis and no subspace of which is isomorphic to  $c_0$  or  $\ell_p$ . Our example is obtained by a modification of their construction. Before passing to the proof of the theorem we need some definitions and notations.

**DEFINITION:** Let  $X$  be a Banach space with a symmetric basis  $\{e_n\}_{n=1}^\infty$ . Let  $N_i$   $i = 1, 2, \dots$  be subsets of the set of natural numbers  $N$ , so that  $\bar{N}_i = \bar{N}_j$  for every  $i, j$ ,  $N = \cup_{i=1}^\infty N_i$  and  $N_i \cap N_j = \emptyset$  for all  $i \neq j$ . For any  $0 \neq \alpha = \sum_i \alpha_i e_i \in X$  put  $u_i^{(\alpha)} = \sum_{j=1}^\infty \alpha_j e_{i,j}$  where  $N_i = \{i, j\}_{j=1}^\infty$  for  $i = 1, 2, \dots$ . The sequence  $\{u_i^{(\alpha)}\}_{i=1}^\infty$  is called a basic sequence generated by  $\alpha$ .

Clearly for any  $\alpha \in X$   $\{u_i^{(\alpha)}\}_{i=1}^\infty$  is a symmetric basic sequence in  $X$ . If  $\{u_n\}_{n=1}^\infty$  and  $\{v_n\}_{n=1}^\infty$  are basic sequences in Banach spaces  $X$ , respectively  $Y$ , we say that  $\{u_n\}_{n=1}^\infty$  and  $\{v_n\}_{n=1}^\infty$  are equivalent when a series  $\sum_i \alpha_i u_i$  converges if and only if  $\sum_i \alpha_i v_i$  converges. We write in this case  $\{u_n\} \sim \{v_n\}$ . We say that a basic sequence  $\{u_n\}_{n=1}^\infty$  is bounded if there exists an  $M > 0$  such that  $M^{-1} < \inf_n \|u_n\| \leq \sup_n \|u_n\| < M$ .

The first example of a Banach space which contains no  $\ell_p$  or  $c_0$  is due to Tsirelson [4]. Figiel and Johnson [2], described the dual of this space, which will be denoted by  $T$ , and showed that  $T$  contains no subsymmetric basic sequence. We also recall that the unit vector basis  $\{x_n\}_{n=1}^\infty$  of  $T$  is an unconditional basis.

We are ready now to construct our example. First we define a sequence of norms,  $|\cdot|_n$ , on  $c_0$ , by

$$(1) \quad |\alpha|_n = \sup_j \left[ \sum_{i=1}^j \hat{\alpha}_i \omega_i / (2^n + 2^{-n} s_j) \right] \quad \text{where } \alpha = \{\alpha_i\}_{i=1}^\infty \in c_0$$

$\{\hat{\alpha}_i\}_{i=1}^\infty$  is the rearrangement in non-increasing order of  $\{|\alpha_i|\}_{i=1}^\infty$ ,  $\omega_i = i^{-1}$ , and  $s_j = \sum_{i=1}^j \omega_i$ . Notice that since

$$(2) \quad 2^{-n-1} \sup_j |\alpha_j| \leq |\alpha|_n \leq \left( \sup_j |\alpha_j| \right) \cdot s_j / (2^n + 2^{-n} s_j) \leq 2^n \sup_j |\alpha_j|$$

we have that for all  $n = 1, 2, \dots$ ,  $|\cdot|_n$  is equivalent to the sup norm on  $c_0$ . We put now  $Y = \{y \in c_0; \|\sum_n |y|_n x_n\|_T < \infty\}$  where  $\{x_n\}_{n=1}^\infty$  is the unit vector basis of  $T$ . The space  $Y$  is a subspace (called the diagonal) of the direct sum  $Z = (\sum_{n=1}^\infty \oplus (c_0, |\cdot|_n))_T$ . Since for any unit vector  $e_j$ ,  $j = 1, 2, \dots$  we get  $|e_j|_n = (2^n + 2^{-n})^{-1}$ , we deduce that the sequence of

unit vectors  $\{e_j\}_{j=1}^\infty$  belong to  $Y$ , and they clearly form a symmetric basis, with symmetric constant 1. We also remark that we may assume, without loss of generality, that every symmetric basic sequence in  $Y$  is equivalent to a symmetric block basic sequence of  $\{e_n\}_{n=1}^\infty$ . So in order to prove the theorem it suffices to check the block bases of  $\{e_n\}_{n=1}^\infty$ .

LEMMA 1: *Let  $y_m = \sum_{i=p_m+1}^{p_{m+1}} \alpha_i e_i$  be a normalized block basis in  $Y$ . If  $\lim_{i \rightarrow \infty} \alpha_i = 0$  then there exists a subsequence  $\{y_{m_j}\}_{j=1}^\infty$  of  $\{y_m\}_{m=1}^\infty$  which is equivalent to a block basis of  $\{x_i\}_{i=1}^\infty$ , the unit vector basis of  $T$ .*

PROOF: For fixed  $m$  and  $N$  we have by (2) that

$$\sum_{n=1}^{N-1} |y_m|_n \leq \sum_{n=1}^{N-1} 2^n \cdot \max \{|\alpha_i|; p_m < i \leq p_{m+1}\}.$$

Therefore we can construct inductively two increasing sequences of integers  $\{m_j\}_{j=1}^\infty$ , and  $\{N_j\}_{j=1}^\infty$  such that  $\|\sum_{n=N_j}^\infty |y_{m_j}|_n x_n\|_T < 2^{-j-1}$  for all  $j \geq 1$  and  $\|\sum_{n=1}^{N_{j-1}-1} |y_{m_j}|_n x_n\|_T < 2^{-j-1}$  for all  $j > 1$ . The block basis  $\{y_{m_j}\}_{j=1}^\infty$  can be identified with the basic sequence  $\{\hat{y}_{m_j}\}_{j=1}^\infty$  in  $Z$  where  $\hat{y}_{m_j} = (y_{m_j}, y_{m_j}, \dots, y_{m_j}, \dots) \in Z$   $j = 1, 2, \dots$ . Put  $v_j = (0, 0, \dots, 0, \overset{N_{j-1}}{y_{m_j}}, \overset{N_{j-1}+1}{y_{m_j}}, \dots, y_{m_j}^{N_{j-1}-1}, 0, 0, \dots) \in Z$   $j = 1, 2, \dots$  and notice that for each  $j$ ,

$$\|\hat{y}_{m_j} - v_j\|_Z = \left\| \sum_{n=1}^{N_{j-1}-1} |y_{m_j}|_n x_n + \sum_{n=N_j}^\infty |y_{m_j}|_n x_n \right\|_T < 2^{-j}.$$

Hence the basic sequence  $\{y_{m_j}\}_{j=1}^\infty$  in  $Y$  is equivalent to  $\{\hat{v}_j\}_{j=1}^\infty$  which, in turn, is equivalent to the block basis  $z_j = \sum_{n=N_{j-1}}^{N_j-1} |y_{m_j}|_n x_n$   $j = 1, 2, \dots$  of  $\{x_n\}_{n=1}^\infty$ .

We can already state some consequences of Lemma 1.

PROPOSITION 1: *Let  $Y$  and  $\{e_i\}_{i=1}^\infty$  be as above. Then the following assertions are true:*

- (i) *There is no symmetric block basis  $y_m = \sum_{i=p_m+1}^{p_{m+1}} \alpha_i e_i$   $m = 1, 2, \dots$  of  $\{e_i\}_{i=1}^\infty$  such that the coefficients  $\{\alpha_i\}_{i=1}^\infty$  tend to zero.*
- (ii)  *$Y$  contains no subspace isomorphic to  $c_0$  or  $\ell_p$  for any  $1 \leq p < \infty$ .*

PROOF: The first assertion follows from Lemma 1 and the fact that  $T$  contains no subsymmetric basic sequence. To prove the second assertion we assume first that there is a block basis  $\{u_j\}_{j=1}^\infty$  of  $\{e_i\}_{i=1}^\infty$  which is equivalent to the unit vector basis of  $\ell_p$ , for some  $p \geq 1$ . Since  $\|\sum_{j=1}^n u_j\|_Y \rightarrow \infty$  as  $n \rightarrow \infty$  it is easy to construct a block basis  $\{v_m\}_{m=1}^\infty$  of  $\{u_j\}_{j=1}^\infty$  with coefficients, in the expansion with respect to

$\{e_i\}_{i=1}^\infty$ , tending to zero. The proof of this case can be then completed by using (i).

Suppose now that there is a block basis  $\{u_j\}_{j=1}^\infty$  of  $\{e_i\}_{i=1}^\infty$  which is equivalent to the unit vector basis of  $c_0$ . If the coefficients of the  $y_j$ 's form a sequence tending to zero, then we complete the proof of (ii) by (i). Otherwise, it follows easily that  $\{e_i\}_{i=1}^\infty$  itself is equivalent to the unit vector basis of  $c_0$ , hence for all  $k = 1, 2, \dots$   $\|\sum_{i=1}^k e_i\|_Y \leq M$ , for some  $M > 0$ . On the other hand for any  $k \geq 4$  we pick an integer  $n = n(k)$  such that  $s_k/2 < 2^{2n} \leq 2s_k$ . For these values of  $k$  and  $n = n(k)$  we have

$$\left\| \sum_{i=1}^k e_i \right\|_Y \geq \left| \sum_{i=1}^k e_i \right|_n = s_k / (2^n + 2^{-n} s_k) \geq \sqrt{s_k} / 6 = \left( \sum_{i=1}^k i^{-1} \right)^{1/2} / 6 \rightarrow \infty$$

as  $k \rightarrow \infty$ .

We consider now block bases generated by one vector in  $Y$ .

**LEMMA 2:** *Every block basis  $\{u_i^{(\alpha)}\}_{i=1}^\infty$  of  $\{e_j\}_{j=1}^\infty$  generated by a vector  $\alpha \in Y$ , is equivalent to  $\{e_i\}_{i=1}^\infty$ .*

**PROOF:** Let  $\{u_i^{(\alpha)}\}_{i=1}^\infty$  be a block basis generated by a vector  $0 \neq \alpha = \sum_i \alpha_i e_i \in Y$ . Then, for any  $\beta = \sum_i \beta_i e_i \in Y$ , we have  $\|\sum_i \beta_i u_i^{(\alpha)}\| \geq (\sup_j |\alpha_j|) \|\sum_i \beta_i e_i\|$ , so in order to prove that  $\{u_i^{(\alpha)}\} \sim \{e_i\}$  we have to show that  $\sum_i \beta_i u_i^{(\alpha)}$  converges, for any  $\beta \in Y$ . We first observe that it is enough to prove this for  $\beta = \alpha$  i.e. to show that  $\sum_i \alpha_i u_i^{(\alpha)}$  is a convergent series for any  $0 \neq \alpha = \sum_i \alpha_i e_i \in Y$ . Indeed, if this is true for any  $\alpha = \sum_i \alpha_i e_i \in Y$  then for any  $\beta = \sum_i \beta_i e_i \in Y$  we would get that  $\sum_i (\alpha_i + \beta_i) u_i^{(\alpha+\beta)}$ , and therefore also that  $\sum_i \beta_i u_i^{(\alpha)}$ , is a convergent series. Fix  $\alpha = \{\alpha_i\}_{i=1}^\infty \in c_0$  with  $1 \geq \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_i \geq \dots \geq 0$ , and notice that in order to check whether  $\sum_i \alpha_i u_i^{(\alpha)}$  converges in  $Y$  we have to compute the  $|\cdot|_n$ -norms of the double sequence  $\{\alpha_i \alpha_j\}_{i,j=1}^\infty$ , which is the expansion of  $\sum_i \alpha_i u_i^{(\alpha)}$  with respect to  $\{e_i\}_{i=1}^\infty$ . Let  $\alpha(t)$  be a non-increasing function on  $[1, \infty)$  such that  $\alpha(i) = \alpha_i$  for all  $i$ . If, for some integer  $m$ ,  $i \cdot j = m$  then at least one of the integers  $i$  or  $j$  is greater than or equal to  $m^{1/2}$ , and therefore  $\alpha_i \alpha_j \leq \alpha(m^{1/2})$ . It follows that the non-increasing rearrangement of  $\{\alpha_i \alpha_j\}_{i,j=1}^\infty$  (as a one indexed sequence) is majorated by the sequence  $\beta = \{\beta_i\}_{i=1}^\infty$  whose explicit form is

$$\beta = \underbrace{(\alpha(1^{1/2}))}_{\tau(1) \text{ times}}, \underbrace{(\alpha(2^{1/2}), \alpha(2^{1/2}))}_{\tau(2) \text{ times}}, \dots, \underbrace{(\alpha(m^{1/2}), \dots, \alpha(m^{1/2}))}_{\tau(m) \text{ times}}, \dots,$$

where  $\tau(m)$  is the number of distinct divisors of  $m$ . Thus, for every  $n$ , we have  $|\sum_i \alpha_i u_i^{(\alpha)}|_n \leq |\beta|_n$ .

For each integer  $m$ , let  $\varphi(m)$  be the first place where  $\alpha(m)$  appears in the sequence  $\beta$ . Then, for  $\varphi(m) \leq k < \varphi(m + 1)$  we have

$$\begin{aligned} \left(\sum_{i=1}^k \beta_i i^{-1}\right) / (2^n + 2^{-n} s_k) &\leq \left(\sum_{j=1}^m \sum_{i=\varphi(j)}^{\varphi(j+1)-1} \beta_i i^{-1}\right) / (2^n + 2^{-n} s_k) \\ &\leq \left(\sum_{j=1}^m \beta_{\varphi(j)} \varphi(j)^{-1} (\varphi(j+1) - \varphi(j))\right) / (2^n + 2^{-n} s_{\varphi(m)}) \\ &\leq \left(\sum_{j=1}^m \alpha_j \varphi(j)^{-1} (\varphi(j+1) - \varphi(j))\right) / (2^n + 2^{-n} s_{\varphi(m)}) \end{aligned}$$

Since  $s_{\varphi(m)} \geq \log \varphi(m)$  we get that

$$(3) \quad |\beta|_n \leq \sup_m \left[ \sum_{j=1}^m \alpha_j \varphi(j)^{-1} (\varphi(j+1) - \varphi(j)) \right] / (2^n + 2^{-n} \log \varphi(m))$$

for all  $n$ .

To estimate further the norm of  $\beta$  we use the quite known fact (see e.g. [3, p. 118]) that  $\sum_{i=1}^k \tau(i) = k \log k + (2\gamma - 1)k + O(k^{1/2})$  where  $\gamma = 0.57721 \dots$  is the Euler constant. Notice that  $\varphi(1) = 1$ , and for  $j > 1$  we have  $\varphi(j) = \sum_{i=1}^{j^2-1} \tau(i) = (j^2 - 1) \log(j^2 - 1) + (2\gamma - 1)(j^2 - 1) + O(j)$ , consequently

$$(4) \quad \varphi(j) \geq 1 + c_1 j^2 \cdot \log j \quad \text{for all } j \geq 1, \text{ and some constant } c_1 > 0.$$

We also have

$$\begin{aligned} (5) \quad \varphi(j+1) - \varphi(j) &= \sum_{i=1}^{(j+1)^2-1} \tau(i) - \sum_{i=1}^{j^2-1} \tau(i) \\ &\leq (j^2 + 2j) \log(j^2 + 2j) - (j^2 - 1) \log(j^2 - 1) + (2\gamma - 1)(2j + 1) + O(j) \\ &\leq c_2(1 + j \log j) \quad \text{for some constant } c_2 > 0. \end{aligned}$$

Using the fact that  $s_m$  behaves asymptotically as  $\log m$  and substituting (4) and (5) in (3) we deduce that

$$|\beta|_n < c_3 \sup_m \left( \left( \sum_{j=1}^m \alpha_j j^{-1} \right) / (2^n + 2^{-n} s_m) \right) = c_3 |\alpha|_n,$$

for all  $n$  and some  $c_3 > 0$ .

Since  $\alpha \in Y$  implies that  $\alpha \in c_0$  we get that  $\|\beta\|_Y \leq c_3 \|\alpha\|_Y$ , i.e.  $\|\sum_i \alpha_i u_i^{(\alpha)}\|_Y \leq c_3 \|\alpha\|_Y$  for all  $\alpha \in Y$ .

We are ready to give the proof of the theorem. Let  $y_m = \sum_{i=p_m+1}^{p_{m+1}} \alpha_i e_i$  be a symmetric normalized block basic sequence in  $Y$ . We may assume without loss of generality that  $\alpha_{p_m+1} \geq \alpha_{p_m+2} \geq \dots \geq \alpha_{p_{m+1}} \geq 0$ , for all  $m = 1, 2, \dots$ . If  $\sup_m (p_{m+1} - p_m) < +\infty$  then clearly  $\{y_m\} \sim \{e_m\}$ ,

hence we may assume also that  $\sup_m (p_{m+1} - p_m) = +\infty$ .

Suppose now that for any  $\epsilon > 0$  there exists an integer  $N = N(\epsilon)$  such that  $\|\sum_{i=p_m+1}^{p_{m+1}} \alpha_i e_i\| < \epsilon$  for all  $m$  with  $p_{m+1} - p_m \geq N$ . In this case  $\{y_m\}_{m=1}^\infty$  is equivalent to a block basis generated by one vector and thus, by Lemma 2,  $\{y_m\} \sim \{e_m\}$ . Indeed, for any  $\epsilon > 0$ , let  $u_m = \sum_{i=1}^{p_{m+1}-p_m} \alpha_{i+p_m} e_i$  and  $u'_m = \sum_{i=1}^{N(\epsilon)} \alpha_{i+p_m} e_i$   $m = 1, 2, \dots$ . Using the fact that for all  $m = 1, 2, \dots$   $u'_m$  have at most the first  $N(\epsilon)$  coordinates distinct from zero, and  $\|u_m - u'_m\| < \epsilon$ , we deduce that  $\{u_m\}_{m=1}^\infty$  is a relatively compact set in  $Y$ . Hence there exists a subsequence  $\{u''_m\}_{m=1}^\infty$  of  $\{u_m\}_{m=1}^\infty$  such that  $\lim_{m \rightarrow \infty} u''_m = \beta = \sum_i \beta_i e_i \in Y$ . Clearly, a subsequence  $\{u_{m_j}\}_{j=1}^\infty$  of  $\{u''_m\}$  can be chosen such that  $\|u_{m_j} - \beta\| < 2^{-j}$ , and since  $\|y_m\|_Y = \|u_m\|_Y = 1$ , for all  $m$ , we have  $\|\beta\|_Y = 1$ . Notice that  $\{u_{m_j}\}_{j=1}^\infty$  is a "translation" of  $\{y_{m_j}\}_{j=1}^\infty$ . Hence  $\{y_{m_j}\}_{j=1}^\infty$  is equivalent to a block basis generated by  $\beta$ .

We treat now the case when such an  $N(\epsilon)$  does not exist for all  $\epsilon > 0$ . In this case there exists an  $\epsilon > 0$  and an increasing sequence of integers  $\{p_{m_j}\}_{j=1}^\infty$  such that  $p_{m_{j+1}} - p_{m_j} > j$  and  $\|\sum_{i=p_{m_j}+1}^{p_{m_{j+1}}} \alpha_i e_i\|_Y \geq \epsilon$  for all  $j$ . Put  $v_j = \sum_{i=p_{m_j}+1}^{p_{m_{j+1}}} \alpha_i e_i$  and  $u_j = \sum_{i=p_{m_j}+1}^{p_{m_{j+1}}} \alpha_i e_i$ . Notice that  $\alpha_{p_{m_j}+j} > c$  for some constant  $c$  and every  $j$  imply  $1 \geq \|u_j\|_Y \geq c \|\sum_{i=1}^j e_i\|$   $j = 1, 2, \dots$  i.e.  $c = 0$ . Thus,  $\lim_{j \rightarrow \infty} \alpha_{p_{m_j}+j} = 0$  which means that  $\{v_j\}_{j=1}^\infty$  is a bounded block basis of  $\{e_i\}_{i=1}^\infty$  with coefficients tending to zero. By Lemma 1 (and passing to a subsequence if necessary) we can assume that  $\{v_j\}_{j=1}^\infty$  is equivalent to a block basis  $\{z_j\}_{j=1}^\infty$  of  $\{x_n\}_{n=1}^\infty$ , the unit vector basis of  $T$ . The definition of the norm in  $T$  implies the existence of a constant  $A_1 > 0$  such that, for every  $k$ , we have  $\|\sum_{i=k+1}^{2k} z_i\|_T \geq A_1 k$ . It follows that for every integer  $k$  and some constant  $A_2 > 0$   $\|\sum_{j=1}^{2k} y_{m_j}\|_Y \geq \|\sum_{j=1}^{2k} v_j\|_Y \geq A_2 \|\sum_{j=1}^{2k} z_j\|_T \geq A_1 A_2 k$ . Since  $\{y_{m_j}\}_{j=1}^\infty$  is a symmetric basic sequence we get that  $\{y_{m_j}\}_{j=1}^\infty$ , and therefore  $\{y_m\}_{m=1}^\infty$ , is equivalent to the unit vector basis of  $\ell_1$ , contrary to Proposition 1. This completes the proof of the theorem.

REMARK: One can check that the unit balls determined by the norms  $\|\cdot\|_n$   $n = 1, 2, \dots$ , are the sets  $2^{-n}B_0 + 2^n B_d$ , where  $B_0$  and  $B_d$  are the unit balls of  $c_0$ , respectively of the Lorentz space  $d(i^{-1}, 1)$ . (Recall that  $d(i^{-1}, 1)$  is the space of all sequences  $\{\alpha_i\}_{i=1}^\infty \in c_0$  such that  $\|\alpha\|_d = \sum_{i=1}^\infty \hat{\alpha}_i i^{-1} < \infty$ , where  $\{\hat{\alpha}_i\}_{i=1}^\infty$  is the non-increasing rearrangement of  $\{\{\alpha_i\}_{i=1}^\infty\}$ ). Similarly, it can be shown that the sequence of norms  $\|\cdot\|_n$   $n = 1, 2, \dots$  defined by Figiel and Johnson in [2] can be given explicitly by the formulas

$$\|\alpha\|_n = \sup_j \left[ \left( \sum_{i=1}^j \hat{\alpha}_i \right) / (2^n + 2^{-n}j) \right] \quad n = 1, 2, \dots$$

In this case it is not true any more that for any  $\alpha = \{\alpha_i\}_{i=1}^\infty \in c_0$  we have  $\|\{\alpha_i\}_{i=1}^\infty\|_n \leq c \|\{\alpha_i \alpha_j\}_{i,j=1}^\infty\|_n$  for all  $n$  and some  $c > 0$ . Indeed, for the sequence  $\alpha_i = i^{-1/2}$  we can find constants  $A, B > 0$  such that  $\|\{i^{-1/2}\}_{i=1}^\infty\|_n \leq A$  but  $\|\{(ij)^{-1/2}\}_{i,j=1}^\infty\|_n \geq Bn^{1/2}$  for all  $n = 1, 2, \dots$

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