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SOME PROPERTIES OF THE RING OF GERMS OF C^∞-FUNCTIONS

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1. Introduction and summary

Let k denote the field $\mathbb{R}$ or $\mathbb{C}$. Let $\mathbf{X} = (X_1, \ldots, X_n)$ denote n-variables and the $C^\infty(\mathbf{X}) = C^\infty(X_1, \ldots, X_n)$ denote the ring of germs of C^∞-functions in the n-variables $\mathbf{X}$. The ringhomomorphism $\pi : C^\infty(\mathbf{X}) \to k[[\mathbf{X}]] = k[[X_1, \ldots, X_n]]$ which assigns to a C^∞-germ its Taylor expansion, is surjective.

An interesting question is whether there exists a ringhomomorphism $\varphi : k[[\mathbf{X}]] \to C^\infty(\mathbf{X})$ with $\pi\varphi = id$. In dimension one ($n = 1$) the existence of many such φ's is proved (by K. Reichard [5], M. Shiota [7] and in §3). For $n > 1$ we can only show that for some subrings A of $k[[\mathbf{X}]]$ (e.g. A/k finitely generated or A an analytic subring) the existence of a ringhomomoprhism $\varphi : A \to C^\infty(\mathbf{X})$ with $\pi\varphi = id_A$. The theorems of this type have the form of the well-known Artin-approximation theorem (§4).

The problem of the existence of a ringhomomorphism $\varphi : k[[\mathbf{X}]] \to C^\infty(\mathbf{X})$ with $\pi\varphi = id$ resembles the question whether a given local ring has a coefficient field. The latter depends on some form of completeness of the local ring. We also start (§2) by discussing several properties of $C^\infty(\mathbf{X})$ that are related to completeness.

I am grateful to F. Takens who interested me in germs of C^∞-functions and provided counter examples to some of my rash statements. Thanks are due to the referee, K. Reichard, and to M. Shiota for correcting some errors and for some commentary.

2. The Hensel property and related statements

Let R denote the ring $C^\infty(\mathbf{X})$, let $\underline{m}$ denote its maximal ideal, $\underline{m}^\infty = \bigcap_{n \geq 1} \underline{m}^n$ = the ideal of the "flat" functions. Then $R/\underline{m} = k$ and $R/\underline{m}^\infty = k[[X_1, \ldots, X_n]]$ (See [2]).

(2.1) PROPOSITION: *R is a Hensalian ring.*

PROOF: Given a monic polynomial $P \in R[T]$ such that its residue $\bar{P} \in k[T]$ splits into $\bar{P} = F_1, F_2$ with F_1, F_2 monic and g.c.d. $(F_1, F_2) = 1$. We have to show the existence of monic polynomials $P_i \in R[T]$ such that $P = P_1P_2$ and $\bar{P}_i = F_i$ $(i = 1, 2)$.

Since $R/\underline{m}^\infty$ is Henselian there are monic polynomials Q_1, $Q_2 \in R[T]$ with $\bar{Q}_i = F_i$ $(i = 1, 2)$ and $Q_1Q_2 \equiv P$ modulo $\underline{m}^\infty$. We have to change the Q_i a little in order to obtain the P_i. This we will do in the next lemmata.

(2.2) LEMMA: *$\{Q_1, Q_2\}$ generate the unit ideal in $R[T]$.*

PROOF: Suppose that Q_1, Q_2 are contained in a maximal ideal N of $R[T]$. Put $\underline{P} = N \cap R$ and let t denote the image of T in $R[T]/N$. Then $R[T]/N = R/\underline{P}[t]$ and t is integral over $R/\underline{P}$ since $Q_1(t) = 0$. Hence $R/\underline{P}$ is a field and $\underline{P} = \underline{m}$. Then we find the contradiction: $N \supseteq (Q_1, Q_2, \underline{m}) \supseteq (F_1, F_2) \ni 1$.

(2.3) LEMMA: *Let $a_i = \deg Q_1$ $(i = 1, 2)$ and let $R[T]_a$ denote the set of polynomials of degree $< a$. Then the map $\tau: R[T]_{a_1} \oplus R[T]_{a_2} \to R[T]_{a_1+a_2}$, given by $\tau(f, g) = gQ_1 + fQ_2$ is R-linear and bijective.*

PROOF: Any $h \in R[T]_{a_1+a_2}$ has, according to (2.2), the form $g_0Q_1 + f_0Q_2$. The Q_i's are monic. So we can write uniquely $g_0 = qQ_2 + g$ and $f_0 = \tilde{q}Q_1 + f$ with $f \in R[T]_{a_1}$, $g \in R[T]_{a_2}$. Hence (f, g) is the unique element with $\tau(f, g) = h$.

(2.4) LEMMA: *There are flat functions (i.e. elements of $\underline{m}^\infty$) $\varepsilon_0, \ldots, \varepsilon_{a_1-1}$, $\delta_0, \ldots, \delta_{a_2-1}$ such that $(Q_1 + \Sigma_{i<a_1} \varepsilon_i T^i)(Q_2 + \Sigma_{i<a_2} \delta_i T^i) = P$.*

PROOF: Put $f = \Sigma\, \varepsilon_i T^i$ and $g = \Sigma\, \delta_i T^i$. Then the required equation becomes $\tau(f, g) + fg = P - Q_1Q_2 =$ a polynomial of degree $< a_1 + a_2$ with flat functions as coefficients. By the implicite function theorem and (2.3) there is a unique solution (f_0, g_0). Further $(f_0, g_0) + \underline{m}^\infty$ is the unique solution of $\tau(f, g) + fg \equiv 0 \bmod \underline{m}^\infty$. So f_0, $g_0 \in \underline{m}^\infty$.

(2.5) REMARK: (a) The Hensel-property is a form of completeness of the ring. We will now discuss some related properties.

A subset of a ring R will be called *convex* if it has the form $a + I$, where I is some ideal of R. A filter $\mathcal{F}$ on R will be called *convex* if $\mathcal{F}$ has a base of convex sets. If R is a local noetherian complete ring, or if R is a maximally complete valuation ring, then every convex filter on R has a non-empty intersection.

Our guess that $C^\infty(\mathbf{X})$ also has this filter-property is false, as the counter example (2.6) constructed by F. Takens shows.

(b) The referee, K. Reichard, remarks: a small variation of the proof (2.1)–(2.4) shows: "The local ring $C^N(X)$, $N<\infty$, of germs at o of N-times differentiable functions, is Henselian". In the proof above one has to replace $\underline{m}^\infty$ by $\{f \in C^N(X) | f^{(\alpha)}(0)=0 \text{ for all } \alpha \text{ with } |\alpha| \le N\}$.

(2.6) *Counter example to the filter-property*

The family of all flat functions of one variable X has the same cardinal as $\mathbb{R}$, so we can denote this family by $\{\varepsilon_\lambda | \lambda \in [-1,+1]\}$. In the ring $C^\infty(X, Y)$ we consider the family of convex sets $C_\lambda = \dot{\varepsilon}_\lambda + (Y-\lambda X)C^\infty(X, Y)$, (here $\dot{\varepsilon}_\lambda$ denotes the germ at $X=0$ of ε_λ). This family generates a convex filter $\mathcal{F}$ on $C^\infty(X, Y)$. Indeed, for different points $\lambda_1, \ldots, \lambda_n \in [-1,+1]$ the set $C_{\lambda_1} \cap \cdots \cap C_{\lambda_n}$ contains the germ of the function

$$\sum_{i=1}^{n} \varepsilon_{\lambda_i}(X) \prod_{j \ne i} \frac{(Y-\lambda_j X)}{(\lambda_i X - \lambda_j X)}.$$

We claim that: $\cap \mathcal{F} = \emptyset$.

Suppose that $\cap\mathcal{F}$ contains the germ of the C^∞-function $f: \mathbb{R}^2 \to k$. Define a function F by $F(t) = \max\{|f(x, xy)| \,\big|\, |x| \le t, \ |y| \le 1\}$. The function F is continuous, decreasing and satisfies $F(t) = 0(t^n)$ for all $n \ge 1$. To see the last statement, we observe that f is flat at $(0,0)$. Indeed, $\cap_\lambda (Y-\lambda X)k[[X, Y]] = 0$. So $f(X, XY) = X^n g(X, Y)$ for some C^∞-function g. Then $F(t) \le t^n \max\{|g(x, y)| \,\big|\, |x| \le 1, \ |y| \le 1\}$ for $0 \le r \le 1$.

We now want to construct a flat function ε such that the set $\{t \ge 0 | F(t) < \varepsilon(t)\}$ has 0 in its closure. If ε exists then we find a contradiction in the following way:

For some $\lambda \in [-1, 1]$, $\varepsilon_\lambda = 2\varepsilon$. Then $F(t) \ge |f(t, \lambda t)|$ and for t small enough (depending on λ) $|f(t, \lambda t)| = \varepsilon_\lambda(t) = 2\varepsilon(t)$. In order to construct ε, we consider a sequence of points $t_k > 0$ with limit zero and such that $F(t_k) = t_k^k \alpha_k$ and $\Sigma\, \alpha_k^{k!} < \infty$. There is a C^∞-function ε_k (see [2]) with the properties:

(a) $\varepsilon_k \ge 0$ everywhere; $0 \notin \operatorname{supp}(\varepsilon_k)$.

(b) $\varepsilon_k(t_k) = 2F(t_k)$.

(c) the C^k-norm of ε_k, $\|\varepsilon_k\|_{(k)} \le 4\alpha_k^{k!}$.

Then $\varepsilon = \Sigma\, \varepsilon_k$ has the required properties.

(2.7) In spite of (2.6), convex subsets of $C^\infty(\mathbf{X})$ have an interesting intersection property. Let a C^∞-function $\varphi: \{t \in \mathbb{R} | t \ge 0\} \to [0, 1]$ be given such that $\varphi(t) = 1$ in a neighbourhood of 0 and $\varphi(t) = 0$ for $t \geq 1$.

An ideal I in $C^\infty(\mathbf{X})$ will be called s-closed if it has the following property:

For any sequence $A_1, A_2, A_3, \ldots$ of C^∞-functions with lim (order of A_k at 0) $= \infty$ defined on some neighbourhood of 0, with all germs in I, there exists a sequence $\lambda_1, \lambda_2, \lambda_3, \ldots$ of positive real numbers such that for any sequence $\mu_1, \mu_2, \mu_3, \ldots$ with $0 < \mu_i \leq \lambda_i$ for all i, the sum $\Sigma\, A_i\varphi(\mu_i^{-1}(x_1^2 + \cdots + x_n^2))$ converges in the C^∞-topology (see [2]) and defines a germ in I.

PROPOSITION:

(2.7.1) *If a filter $\mathscr{F}$ on $C^\infty(\mathbf{X})$ is generated by a countable family $\{a_n + I_n | n \geq 1\}$ and all I_n's are s-closed, then $\cap \mathscr{F} \neq \emptyset$.*

(2.7.2) *If $f \in C^\infty(\mathbf{X})$ is not flat then $fC^\infty(\mathbf{X})$ is a s-closed ideal.*

(2.7.3) *Any ideal containing $\underline{m}^\infty$ is s-closed.*

(2.7.4) *For $f \in C^\infty(\mathbf{X})$ we denote the germ (at 0) of the support of f by $S(f)$. For any germ F of a closed set the ideal $I(F) = \{f \in C^\infty(\mathbf{X}) | S(f) \subseteq F\}$ is s-closed.*

(2.7.5) *For $s \in C^\infty(\mathbf{X})$ the ideal $I = \{f \in C^\infty(\mathbf{X}) | fs = 0\}$ is s-closed.*

(2.7.6) *The countable intersection of closed ideals is again s-closed.*

PROOF:

(2.7.1) We may suppose $a_1 + I_1 \supseteq a_2 + I_2 \supseteq a_3 + I_3 \supseteq \cdots$. Modulo $\underline{m}^\infty$ there is an element f in the intersection. After subtracting f, we arrive at the situation where all a_i are in $\underline{m}^\infty$. Let the C^∞-functions $A_1, A_2, A_3, \ldots$ have germs a_1, a_2, a_3, at 0. For a suitable sequence $\lambda_1, \lambda_2, \lambda_3, \ldots$ of positive real numbers $A = A_1\varphi(\lambda_1^{-1}(x_1^2 + \cdots + x_n^2)) + \Sigma_{m=1}^\infty (A_{m+1} - A_m)\varphi(\lambda_m^{-1}(x_1^2 + \cdots + x_n^2))$ converges in the C^∞-topology and for all N, the germ of $\Sigma_{m \geq N} (A_{m+1} - A_m)\varphi(\lambda_m^{-1}(x_1^2 + \cdots + x_n^2))$ belongs to I_N. Hence a, the germ of A, belongs to $\cap a_n + I_n$.

(2.7.2) Let (A_i) be the given sequence and let F represent the germ f. For suitable positive real numbers λ_i, we have $A_i \cdot \varphi(\lambda_i^{-1}(x_1^2 + \cdots + x_n^2)) = B_iF$ and

(a) lim (order of B_i at 0) $= \infty$

(b) $\Sigma\, B_i$ converges in the C^∞-topology.

The same holds for any sequence (μ_i) with $0 < \mu_i \leq \lambda_i$. So $\Sigma\, A_i\varphi(\mu_i^{-1}(x_1^2 + \cdots + x_n^2)) = (\Sigma\, B_i\varphi(\mu_i^{-1}(x_1^2 + \cdots + x_n^2)))F$ and its germ lies in $fC^\infty(\mathbf{X})$.

(2.7.3) Trivial.

(2.7.4) Let (A_i) be the given sequence and let F represent the germ of a closed set. Choose the λ_i such that $\Sigma\, A_i\varphi(\mu_i^{-1}(x_1^2+\cdots+x_n^2))$ converges in C^∞-topology for all $0<\mu_i\le\lambda_i$ and such that $\operatorname{supp}(A_i\varphi(\lambda_i^{-1}(x_1^2+\cdots+x_n^2))\subseteq F$.

(2.7.5) Similar to (2.7.4).

(2.7.6) Obvious.

(2.7.7) PROPOSITION: *If $f\in C^\infty(\mathbf{X})$, $f\neq 0$, is a flat function then the ideal $fC^\infty(X)$ is not s-closed.*

PROOF: Let f be represented by F and choose $A_i = F$ for all i. For any sequence (λ_i) such that $A=\Sigma\, A_i\varphi(\lambda_i^{-1}(x_1^2+\cdots+x_n^2))$ converges in the C^∞-topology, the germ a of A does not belong to $fC^\infty(\mathbf{X})$.

(2.7.8) REMARKS:

(1) The definition of s-closed ideal seems to depend on the choice of φ. However for any other function ψ of that type we have $\varphi(\lambda_1 t)\le\psi(t)\le\varphi(\lambda_2 t)$ (all $t\ge 0$ and suitable $\lambda_1,\ \lambda_2>0$).

(2) Returning to the counter example (2.6), we see by (2.7.1) and (2.7.2) that every countable intersection $\cap_{i=1}^{\infty} C_{\lambda_i}$ is non-empty.

3. Dimension one

Let $C^\infty(X)$ denote the ring of germs of C^∞-functions in one variable X. Again $\pi: C^\infty(X)\to k[[X]]$ denotes the canonical ringhomomorphism.

(3.1) PROPOSITION (see also [5] and [7]): *Let a subalgebra A of $k[[X]]$ and a k-algebrahomomorphism $\varphi: A\to C^\infty(X)$ be given such that $\pi\varphi = id_A$. Then there exists a k-algebrahomomorphism $\psi: k[[X]]\to C^\infty(X)$ extending φ and such that $\pi\psi = id$.*

PROOF: Let for a domain R its quotient field be denoted by $Qt(R)$. Choose a transcendance base $\{t_i \mid i\in I\}\subset k[[X]]$ of $Qt(k[[X]])$ over $Qt(A)$. Choose further elements $t_i^*\in C^\infty(X)$ with $\pi(t_i^*) = t_i$ and extend φ to $\varphi_1: A_1 = A[t_i \mid i\in I]\to C^\infty(X)$ by $\varphi_1(t_i) = t_i^*$. Let $A_2 = Qt(A_1)\cap k[[X]]$ and let $c = a/b\in A_2$ with $a,\ b\in A_1$. Then we have $a = x^{n_1}u_1$, $b = x^{n_2}u_2$; $n_1\ge n_2$; $u_1,\ u_2\in k[[X]]^*$. Then also $\varphi_1(a) = x^{n_1}\tilde{u}_1$,

$\varphi_1(b) = x^{n_2}\tilde{u}_2$ with $\tilde{u}_1$, $\tilde{u}_2 \in C^\infty(X)^*$. Hence there is a unique element $\tilde{c} \in C^\infty(X)$ with $\varphi_1(a) = \tilde{c}\varphi_1(b)$. Define now $\varphi_2 : A_2 \to C^\infty(X)$ by $\varphi_2(c) = \tilde{c}$; φ_2 is a ringhomomorphism and $\pi\varphi_2 = id_{A_2}$.

We may suppose without loss of generality that A_2 contains an element t of order 1. It follows that A_2 is a discrete valuation ring and $\hat{A}_2 = k[[X]]$. Let A_3 denote the henselisation of A_2, then $A_2 \subset A_3 \subset k[[X]]$ and $\hat{A}_3 = k[[X]]$. (See [4], Ch. VIII). The map $\varphi_2 : A_2 \to C^\infty(X)$ extends uniquely to a map $\varphi_3 : A_3 \to C^\infty(X)$, since $C^\infty(X)$ is Henselian. Let L be a field which is a finite extension of $Qt(A_3)$ and contained in $Qt(k[[X]])$. The integral closure A_4 of A_3 in L is a finitely generated A_3-module; it is a local ring since A_3 is Henselian and $A_4 \subseteq k[[X]] \cap L$. Further $A_4/tA_4 = k$, so by Nakayama's lemma it follows that $A_3 = A_4$ and consequently $Qt(A_3) = Qt(k[[X]])$.

Then clearly $A_3 = k[[X]]$ and the proof is finished.

(3.2) Corollary (see also [5] and [7]).

(3.2.1). *There is a k-algebrahomomorphism* $\varphi : k[[X]] \to C^\infty(X)$ *with* $\pi\varphi = id$. *The k-algebrahomomorphism* $\varphi\pi : C^\infty(X) \to C^\infty(X)$ *is a quasi-finite map but not a finite map. The map* $\varphi\pi$ *is not induced by a* C^∞*-map.* (Compare with questions in [6]).

(3.2.2). *Given a family of polynomials* $P_i(T_1, \ldots, T_n)$ $(i = 1, \ldots, N)$ *with coefficients in* $C^\infty(X)$. *Let* A' *denote the k-subalgebra of* $C^\infty(X)$ *generated by the coefficients of all the* P_i*'s. Suppose that* $A' \cap \underline{m}^\infty = 0$. *Then any formal solution* $t_1, \ldots, t_n \in k[[X]]$; $P_i(t_1, \ldots, t_n) \equiv 0$ *modulo* $\underline{m}^\infty$ *lifts to a* C^∞*-solution* $\tilde{t}_1, \ldots, \tilde{t}_n \in C^\infty(X)$ *i.e.* $\pi(\tilde{t}_j) = t_j$ *(all j) and* $P_i(\tilde{t}_1, \ldots, \tilde{t}_n) = 0$ *(all i).*

Proof: The k-algebrahomomorphism $\varphi\pi$ is quasi-finite since $\varphi\pi(x) = xu$, $u \in C^\infty(X)^*$. The map is not finite. Indeed if $\varphi\pi$ is a finite map then by the lemma of Nakayama $\varphi\pi$ would be surjective. But $\varphi\pi$ is clearly not surjective. If $\varphi\pi$ were induced by a C^∞-map f then $f = \varphi\pi(X)$ and $\varphi\pi(g) = g \circ f$ for any g. This is however false since $\varphi\pi(g) = 0$ for every $g \in \underline{m}^\infty$. The second part (3.2.2) follows easily from (3.1).

(3.3) Remarks: (3.3.1) The statement (3.2.2) can be generalized a little: Let $I \subseteq \underline{m}^\infty$ be an ideal with $XI = I$ and $A' \cap \underline{m}^\infty \subseteq I$. Then any formal solution can be lifted to elements $\tilde{t}_1, \ldots, \tilde{t}_n \in C^\infty(X)$ such that $\pi(\tilde{t}_i) = t_i$ (all i) and $P_j(\tilde{t}_1, \ldots, \tilde{t}_n) \equiv 0$ (modulo I) for all j.

Indeed the ring $C^\infty(X)/I$ is again Henselian and one can prove the statement (3.1) also for this ring.

(3.3.2) M. Shiota recently obtained the following (yet unpublished) result: There is a k-algebrahomomorphism $\varphi : k[[X]] \to C^\infty(X)$ satisfying $\pi\varphi = id$ and $\varphi(f') = \varphi(f')$ for all f. ($' =$ differentiation). In his proof he uses a result of B. Malgrange that states the possibility of lifting a formal solution of certain differential equation to a C^∞ solution.

4. Artin-approximation for germs of C^∞-functions

The ring of germs of C^∞-functions in the n variables $\mathbf{X} = (X_1, \ldots, X_n)$ will be denoted by $C^\infty(\mathbf{X})$ or $C^\infty(X_1, \ldots, X_n)$. The surjective ringhomomorphism $C^\infty(\mathbf{X}) \to k[[\mathbf{X}]] = k[[X_1, \ldots, X_n]]$ will again be denoted by π. For some k-subalgebra's $A \subseteq k[[\mathbf{X}]]$ we will prove the existence of a k-algebrahomomorphism $\varphi : A \to C^\infty(\mathbf{X})$ with $\pi\varphi = id_A$. We allow two types of subalgebra's A, namely

(1) A is *finitely generated over k.*
(2) A is *an analytic subalgebra* of $k[[\mathbf{X}]]$, i.e. A is the image of a k-algebrahomomorphism $k\{T_1, \ldots, T_d\} \to k[[\mathbf{X}]]$ (any $d \geq 1$). Here $k\{T_1, \ldots, T_d\}$ denotes the local ring of convergent power series in d variables.

Of course a local map $\varphi : A \to C^\infty(\mathbf{X})$, say A of type (1), can be extended somewhat by taking a localisation and a Henselisation of A. The proof in case (1) will be given explicitely. The proof in case (2) is very much the same as in case (1) and we will leave this to the imagination of the reader (see also [1] and [3]).

We consider two statements; the first of which covers case (1).

(4.1.n) PROPOSITION: *Let* $\mathbf{X}$ *denote n variables* $(X_1, \ldots, X_n)$ *and* $\mathbf{Y}$ *denotes N variables* $(Y_1, \ldots, Y_N)$. *Given an ideal* $F \subset k[\mathbf{X}, \mathbf{Y}]$ *and a* $k[\mathbf{X}]$*-homomorphism* $\varphi : k[\mathbf{X}, \mathbf{Y}] \to k[[\mathbf{X}]]$ *with* $\varphi(F) = 0$, *there exists a* $k[\mathbf{X}]$*-homomorphism* $\psi : k[\mathbf{X}, \mathbf{Y}] \to C^\infty(\mathbf{X})$ *with* $\pi\psi = \varphi$ *and* $\psi(F) = 0$.

(4.2.n) PROPOSITION: $\mathbf{X}$ *and* $\mathbf{Y}$ *are as in* (4.1.n). *Given an ideal* $F \subset k[\mathbf{X}, \mathbf{Y}]$, *an element* $G \in k[\mathbf{X}, \mathbf{Y}]$ *and a* $k[\mathbf{X}]$*-homomorphism* $\varphi : k[\mathbf{X}, \mathbf{Y}] \to k[[\mathbf{X}]]$ *such that* $\varphi(F) \subseteq \varphi(G)k[[\mathbf{X}]]$ *and* $\varphi(G) \neq 0$. *Then there exists a* $k[\mathbf{X}]$*-homomorphism* $\psi : k[\mathbf{X}, \mathbf{Y}] \to C^\infty(\mathbf{X})$ *with* $\pi\psi = \varphi$ *and* $\psi(F) \subseteq \psi(G)C^\infty(\mathbf{X})$.

PROOF: We will show (4.1.n − 1)⇒(4.2.n) and (4.2.n)⇒(4.1.n).

(4.3) LEMMA: (4.1.n − 1) *implies* (4.2.n).

PROOF: After a linear change of the variables X we may suppose that $\varphi(G)$ is general in X_n of order d (i.e. $\varphi(G)(0,0,\ldots,0,X_n)$ has order d). We introduce new variables $T_0,\ldots, T_{d-1}$; $Y_{ij}(i=1,\ldots, N: 0\le j<d)$; $\tilde{Y}_i(i=1,\ldots, N)$ and a $k[\mathbf{X}]$-homomorphism $\tau: k[\mathbf{X},\mathbf{Y}]\to k[\mathbf{X}, Y\ldots, \tilde{Y},,.T.]$ given by

$$\tau(Y_i)=\sum_{j=0}^{d-1} Y_{ij}X_n^j+\tilde{Y}_i(X_n^d+T_{d-1}X_n^{d-1}+\cdots+T_0).$$

Let $\mathbf{X}'$ denote the variables $X_1,\ldots, X_{n-1}$. There is a $k[\mathbf{X}]$-homomorphism $\tilde{\varphi}: k[\mathbf{X}, Y.., \tilde{Y}., T.]\to k[[X]]$ with $\varphi=\tilde{\varphi}\circ\tau$ and $\tilde{\varphi}(T.), \tilde{\varphi}(Y..)\in k[[\mathbf{X}']]$. Namely:

(a) $\tilde{\varphi}(T_i)$ are the unique elements in $k[[\mathbf{X}']]$ such that $\varphi(G)=(X_n^d+\tilde{\varphi}(T_{d-1})X_n^{d-1}+\cdots+\tilde{\varphi}(T_0))\cdot$ (unit of $k[[\mathbf{X}]]$).

(b) $\tilde{\varphi}(Y_{ij})\in k[[\mathbf{X}']]$ and $\tilde{\varphi}(\tilde{Y}_i)\in k[[\mathbf{X}]]$ are the unique elements satisfying

$$\varphi(Y_i)=\sum_{j=0}^{d-1}\tilde{\varphi}(Y_{ij})X_n^j+\tilde{\varphi}(\tilde{Y}_i)(X_n^d+\tilde{\varphi}(T_{d-1})X_n^{d-1}+\cdots+\tilde{\varphi}(T_0)).$$

For generators $F_1,\ldots, F_s$ of the ideal F, we write

$$\tau(F_i)=\sum_{j=0}^{d-1}F_{ij}(\mathbf{X}', Y.., T.)X_n^j + \tilde{F}_i(\mathbf{X}, Y.., T.\ \tilde{Y}.)(X_n^d+T_{d-1}X_n^{d-1}+\cdots+T_0)$$

$$\tau(G)=\sum_{j=0}^{d-1}G_j(\mathbf{X}', Y.., T.)X_n^j + \tilde{G}(\mathbf{X}, Y.., T., \tilde{Y}.)(X_n^d+T_{d-1}X_n^{d-1}+\cdots+T_0)$$

This is simply "division with remainder" by $(X_n^d+T_{d-1}X_n^{d-1}+\cdots+T_0)$. Since $\varphi(F)\subseteq\varphi(G)k[[X]]$ we have $\ker\tilde{\varphi}\supseteq(F.., G.)$.

The map $\varphi_0=\tilde{\varphi}|k[\mathbf{X}', Y.., T.]\to k[[\mathbf{X}']]$ is $k[\mathbf{X}']$-linear and $\ker\varphi_0\supseteq(F.., G.)$.

According to (4.1.n − 1) the map φ_0 lifts to $\psi_0: k[\mathbf{X}', Y.., T.]\to C^\infty(\mathbf{X}')$ with $\ker\psi_0\supseteq(F.., G.)$. Define $\tilde{\psi}: k[\mathbf{X}, Y.., T., \tilde{Y}.]\to C^\infty(\mathbf{X})$ by $\tilde{\psi}(X_n)=X_n$; $\tilde{\psi}(Y_{ij})=\psi_0(Y_{ij})$; $\tilde{\psi}(T_i)=\psi_0(T_i)$ and $\tilde{\psi}(\tilde{Y}_i)=\alpha_i$ with $\pi(\alpha_i)=\tilde{\varphi}(\tilde{Y}_i)$.

Then $\pi\tilde{\psi}=\tilde{\varphi}$; and $\psi=\tilde{\psi}\tau$ satisfies $\pi\psi=\varphi$ and

$$\psi(F_i)=\tilde{\psi}(\tilde{F}_i(\mathbf{X}, Y.., T., \tilde{Y}.))(X_n^d+\tilde{\psi}(T_{d-1})X_n^{d-1}+\cdots+\tilde{\psi}(T_0))$$

$$\psi(G)=\tilde{\psi}(\tilde{G}(\mathbf{X}, Y.., T., \tilde{Y}.))(X_n^d+\tilde{\psi}(T_{d-1})X_n^{d-1}+\cdots+\tilde{\psi}(T_0)).$$

Since $\tilde{\psi}(\tilde{G}(\mathbf{X}, Y.., T., \tilde{Y}..))$ is a unit in $C^\infty(\mathbf{X})$, it follows that $\psi(F)\subseteq\psi(G)C^\infty(\mathbf{X})$.

(4.4) LEMMA: (4.2.n) *implies* (4.1.n).

PROOF: The kernel F of the given map $\varphi : k[\mathbf{X}, \mathbf{Y}] \to k[[\mathbf{X}]]$ is a prime ideal of some height r. There are elements $F_1, \ldots, F_r \in F$ generating $Fk[\mathbf{X}, \mathbf{Y}]_F$. The usual Jacobi-criterium for simple points implies that $\Delta = \dfrac{\partial F_1, \ldots, F_r}{\partial Y_1, \ldots, Y_r}$ (suitable variables $Y_1, \ldots, Y_r$) does not belong to F. Hence $\varphi(\Delta) \neq 0$ and according to (4.2.n) there exists a lifting $\psi : k[\mathbf{X}, \mathbf{Y}] \to C^\infty(\mathbf{X})$ of φ such that $\psi(F) \subseteq \psi(\Delta)^2 C^\infty(X)$.

We want to change the map ψ a little such that at least $\psi(F_1) = \cdots = \psi(F_r) = 0$. The new map $\tilde{\psi} : k[\mathbf{X}, \mathbf{Y}] \to C^\infty(\mathbf{X})$ will have the form

$$\tilde{\psi}(Y_i) = \psi(Y_i) + h_i \psi(\Delta) \qquad (i = 1, \ldots, r)$$

$$\tilde{\psi}(Y_i) = \psi(Y_i) \qquad (i = r+1, \ldots, N)$$

Here $h = (h_1, \ldots, h_r)$ are flat functions in $C^\infty(X)$. They should solve the equations

$$F_i(y_1 + h_1\Delta(y), \ldots, y_r + h_r\Delta(y), y_{r+1}, \ldots, y_N) = 0 \quad (i = 1, \ldots, r),$$

where

$$y_i = \psi(Y_i) \text{ and } y = (y_1, \ldots, y_N).$$

Equivalently:

$$F_i(y) + \sum_{j=1}^{r} \Delta(y) h_j \frac{\partial F_i}{\partial Y_j}(y) + \Delta(y)^2 \sum h_j h_k G_{jk}(y, h) = 0 \qquad (i = 1, \ldots, r).$$

Here we view $h_1, \ldots, h_r$ as new variables and the $G_{jk}(y, h)$ are C^∞-functions in y and h. After inverting the matrix$([\partial F_i/\partial Y_j](y))$ (i.e. multiply the equations by the matrix of minors of $([\partial F_i/\partial Y_j](y))$ and divide them by $\Delta(y)^2$) the equations become (*) $h_i + \Sigma\, h_j h_k H_{jk}(y, h) = k_i(y)$ $(i = 1, \ldots, r)$. Here H_{jk} are C^∞-functions and the K_i are flat functions.

The inverse function theorem yields a unique solution h of (*) and the h_i are then flat functions.

So we have arrived at a lifting $\tilde{\psi}$ of φ satisfying $\tilde{\psi}(F_i) = 0$ $(i = 1, \ldots, r)$. From the choice of the $F_1, \ldots, F_r$ it follows that for some $a \in k[\mathbf{X}, \mathbf{Y}] - F$ we have a $F \subseteq (F_1, \ldots, F_r)$. So $\tilde{\psi}(a)\tilde{\psi}(F) = 0$. From the next lemma it follows that $\tilde{\psi}(F) = 0$ and we constructed the required map in (4.1.n).

(4.5) LEMMA: *Let $a \in C^\infty(\mathbf{X})$ such that $\pi(a) \in k[[\mathbf{X}]]$ is non-zero. Then a is a non-zero-division on $C^\infty(\mathbf{X})$.*

PROOF: Let the C^∞-function $A: U \to k$, U an open neighbourhood of 0, have germ a at 0. Let V be the interior of $\{x \in U | A(x) = 0\}$. Then A and all its derivatives are zero on V. If $0 \in \bar{V}$ then all derivatives of A are zero in 0. However, $\pi(a) \neq 0$, so $0 \notin \bar{V}$.

Let $ab = 0$ and let B represent the germ b. After a possible shrinking of U, we have $\{u \in U | B(u) \neq 0\} \subseteq V \subseteq \bar{V}$. Hence B is zero in a neighbourhood of 0. So $b = 0$.

(4.6) *Concluding remarks*

The ring $k[[\mathbf{X}]] = \varinjlim A$ = the direct limit of finitely generated $k[X]$-algebra's A. Let $\Phi(A)$ denote the non-empty set of $k[\mathbf{X}]$-homomorphisms $\varphi: A \to C^\infty(\mathbf{X})$ with $\pi\varphi = id_A$. The existence of a $\varphi: k[[\mathbf{X}]] \to C^\infty(\mathbf{X})$ with $\pi\varphi = id$, is equivalent with $\varprojlim \Phi(\mathrm{A}) \neq \emptyset$. Using (2.7) one can show for any countable subfamily $(A_n)_{n\in N}$ that $\varprojlim \Phi(A_n) \neq \emptyset$.

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