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NEW UNIQUE MODELS OF REPRESENTATIONS OF UNITARY GROUPS

Mark E. Novodvorsky

Introduction

Let π be an irreducible admissible representation of a unitary group G over a p -adic field, with sufficiently small anisotropic kernel. This paper introduces a family of Z -eigen-functionals on the space of π , Z a subgroup of G , which are unique up to a scalar multiplier. This is equivalent to the uniqueness of corresponding Whittaker models of the representation π , and allows one to obtain Euler products for some meromorphic functions attached to automorphic forms on orthogonal groups – the details of the latter will appear in a forthcoming paper.

When G is a split orthogonal group of type B_2 realized as the group of 4×4 symplectic matrices, the functionals of this paper are essentially the same as those considered by I. I. Pyatetsky-Shapiro and the author in [5]. The proof follows the modification of arguments of I. M. Gelfand and D. A. Kajdan [3] introduced by the author in [6].

1. The result and its reduction

Let k be a locally compact non-discrete disconnected field, ℓ an extension of it,

$$(1) \quad \lambda \rightarrow \bar{\lambda}, \quad \lambda \in \ell,$$

an involution of ℓ trivial on k only (particularly, $\ell : k \leq 2$). Suppose the characteristic of k different from 2.

Algebraic groups defined over k will be identified throughout this paper with the groups of their k -points in the natural locally compact topology. The linear span of a set of vectors $\{e_i : 1 \leq i \leq n\}$ in a linear space over ℓ will be denoted $\{e_i | 1 \leq i \leq n\}$.

Let G denote the group of all automorphisms of a Hermitian form (\dots, \dots) on a linear space over ℓ with base $\{e_i : 1 \leq i \leq n\}$. Let e be a linear combination of vectors e_i , $r \leq i \leq n - r$, with coefficients from k .

Suppose

$$(2) \quad \begin{cases} (e_i, e_j) = (e_j, e_i) \forall_{i,j} \leq n; \\ (e_i, e_j) = \begin{cases} 0 & \text{if } i+j \neq n+1, \\ 1 & \text{if } i+j = n+1, \end{cases} & 1 \leq i \leq r, 1 \leq j \leq n; \\ (e, e) \neq 0. \end{cases}$$

Define

$$(3) \quad \begin{aligned} T &= \{g \in G : ge = e; ge_i = e_i \text{ if } i \leq r \text{ or if } i > n - r\}, \\ U &= \{g = (a_{ij}) \in G : a_{ii} = 1 \forall_i; a_{ij} = 0 \text{ if } j < i \text{ or} \\ &\quad Z = T \times U. \hspace{15em} \text{if } r < i < j \leq n - r\}, \end{aligned}$$

T and U are subgroups of G ; T normalizes U ; hence, Z is a subgroup too.

Take a non-trivial character χ of the additive group of the field k and a character τ of the group T trivial on the subgroup of unimodular matrices from T if $\ell \neq k$ and of order 2 if $\ell = k$.

Define

$$(4) \quad \begin{cases} \theta(u) = \chi \left(\text{tr} \left(\sum_{i=1}^{r-1} a_{i,i+1} + (e_{n-r+1}, ue) \right) \right), & u = (a_{ij}) \in U; \\ \alpha(z) = \tau(t) \cdot \theta(u), & z = t \cdot u \in Z. \end{cases}$$

Then θ and α are characters of the subgroups U and Z , respectively.

Assume that *the dimension r' of a maximal isotropic subspace of the form (\dots, \dots) does not exceed $r + 1$.*

THEOREM: *For any irreducible admissible representation π of the group G the dimension $\dim(\pi, \alpha)$ of all linear Z -eigenfunctionals with character α is, at most, 1.*

Define an involution of the group G :

$$(5) \quad \begin{aligned} \tilde{g} &= I^{-1}(\bar{a}_{ij})^{-1}I, \quad g = (a_{ij}) \in G, \\ Ie_i &= \begin{cases} (-1)^i e_i & \text{if } i \leq r, \\ (-1)^{r+1} e_i & \text{if } r < i \leq n - r, \\ (-1)^{n+1-i} e_i & \text{if } n - r < i. \end{cases} \end{aligned}$$

(Note that this involution preserves the subgroups U and T and the characters θ and τ .)

The main technical step of the proof is the following:

LEMMA: *For every $g \in G$ there exist $z_1, z_2 \in Z$ such that either both z_1, z_2 are unipotent and*

$$(6) \quad z_1 g \tilde{z}_2 = g, \quad \alpha(z_1 z_2) \neq 1,$$

or

$$(7) \quad z_1 g \tilde{z}_2 = \tilde{g}, \quad \alpha(z_1 z_2) = 1.$$

This lemma, in view of Theorem 1 from [6], provides the inequality:

$$(8) \quad \dim(\pi, \alpha) \times \dim(\pi^*, \hat{\alpha}) \leq 1, \quad \hat{\alpha}(z) = \alpha(\tilde{z}^{-1}),$$

where π^* denotes the contragradient representation to π .

Now the theorem follows directly from the inequality (8) and the isomorphism $\hat{\pi} \approx \pi^*$, $\hat{\pi}(g) = \pi(\tilde{g}^{-1})$, proved by I. M. Gelfand and A. D. Kajdan [2] providing that the elements g and \tilde{g} are conjugate in G for all $g \in G$; the last fact is an immediate consequence of Milnor's classification of conjugate classes of G in [4].

The theorem is true for finite field k too; in that case it follows directly from the lemma, which, together with its proof, is true for any field k of characteristic different from 2.

2. The proof of the lemma

1. Note that if (1.6) is true for $g \in G$, $z_1(g), z_2(g) \in U$, then for $g' = z' g z''$, $z', z'' \in Z$ the same formulas are true when $z_1(g') = z' z_1(g)(z')^{-1}$ and $z_2(g') = \tilde{z}'' z_2(g)(\tilde{z}'')^{-1}$; similarly, if (1.7) is true for $g \in G$, $z_1(g), z_2(g) \in Z$, then it is true for $g' = z' g z''$, $z', z'' \in Z$, when $z_1(g') = \tilde{z}'' z_1(g)(z')^{-1}$, $z_2(g') = z' z_2(g)(\tilde{z}'')^{-1}$. Therefore, it is enough to consider any complete set of representatives of the double cosets $Z \backslash G / Z$.

The definition of the groups U, T , and Z and the characters θ and α does not depend on the base $\{e_i : r < i \leq n - r\}$ of the subspace $\{e_i | r < i \leq n - r\}$. Changing it, if necessary, we may assume that

$$(1) \quad (e_i, e_j) = \begin{cases} 0 & \text{if } i + j \neq n + 1, \\ 1 & \text{if } i + j = n + 1, \\ 0 & \text{if } i \neq j, r' < i \leq n - r', 1 \leq j \leq n; \end{cases} \quad 1 \leq i \leq r', 1 \leq j \leq n;$$

and that the restriction of the form (\dots, \dots) to the subspace $\{e_i | r' < i \leq n - r'\}$ is anisotropic; if $r' = r + 1$ we may assume that vectors e and e_{r+1} are not orthogonal. Then the representatives can be chosen in the form

$$(2) \quad g = m_1 w m_2, \quad w e_i = \omega_i e_i, \quad \omega_i \in \ell, \quad w e_i = e_i \quad \text{if both } e_i, \\ w e_i \in \{e_j | r < j \leq n - r\}$$

$$m_1, m_2 \in M = \{g \in G : g e_i = e_i, \quad \text{if } i \leq r \quad \text{or} \quad i > n - r\}.$$

Take an element g of the form (2) for which (1.6) is wrong for all unipotent $z_1, z_2 \in Z$. Define a function $\varphi(i), 1 \leq i \leq n$, by the equality

$$(3) \quad we_i = \omega_i e_{\varphi(i)}.$$

2. If $i < r$ then $\varphi(i) \neq r$.

PROOF: The vectors $\{m_1 e_j : r < j \leq n - r\}$ form a base in the space $\{e_j | r < j \leq n - r\}$. Let

$$(4) \quad e = \sum_j \lambda_j m_1 e_j, \lambda_j \in \ell, r < j \leq n - r.$$

Since $(e, e) \neq 0$ there exists an index $j = s$ such that either $\lambda_s \neq 0$ and $(e_s, e_s) \neq 0$ or $\lambda_s \neq 0$ and $\lambda_{n+1-s} \neq 0$. Therefore, putting either $\bar{e} = m_1 e_s$ or $\bar{e} = m_1 e_{n+1-s}$, we obtain

$$(5) \quad g^{-1} \bar{e} \in \{e_j | 1 \leq j \leq n - r\}, (\bar{e}, e) \neq 0.$$

Define

$$(6) \quad z_1 e_j = \begin{cases} e_j & \text{if } 1 \leq j \leq r \text{ or } n + 2 - r \leq j \leq n; \\ e_j - \lambda \bar{e} - \mu e_r, \mu = \lambda \cdot \bar{\lambda} \cdot (\bar{e}, \bar{e})/2, & \text{if } j = n + 1 - r; \\ e_j + (e_j, \lambda \bar{e}) e_r & \text{if } r < j \leq n - r; \end{cases}$$

$$\tilde{z}_2 = g^{-1} z_1^{-1} g, \lambda \in \ell.$$

Explicit formulas for \bar{e}, z_1 , and z_2 allow one to check that if $we_i = e_r, i < r$, then $z_2 \in U$ and $\theta(z_2) = 1$; since $\theta(z_1) = \chi(\text{tr}(\lambda \bar{e}, e))$, the elements g, z_1, z_2 satisfy (1.6) for a suitable λ , which contradicts our choice of g .

3. If $\varphi(i) > r, i \leq r$, and $i' < i$, then $\varphi(i') > r$.

PROOF: Otherwise there exist $i \leq r, i \geq 2$, such that $\varphi(i) > r$ and $\varphi(i - 1) \leq r$; hence, $\varphi(i - 1) < r$. Then g would have satisfied (1.6) when

$$(7) \quad z_2 e_j = \begin{cases} e_j & \text{if } j \neq i, j \neq n + 2 - i, \\ e_j + \lambda e_{j-1}, & \text{if } j = i, \\ e_j - \bar{\lambda} e_{j-1} & \text{if } j = n + 2 - i; \end{cases}$$

$$z_1 = g(\tilde{z}_2)^{-1} g^{-1}, \chi(\text{tr} \lambda) \neq 1, \lambda \in \ell.$$

4. If the set $\{i : i \leq r, r < \varphi(i) \leq n - r\}$ is empty, then the set $\{i : i > n - r, r < \varphi(i) \leq n - r\}$ is empty too, and, consequently, the set $\{i : r < i \leq n - r\}$ is invariant, hence, stable under φ (cf. no. 1). Therefore,

$$(8) \quad g = mw = wm, m = m_1 \cdot m_2; \tilde{g} = \tilde{w} \tilde{m} = \tilde{m} \tilde{w}.$$

Arguments of R. Steinberg [2], proof of Theorem 49, show that $w = \bar{w}$. Since

$$(9) \quad (me, e) = (e, m^{-1}e) = (\overline{m^{-1}e}, e) = (\tilde{m}e, e),$$

Witt's theorem (cf. [1], chap. 9, no. 3) provides an element

$$(10) \quad t \in T, \quad tme = \tilde{m}e = \tilde{m}\tilde{t}e.$$

Changing g into tg (cf. no. 1) we obtain

$$(11) \quad me = \tilde{m}e.$$

Now g satisfies (1.7) with

$$(12) \quad z_1 = \tilde{m}m^{-1}, \quad z_2 = id.$$

(When $k = \ell$, $\tilde{m} = m^{-1}$, so z preserves both vectors e and me and coincides with m^{-2} on their orthogonal complement; if $k \neq \ell$, $\det z_1 = 1$; in both cases our restrictions on the character τ guarantee the equality $\alpha(z_1z_2) = 1$.)

Now consider the case when the set $\{i : i \leq r, r < \varphi(i) \leq n - r\}$ is non-empty; then it consists of only one element. Denote it i_0 . Necessarily $r' = r + 1$, and either $\varphi(i_0) = r + 1$ or $\varphi(i_0) = n - r$.

5. If $\varphi(i) > r$ and $i \geq i_0$, then $i = i_0$. Otherwise, in view of no. 3, $\varphi(i_0 + 1) > n - r$. Then g satisfies (1.6) with

$$(13) \quad z_2e_j = \begin{cases} e_j & \text{if } j \neq i_0 + 1, j \neq n + 1 - i_0; \\ e_j + \lambda e_{j-1} & \text{if } j = i_0 + 1; \\ e_j - \bar{\lambda} e_{j-1} & \text{if } j = n + 1 - i_0; \end{cases}$$

$$z_1 = g(\tilde{z}_2)^{-1}g^{-1}, \quad \lambda \in \ell, \quad \chi(\text{tr } \lambda) \neq 1,$$

which contradicts our choice of g .

6. If one of the formulas (1.6), (1.7) is valid for an element $g \in G$, it is valid for \tilde{g} too.

PROOF: Applying the antiautomorphism (1.5) to the equations (1.6) and (1.7) we obtain the statement immediately with

$$(14) \quad z_1(\tilde{g}) = z_2(g), \quad \widetilde{z_2(\tilde{g})} = \widetilde{z_1(g)}.$$

Applying the statements of nos. 2-5 to \tilde{g} we obtain:

$$\varphi(r) \geq r \text{ (cf. no. 2);}$$

$\varphi^{-1}(i) > n - r$ if and only if $i < i_0$ (cf. no. 3 and no. 5);

$\varphi^{-1}(i_0)$ equals either $r' = r + 1$ or $n - r$ (cf. no. 4).

7. Matrices m_1 and m_2 can be chosen so that $w = \bar{w}$.

PROOF: Changing, if necessary, m_1 and m_2 obtain:

$$(15) \quad w(e_{r+1}) = -w^{-1}(e_{r+1}) = e_{i_0}.$$

Now the subspaces $\{e_i | i < i_0 \text{ or } i > n + 1 - i_0\}$ and $\{e_i | i_0 < i \leq r \text{ or } n - r < i \leq n - i_0\}$ are invariant under w . Applying Steinberg's arguments (loc. cit.) to the restrictions of w on these subspaces, we complete the proof.

8. Suppose w , m_1 , and m_2 are chosen as in no. 7. If $i_0 < r$, then in view of no. 2 $\varphi(r) = r$, and $\varphi(n - r + 1) = \varphi(n - r + 1)$. Consequently, g satisfies (1.6) with

$$(16) \quad z_1 e_i = \begin{cases} e_i & \text{if } i \neq n - r, i \neq n + 1 - r; \\ e_i + \lambda e_{r+1} & \text{if } i = n + 1 - r; \\ e - \bar{\lambda} e_r & \text{if } i = n - r; \end{cases}$$

$$z_2 = g^{-1} z_1 g$$

for a suitable $\lambda \in \ell$. Hence, $i_0 = r$.

If

$$(17) \quad (m_1 e_{r+1}, e) \neq 0.$$

Witt's theorem (Bourbaki, loc. cit.) provides an element $t \in T$ such that

$$(18) \quad t m_1 e_{r+1} = \lambda e_{r+1}, \lambda \in \ell.$$

Replacing g by tg we may suppose m_1 to be identity. If, in addition,

$$(19) \quad (m_2^{-1} e_{r+1}, e) \neq 0,$$

similar arguments lead us to equalities

$$(20) \quad g = wm, m e_{r+1} = \mu e_{r+1}, m e_{n-r} = \mu^{-1} e_{n-r}, \mu \in \ell.$$

Moreover, $\mu = 1$. Otherwise, g would have satisfied (1.6) with z_1, z_2 given by (16) for a suitable $\lambda \in \ell$ (remember that we assumed $(e_{r+1}, e) \neq 0$, cf. no. 1)).

Denote e' the orthogonal projection of the vector e on the subspace $\{e_i | r + 1 < i < n - r\}$. According to our assumption (cf. no. 1) this subspace is anisotropic. Therefore, $(e', e') \neq 0$, and the arguments similar to those of no. 4 prove (1.7) for the element g .

If

$$(21) \quad (m_2^{-1} e_{r+1}, e) = 0$$

(while (17) is valid, i.e. $m_1 = id$), g satisfies (1.6) with z_1, z_2 given by (16) again.

Similarly, if

$$(22) \quad (m_1 e_{r+1}, e) = 0, (m_2^{-1} e_{r+1}, e) \neq 0.$$

At last, let both

$$(23) \quad (m_j e_{r+1}, e) = 0, j = 1, 2.$$

Then the subspace $\{e_i | r+1 < i < n-r\}$ contains a vector \bar{e} for which

$$(24) \quad (m_1 \bar{e}, e) \neq 0$$

(otherwise e would have been proportional to $m_1 e_{r+1}$, which is impossible, for $(e, e) \neq 0$). Therefore, g satisfies (1.6) with

$$(25) \quad m_1 z_1 (m_1)^{-1} e_i = \begin{cases} e_i & \text{if } i \leq r \text{ or } i > n+1-r \\ e_i + \lambda \bar{e} - \mu e_r, \mu = \lambda \cdot \bar{\lambda} \cdot (\bar{e}, \bar{e})/2, & \text{if } i = n+1-r \\ e_i - (e_i, \lambda \bar{e}) e_r & \text{if } r < i \leq n-r, \end{cases}$$

$$\tilde{z}_2 = g^{-1} z_1^{-1} g$$

for a suitable $\lambda \in \ell$.

This completes the proof of the lemma and, therefore, of the theorem.

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