

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 30, n° 3 (1975), p. 293-297

http://www.numdam.org/item?id=CM_1975__30_3_293_0

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AN ISOMORPHIC CHARACTERIZATION OF THE SCHMIDT CLASS

D. R. Lewis¹

This note gives an isomorphic characterization of the Hilbert-Schmidt norm within the class of unitarily invariant crossnorms on the space R of finite rank operators on l_2 . Specifically if the completion of R under α has an unconditional basis (or more generally local unconditional structure) than α is equivalent to the Hilbert-Schmidt norm.

For convenience only real Banach spaces are considered. Notation and terminology is standard, with the following possible exceptions.

For E finite dimensional the unconditional basis constant of E is denoted by $\chi(E)$, and the parameter η is defined by $\eta(E) = \inf \|u\| \|v\| \chi(F)$, where the infimum is taken over all spaces F and pairs of operators $u : E \rightarrow F, v : F \rightarrow E$ satisfying $vu = \text{identity}$. The definition of η is extended to infinite dimensional spaces by setting $\eta(G) = \inf_{\mathcal{D}} \sup_{E \in \mathcal{D}} \eta(E)$, with the infimum taken over all confinal collections of finite dimensional subspaces of G . G has *local unconditional structure* if $\eta(G) < \infty$.

R (respectively, $R(n)$) is the space of finite rank operators on l_2 (resp., l_2^n). A norm α on R (or $R(n)$) is called a *unitarily invariant crossnorm* if, for each $u \in R$, $\|u\| \leq \alpha(u) \leq i_1(u)$ and $\alpha(guh) = \alpha(u)$ for all isometries g and h . This coincides with Schatten's definition [7]. The completion of R under α is written $R(\alpha)$, and $R(\alpha, n)$ is $R(n)$ with norm α . Finally, π_p and i_p denote the p -absolutely summing and p -integral norms, respectively ([6]).

THEOREM: *If α is a unitarily invariant crossnorm on l_2 and if $R(\alpha)$ has local unconditional structure, then α is equivalent to the Hilbert-Schmidt norm.*

PROOF: Fix n and let P (respectively, Q) be the natural inclusion of $R(\alpha, n)$ (resp., $R(\alpha, n)$) into $R(\pi_2, n)$. G denotes the group of isometries of l_2^n and dg is the normalized Haar measure on G . Let $\omega \in R(n)$ with $\omega \neq 0$, and write

¹ The author was supported in part by NSF GP-42666.

$(\omega^*\omega)^{\frac{1}{2}} = \sum_{i \leq n} \lambda_i e_i \otimes e_i$, where (e_i) is some orthonormal basis for l_2^n and each $\lambda_i \geq 0$. The dual of $R(\alpha, n)$ is naturally identified with $R(\alpha', n)$, where α' is the associate crossnorm of α ([7]) and the action is the trace of the composition.

Define a probability measure μ on the closed unit ball of $R(\alpha, n)$ by setting

$$\mu(f) = \int_G \int_G f(\alpha(\omega)^{-1} g \omega h^*) dg dh.$$

Then for $u \in R(\alpha, n)$

$$\begin{aligned} \alpha(\omega)\mu(|\langle u, \cdot \rangle|) &= \iint |\text{trace}(ug(\omega^*\omega)^{\frac{1}{2}}h^*)| dg dh \\ &= \iint \left| \sum_{i \leq n} \lambda_i (h(e_i), ug(e_i)) \right| dg dh, \end{aligned}$$

the first since ω may be translated to $(\omega^*\omega)^{\frac{1}{2}}$ by an isometry. Now in the last integral replace g by gg_ε , where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ is an n -tuple of signs and $g_\varepsilon(e_i) = \varepsilon_i e_i$. Averaging over all such n -tuples ε and applying Khinchin's inequality yields

$$\begin{aligned} \alpha(\omega)\mu(|\langle u, \cdot \rangle|) &\geq 3^{-\frac{1}{2}} \iint \left(\sum_{i \leq n} \lambda_i^2 |(h(e_i), ug(e_i))|^2 \right)^{\frac{1}{2}} dg dh \\ &\geq 3^{-\frac{1}{2}} \left[\sum_{i \leq n} \lambda_i^2 \left(\iint |h(e_i), ug(e_i)| dg dh \right)^2 \right]^{\frac{1}{2}} \\ &\geq 4^{-1} \pi_2(\omega) \iint |(h(e), ug(e))| dg dh \end{aligned}$$

for any vector $e \in l_2^n$ with norm one. Let dm be the normalized $(n-1)$ -dimensional rotational invariant measure on the unit sphere of l_2^n and c_n the constant satisfying

$$\|x\| = c_n \int |(x, z)| m(dz), \quad x \in l_2^n.$$

This equality shows that for $\|e\| = 1$

$$\iint |(h(e), ug(e))| dg dh = \int c_n^{-1} \|u(x)\| m(dx)$$

$$\begin{aligned}
&= c_n^{-1} \int \left(\sum_{i \leq n} |(x, u^*(e_i))|^2 \right)^{\frac{1}{2}} m(dx) \\
&\geq c_n^{-1} \left[\sum_{i \leq n} \left(\int |(x, u^*(e_i))| m(dx) \right)^2 \right]^{\frac{1}{2}} \\
&= c_n^{-2} \left(\sum_{i \leq n} \|u^*(e_i)\|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

and the last is nothing but $c_n^{-2} \pi_2(u)$. Combining inequalities now gives

$$\pi_2(u) \leq 4c_n^2 \alpha(\omega) \pi_2(\omega)^{-1} \mu(|\langle u, \cdot \rangle|)$$

so that

$$\pi_1(Q) \leq 4c_n^2 \alpha(\omega) \pi_2(\omega)^{-1}.$$

To estimate $\pi_1(P)$ let γ be the probability measure on the closed unit ball of $R(\alpha, n)$ defined by

$$\gamma(f) = \int_G \int_G f(g(e) \otimes h(e)) dg dh,$$

where e is some fixed vector of norm one. The previous inequality shows

$$\pi_2(u) \leq c_n^2 \gamma(|\langle \cdot, u \rangle|)$$

for all $u \in R(n)$, and consequently $\pi_1(P) \leq c_n^2$. By the proof of Theorem 3.5 of [2]

$$n^2 \leq \eta(R(\alpha, n)) \pi_1(P) \pi_1(Q) \leq \eta(R(\alpha, n)) 4c_n^4 \alpha(\omega) \pi_2(\omega)^{-1}$$

and by [1] $c_n \leq (\pi n/2)^{\frac{1}{2}}$, so

$$\pi_2(\omega) \leq \pi^2 \eta(R(\alpha, n)) \alpha(\omega).$$

This first inequality is also valid if α is replaced by its associate α' , and hence by duality (using $\pi'_2 = \pi_2$)

$$\alpha(\omega) \leq \pi^2 \eta(R(\alpha, n)) \pi_2(\omega).$$

The constants in the last two inequalities are both dominated by $\pi^2 \eta(R(\alpha))$, so that α and π_2 are $\pi^4 \eta(R(\alpha))^2$ -equivalent on R . This completes the proof.

REMARK 1: An absolutely summing operator on a subspace of an $L_1(\mu)$ -space or a quotient of a $C(K)$ -space must factor through an L_1 -space [2]. Thus the same proof as above shows that α is equivalent to the Hilbert-Schmidt norm if $R(\alpha)$ is isomorphic to a subspace of an L_1 -space or to a quotient of a $C(K)$ -space. In addition, by applying a basis selection theorem of Kadec-Pelczynski [3] and the result of McCarthy [5] that $R(c_p) \not\subset L_p$, it can be shown that α is equivalent to π_2 if $R(\alpha) \subset L_p$ for some $p > 2$. It seems a reasonable conjecture that α is equivalent to π_2 whenever $R(\alpha)$ embeds in a space with an unconditional basis.

REMARK 2: Given $\lambda \geq 1$ there is a unitarily invariant crossnorm α on R which is equivalent to π_2 and such that the unconditional basis constant of $R(\alpha)$ is at least λ . To see this fix n and let ϕ_n be the symmetric gauge function ([7], Chapter V) on the space of finitely non-zero scalar sequences defined by

$$\phi_n(x) = \max \left\{ \left(\sum_{k \geq 1} |x_k|^2 \right)^{\frac{1}{2}}, \max_{|\sigma|=n} \sum_{k \in \sigma} |x_k| \right\}.$$

The crossnorm α on R induced by ϕ_n is clearly $n^{\frac{1}{2}}$ -equivalent to π_2 and by the inequalities of the proof $\eta(R(\alpha)) \geq \pi^{-2} n^{\frac{1}{2}}$. Similarly it is possible to produce unitary crossnorms β for which $\eta(R(\beta, n))$ increases very slowly. For instance, the crossnorm β induced by the symmetric gauge function

$$\psi(x) = \max_{\sigma} |\sigma|^{-\frac{1}{2}} \sum_{k \in \sigma} |x_k|$$

is, asymptotically, $(\log n)^{\frac{1}{2}}$ -equivalent to π_2 on l_2^n .

REMARK 3: Suppose $R(\alpha, n)$ has a basis with the property that each permutation of the basis vectors naturally defines an isomorphism of norm at most λ . By [4] $\eta(R(\alpha, n)) \leq 3\lambda$, so α is $9\pi^4 \lambda^2$ -equivalent to π_2 on l_2^n .

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(Oblatum 11–XI–1974)

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