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PROBABILITY MEASURES ON COMPACT SEMITOPOLOGICAL SEMIGROUPS

H. L. Chow

1. Introduction

In what follows S is always a compact semitopological semigroup i.e. the multiplication is separately continuous. It is well known that the set $P(S)$ of probability measures on S is a compact semitopological semigroup under convolution and the weak* topology, [4]. In case S is *jointly continuous*, Collins [2] has given necessary and sufficient conditions that a measure $\mu \in P(S)$ is idempotent. More precisely, it is shown in [2] that, for $\mu \in P(S)$ with support F, the following are equivalent:

1) $\mu^2 = \mu$.

2) (i) F is a simple semigroup i.e. F has no proper ideal.
   (ii) $f \in C(S)$(the set of continuous functions on S) implies f' is constant on each minimal left ideal of F where

$$f'(x) = \mu(f^x) = \int f(tx) d\mu(t) \text{ for } x \in S.$$

   (iii) $\mu^3 = \mu$.

3) (i) $\mu^2 = \mu$ on the F-translate space i.e. $\mu^2(f^x) = \mu(f^x)$ for $f \in C(S)$ and $x \in F$.
   (ii) $\mu^3 = \mu$.

One of the above conditions was improved substantially in ([1] Proposition 3). The other conditions will be extended here, concerning only the separately continuous situation.

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2. Probability measures of finite order

In this section we denote by μ a probability measure of finite order on S, i.e. $\mu^{n+1} = \mu$ for some positive integer n, and F its support. When $n = 1$, it is known [6] that F is a topologically simple subsemigroup of S

(a semigroup is called topologically simple if every ideal is dense). In general, we consider the closed semigroup $S(F)$ generated by F; it is clear that

$$S(F) = \bigcup_{i=1}^{n} \overline{F^i}$$

where the bar denotes closure.

PROPOSITION (1): *$S(F)$ and $\overline{F^n}$ are compact, topologically simple subsemigroups having the same idempotents. Consequently $S(F)$ is a group if and only if $\overline{F^n}$ is.*

PROOF: Since $S(F)$ and $\overline{F^n}$ are supports of idempotent measures $1/n(\mu + \cdots + \mu^n)$ and μ^n, they are topologically simple semigroups by ([6] Lemma 2(iii)). Next take any idempotent $e \in S(F)$ and suppose $e \in \overline{F^i}$, say, for some $i = 1, \cdots, n$. Since $e = e^n \in (\overline{F^i})^n \subset \overline{(F^i)^n} = \overline{F^n}$, we see that $S(F)$ and $\overline{F^n}$ have the same idempotents. Finally, if $S(F)$ is a group, $S(F)$ has a unique idempotent. It follows that $\overline{F^n}$ has a unique idempotent and so is easily seen to be a group by Theorem 2.3 and Corollary 2.5 of [3]. (The semitopological semigroups discussed in [3] have identity elements, but the arguments are still valid for the present case.) The converse can be proved similarly.

We may obtain the following results as in the jointly continuous situation (see [1]) and therefore omit the proofs.

PROPOSITION (2): *If $\mu \in P(S)$ whose support is contained in $\overline{F^j}$ for some $j = 1, \cdots, n$, then*

$$\mu^i \nu \mu^k = \mu^{i+j+k} \text{ for } i, k = 1, \cdots, n.$$

PROPOSITION (3): *For $i, j = 1, \cdots, n$, if $\overline{F^i} \cap \overline{F^j} \neq \phi$, then $\mu^i = \mu^j$. Hence, if F is a semigroup, then μ is idempotent.*

REMARK: By Proposition 3, we see that μ is idempotent if $F = \overline{F^n}$. This property can be extended in the following sense. Consider two probability measures ν and ν' on a locally compact semitopological semigroup with the same support H such that $\nu\nu' = \nu'\nu = \nu$. Then $\nu(Bx^{-1}y^{-1}) = \nu(Bx^{-1})$ for every Borel set $B \subset S$, x, $y \in H$ where $Bx^{-1} = \{t : tx \in B\}$(see Lemma 3 of [5], the proof of which also applies to the separately continuous case). Thus

$$\nu(Bx^{-1}) = \int \nu(Bx^{-1})d\nu(y) = \int \nu(Bx^{-1}y^{-1})d\nu(y) = \nu^2(Bx^{-1}).$$

This together with the facts that $\nu\nu' = \nu$ and $\nu^2\nu' = \nu^2$ gives $\nu(B) = \nu^2(B)$, i.e. ν is idempotent. The particular case in Proposition 3 follows by taking $\nu' = \nu^n$.

The author wishes to thank the referee for suggesting this remark.

COROLLARY (4): *μ is idempotent if $S(F)$ is connected.*

Suppose e is an idempotent in the minimal ideal of $\overline{F^n}$. Let X and Y be the sets of idempotents in $\overline{F^n}e$ and $e\overline{F^n}$, respectively, and let $G = e\overline{F^n}e$. Then X and Y are compact semigroups and G is a compact group so that $\mu^n = (\mu^n)_X(\mu^n)_G(\mu^n)_Y$ where the measures $(\mu^n)_X$, $(\mu^n)_Y$ are projections of μ on X, Y respectively, and $(\mu^n)_G$ is the Haar measure of G; see [6].

PROPOSITION (5): *$\mu = (\mu^n)_X\delta(e)\mu\delta(e)(\mu^n)_Y$, where $\delta(e)$ is the unit point mass at e.*

3. Extending Collins' results

It is a direct consequence of Proposition 3 that a probability measure of finite order on S is idempotent if and only if its support is a semigroup. In order to establish other characterizations of idempotent probability measures, we need the lemma below which extends Theorem 2 of [2]. The proof is omitted since it parallels that in [2].

LEMMA (6): *Suppose $\mu \in P(S)$ with support F such that $\overline{F^{n+1}} = F$ for some positive integer n. Let $S(F)$ be the closed semigroup generated by F. Then the following are equivalent:*

1) *$\mu^2 = \mu$ on the $S(F)$-translate space.*
2) *The set $M(f) = \{x \in S(F) : f'(x) = \max f'(S(F))\}$ is a left ideal of $S(F)$ for each $f \in C_R(S)$(the set of real-valued continuous functions on S).*
3) (i) *$S(F)$ is topologically simple.*
   (ii) *f' is constant on each minimal left ideal of $S(F)$.*

THEOREM (7): *Suppose $\mu \in P(S)$ with support F. Then the following are equivalent:*

1) *μ is idempotent.*
2) *f' is constant on each minimal left ideal of $S(F)$ and $\mu^{n+1} = \mu$ for some positive integer n.*
3) *$\mu^n = \mu$ on the F-translate space and $\mu^{n+1} = \mu$.*

PROOF: That (1) implies (2) and (2) implies (3) are immediate from the preceding lemma. As for (3) implies (1), it is enough, by Proposition 3, to show that $F \cap \overline{F^n} \neq \phi$. Suppose $F \cap \overline{F^n} = \phi$. Then there exists $f \in C_R(S)$ such that $f \geqq 0, f(F) = 0$ and $f(\overline{F^n}) > 0$. Take any $c \in F^n$ and consider f^c. It follows that $\mu(f^c) = \int f(tc)\mathrm{d}\mu(t) = 0$. On the other hand, let $c = ab$ where $a \in F$ and $b \in F^{n-1}$, and let $g(x) = f(xb)$ for $x \in S$. Then $f^c = g^a$, i.e. f^c is an F-translate so that $\mu(f^c) = \mu^n(f^c) = \int f(tc)\mathrm{d}\mu^n(t) > 0$. This contradiction proves the theorem.

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