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# TRANSCENDENCE MEASURES OF CERTAIN NUMBERS WHOSE TRANSCENDENCY WAS PROVED BY A. BAKER

P. L. Cijsouw

## 1. Introduction

In the subsequent paper we continue the investigation of transcendence measures of certain transcendental numbers  $\sigma$ , i.e. positive lower bounds for  $|P(\sigma)|$  in terms of the degree  $N$  and height  $H$  of  $P$ , where  $P$  is an arbitrary polynomial with integral coefficients. For more information about transcendence measures and the type of transcendence measures we will look for, see the earlier paper [4]; see also the authors thesis [3], which includes the results of the present paper.

Let  $\alpha_1, \dots, \alpha_n$  be non-zero algebraic numbers such that, for any (fixed) values of the logarithms,  $\log \alpha_1, \dots, \log \alpha_n$  are linearly independent over  $\mathbb{Q}$ . In this paper, transcendence measures are derived for numbers which can be written in one of the following ways:

- (i)  $\beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n$  with  $n \geq 2$ , where  $\beta_1, \dots, \beta_n$  are algebraic numbers, not all zero
- (ii)  $e^{\beta_0} \alpha_1^{\beta_1} \dots \alpha_n^{\beta_n}$  with  $n \geq 1$  and

$$\alpha_i^{\beta_i} = e^{\beta_i \log \alpha_i}$$

where  $\beta_0, \beta_1, \dots, \beta_n$  are algebraic numbers such that at least one of  $\beta_1, \dots, \beta_n$  is irrational. We prove the transcendence measure

$$\exp \{ -C_1 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon} \}$$

for numbers of the form (1), and the transcendence measure

$$\exp \{ -C_2 N^{n^2+2n+2+\varepsilon} S^{n+1+\varepsilon} \}$$

for numbers of the form (2). Here  $S = N + \log H$ ,  $\varepsilon$  is an arbitrary positive number and  $C_1$  and  $C_2$  are effectively computable numbers, depending on  $\varepsilon, n$ , the  $\alpha$ 's and their logarithms and the  $\beta$ 's.

We remark, that these transcendence measures are the first explicit ones to be published for these numbers in which both the dependence on  $H$  and  $N$  is expressed. If one is interested merely in the height  $H$ , better results can be given. For numbers of the form (i), N. I. FEL'DMAN [6] proved the transcendence measure  $\exp \{ -CS \}$ , where  $C$  is a positive

number, depending on  $N$ , the  $\alpha$ 's and their logarithms and the  $\beta$ 's. For numbers of the form (ii), a recent result of A. BAKER [2] implies the transcendence measure  $\exp \{-C \log H \log \log H\}$  for  $H \geq 4$ , where  $C$  again is a positive number depending on  $N$ , the  $\alpha$ 's and their logarithms and the  $\beta$ 's. An earlier result for the special case of numbers of the form  $e^{\beta_0} \alpha_1^{\beta_1}$  can be found in [7].

The method of proof of our transcendence measures is A. BAKER's one with some improvements introduced by N. I. FEL'DMAN. In the proof we firstly derive a measure for the approximability of numbers of the types (i) and (ii); after that, the transcendence measures are given.

The transcendence of numbers of the form (i) follows immediately from e.g. Theorem 1 of A. BAKER's paper [1]. The transcendence of numbers of the form (ii) was proved by A. BAKER, distinguishing the cases  $\beta_0 = 0$  and  $\beta_0 \neq 0$ ; see the same paper.

## 2. Lemmas

We shall use the same notations (especially for the degree; height and size) as in [4]. For shortness, we use without reformulation the lemmas 3, 6, 7, 8 and 9 of [4]. Further, we need the following lemmas:

LEMMA 1: *Let  $\alpha_1, \dots, \alpha_n$  be non-zero algebraic numbers such that, for any fixed values of the logarithms,  $\log \alpha_1, \dots, \log \alpha_n$  are linearly independent over  $\mathbb{Q}$ . Let  $\varepsilon$  be positive and let  $d$  be a positive integer. Then there exists an effectively computable positive number*

$$\theta(\varepsilon, d) = \theta(\varepsilon, d, \log \alpha_1, \dots, \log \alpha_n)$$

*such that*

$$(1) \quad |\beta_0 + \beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n| > \theta(\varepsilon, d) \exp \{-(\log h)^{1+\varepsilon}\}$$

*for all algebraic numbers  $\beta_0, \beta_1, \dots, \beta_n$ , not all zero, of degrees at most  $d$  and of heights at most  $h$ .*

PROOF: See Theorem 1 of [5].

LEMMA 2: *Let  $\alpha$  and  $\beta$  be algebraic numbers,  $\beta \neq 0$ . Then*

$$(2) \quad s(\alpha\beta^{-1}) \leq 2d(\alpha)d(\beta) + d(\alpha)s(\beta) + d(\beta)s(\alpha).$$

PROOF: From Lemma 3 of [5] it follows that

$$h(\alpha\beta^{-1}) \leq 2^{d(\alpha)d(\beta)} \{h(\alpha)(d(\alpha)+1)\}^{d(\beta)} \{h(\beta)(d(\beta)+1)\}^{d(\alpha)}.$$

Using  $d(\alpha)+1 < e^{d(\alpha)}$  and  $d(\beta)+1 < e^{d(\beta)}$  we get

$$\begin{aligned} s(\alpha\beta^{-1}) &= d(\alpha\beta^{-1}) + \log h(\alpha\beta^{-1}) \\ &\leq d(\alpha)d(\beta)(1 + \log 2) + d(\beta)s(\alpha) + d(\alpha)s(\beta) \end{aligned}$$

from which (2) follows.

LEMMA 3: *Let  $\beta$  be an algebraic number, and let  $k$  and  $\ell$  be integers. Then*

$$(3) \quad h(k + \ell\beta) \leq h(\beta)|2k\ell|^{d(\beta)}.$$

PROOF: If  $a_n z^n + \cdots + a_1 z + a_0$  is the minimal polynomial of  $\beta$ , then

$$a_n(z-k)^n + \cdots + a_1 \ell^{n-1}(z-k) + a_0 \ell^n$$

is a constant multiple of the minimal polynomial of  $k + \ell\beta$ . Thus, the coefficient of  $z^i$  ( $i = 0, 1, \dots, n$ ) of this minimal polynomial is in absolute value at most

$$\begin{aligned} \left| a_n \binom{n}{i} (-k)^{n-i} + a_{n-1} \ell \binom{n-1}{i} (-k)^{n-1-i} + \cdots + a_i \ell^{n-i} \right| \\ \leq h(\beta)|k|^n |\ell|^n \left\{ \binom{n}{i} + \binom{n-1}{i} + \cdots + 1 \right\}. \end{aligned}$$

From the obvious inequality  $\binom{m}{i} \leq 2^{m-1}$  for all positive integers  $m$ , it follows that

$$\binom{n}{i} + \binom{n-1}{i} + \cdots + 1 \leq 2^n,$$

by which the proof is completed.

### 3. The case $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$

First we give a measure for the approximability for numbers of the form (i), in the special case in which  $\beta_n = -1$ .

THEOREM 1: *Let, for  $n \geq 2$ ,  $\alpha_1, \dots, \alpha_n$  and  $\gamma_1, \dots, \gamma_{n-1}$  be non-zero algebraic numbers such that, for any fixed values of the logarithms,  $\log \alpha_1, \dots, \log \alpha_n$  are linearly independent over  $\mathbb{Q}$ . Let  $\varepsilon$  be a positive number. Then there exists an effectively computable number  $S_1 = S_1(\varepsilon, \log \alpha_1, \dots, \log \alpha_n, \gamma_1, \dots, \gamma_{n-1})$  such that*

$$(4) \quad |\gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta| \\ > \exp \{ -N^{n^2+n+\varepsilon} S(\log S)^{n+1+\varepsilon} \}$$

for all algebraic numbers  $\eta$  of degree  $N$  and size  $S \geq S_1$ .

PROOF: Put  $\delta = (2n^3 + 4n^2 + 3n + 7)^{-1}\varepsilon$ . For abbreviation, put

$$\sigma = \gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n$$

and

$$U = N^{n^2+n+(2n^3+4n^2+2n+1)\delta} S(\log S)^{n+1+(2n^2+3n+7)\delta}.$$

It is sufficient to prove that

$$|\sigma - \eta| > \exp \{-U\}$$

if  $S \geq S_1$ . In this proof we may restrict ourselves to the case in which  $\delta$  is rather small. By  $c_1, c_2, \dots$  we denote positive numbers which depend only on  $n, \log \alpha_1, \dots, \log \alpha_n, \gamma_1, \dots, \gamma_{n-1}$ .

Suppose that

$$(5) \quad |\sigma - \eta| \leq \exp \{-U\}$$

for some algebraic number  $\eta$  of degree  $N$  and size  $S$ . We prove that this leads to a contradiction if  $S$  is sufficiently large.

Choose the following integers:

$$\begin{aligned} K &= [N^{n+(n^2-1)\delta}(\log S)^{1+(2n+1)\delta}], \\ M &= [N^{n+(n^2+n)\delta}S(\log S)^{1+(2n+3)\delta}], \\ C &= 2[\frac{1}{2} \exp \{N^{n+(n^2+n)\delta}S(\log S)^{2+(2n+4)\delta}\}], \\ T &= [N^{n+1+(n^2+n)\delta}(\log S)^{1+(2n+3)\delta}], \\ P &= [S(\log S)^{1+2\delta}], \\ R &= \left[ \frac{n-1}{\delta} + 2n^2 + n \right], \\ T' &= [2^{-R}T] \text{ and} \\ P' &= [N^{n^2-1+(n^3-n)\delta}S(\log S)^{n+(2n^2+n+1)\delta}]. \end{aligned}$$

Put

$$\begin{aligned} F(z) = \sum_{k_1=0}^{K-1} \cdots \sum_{k_n=0}^{K-1} \sum_{m=0}^{M-1} \sum_{v=0}^{N-1} C_{k_1 \dots k_n m v} \eta^v z^m \\ \times \exp \{-k_n \eta z + \sum_{i=1}^{n-1} (k_i + k_n \gamma_i)(\log \alpha_i)z\}, \end{aligned}$$

where the numbers  $C_{k_1 \dots k_n m v}$  are integers of absolute values at most  $C$ ; they will be specified later.

For  $t = 0, 1, 2, \dots$  it is easily seen that

$$(6) \quad F^{(t)}(z) = \sum_{\tau_1 + \tau_2 + \dots + \tau_{n-1} = t} \frac{t!}{\tau_1! \tau_2! \cdots \tau_{n-1}!} \\ \times (\log \alpha_1)^{\tau_1} \cdots (\log \alpha_{n-1})^{\tau_{n-1}} F_{\tau_1 \dots \tau_{n-1}}(z)$$

where

$$\begin{aligned} (7) \quad F_{\tau_1 \dots \tau_{n-1}}(z) &= \sum_{k_1} \cdots \sum_{k_n} \sum_m \sum_v C_{k_1 \dots k_n m v} \eta^v \\ &\times \sum_{\kappa=0}^m \binom{\tau}{\kappa} \frac{m!}{(m-\kappa)!} z^{m-\kappa} (-k_n)^{\tau-\kappa} \eta^{\tau-\kappa} \prod_{i=1}^{n-1} (k_i + k_n \gamma_i)^{\tau_i} \\ &\times \exp \left\{ \left( \sum_{i=1}^n k_i \log \alpha_i \right) z + k_n (\sigma - \eta) z \right\}. \end{aligned}$$

Put

$$\begin{aligned} \Phi_{\tau\tau_1 \dots \tau_{n-1}}(z) &= \sum_{k_1} \dots \sum_{k_n} \sum_m \sum_v C_{k_1 \dots k_n m v} \eta^v \\ &\times \sum_{\kappa=0}^m \binom{\tau}{\kappa} \frac{m!}{(m-\kappa)!} z^{m-\kappa} (-k_n)^{\tau-\kappa} \eta^{\tau-\kappa} \prod_{i=1}^{n-1} (k_i + k_n \gamma_i)^{\tau_i} \\ &\times \exp \left\{ \left( \sum_{i=1}^n k_i \log \alpha_i \right) z \right\}. \end{aligned}$$

We estimate the difference

$$|F_{\tau\tau_1 \dots \tau_{n-1}}(p) - \Phi_{\tau\tau_1 \dots \tau_{n-1}}(p)|$$

as follows: by  $|e^z - 1| \leq |z|e^{|z|}$ , one has for  $k_n = 0, 1, \dots, K-1$  and

$$p = 0, 1, \dots, [N^{n^2-1+(2n^3+3n^2+n)\delta} S(\log S)^{n+(2n^2+n+2)\delta}]$$

the inequality

$$|e^{k_n(\sigma-\eta)p} - 1| \leq \exp \left\{ -\frac{1}{2}U \right\}.$$

Hence,

$$(8) \quad |F_{\tau\tau_1 \dots \tau_{n-1}}(p) - \Phi_{\tau\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for  $\tau, \tau_1, \dots, \tau_{n-1} = 0, 1, \dots, T-1$  and

$$p = 0, 1, \dots, [N^{n^2-1+(2n^3+3n^2+n)\delta} S(\log S)^{n+(2n^2+n+2)\delta}].$$

Let

$$P_{\tau\tau_1 \dots \tau_{n-1} p k_1 \dots k_n m}(z_0, z_1, \dots, z_{2n-1})$$

$(\tau, \tau_1, \dots, \tau_{n-1} = 0, 1, \dots, T-1; p = 0, 1, \dots, P-1; k_1, \dots, k_n = 0, 1, \dots, K-1; m = 0, 1, \dots, M-1)$  be the polynomials, chosen in the appropriate way, such that

$$\begin{aligned} \Phi_{\tau\tau_1 \dots \tau_{n-1}}(p) &= \sum_{k_1=0}^{K-1} \dots \sum_{k_n=0}^{K-1} \sum_{m=0}^{M-1} \sum_{v=0}^{N-1} C_{k_1 \dots k_n m v} \eta^v \\ &\times P_{\tau\tau_1 \dots \tau_{n-1} p k_1 \dots k_n m}(\eta, \alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_{n-1}). \end{aligned}$$

We apply Lemma 6 of [4] to these polynomials in the specified points. If  $r, s$  and  $B$  have the same meaning as in this lemma we have

$$\begin{aligned} r &= T^n P \leq N^{n^2+n+(n^3+n^2)\delta} S(\log S)^{n+1+(2n^2+3n+2)\delta}, \\ s &= K^n M \geq \frac{1}{2} N^{n^2+n+(n^3+n^2)\delta} S(\log S)^{n+1+(2n^2+3n+3)\delta}. \end{aligned}$$

Hence,  $s \geq 4rd$  where  $d = [Q(\alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_{n-1}) : Q]$ . Further,

$$B \leq \exp \{c_1 N^{n+(n^2+n)\delta} S(\log S)^{2+(2n+3)\delta}\}.$$

Hence, the right hand side of the second condition in Lemma 6 of [4] is at most

$$\exp \{c_2 N^{n+1+(n^2+n)\delta} S(\log S)^{2+(2n+3)\delta}\} \leq C^N.$$

It follows that the numbers  $C_{k_1 \dots k_{nmv}}$  can be chosen as integers, not all zero, of absolute values at most  $C$ , such that  $\Phi_{\tau_1 \dots \tau_{n-1}}(p) = 0$  for  $\tau, \tau_1, \dots, \tau_{n-1} = 0, 1, \dots, T-1$  and  $p = 0, 1, \dots, P-1$ . Doing so, we certainly have  $\Phi_{\tau_1 \dots \tau_{n-1}}(p) = 0$  for  $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T-1$  and  $0 \leq p \leq P-1$ . From (8) we now get

$$(9) \quad |F_{\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \{-\frac{1}{3}U\}$$

for  $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T-1$  and  $0 \leq p \leq P-1$ .

Define  $T_r$  and  $P_r$  for  $r = 0, 1, \dots, R$  by

$$T_r = [2^{-r}T]$$

and

$$P_r = [(N^{n+1} \log S)^{\delta} P].$$

Remark that

$$(10) \quad P_R \leq [N^{n^2-1+(2n^3+3n^2+n)\delta} S(\log S)^{n+(2n^2+n+2)\delta}].$$

LEMMA: For  $r = 0, 1, \dots, R$  the inequality

$$(11) \quad |F_{\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \{-\frac{1}{3}U\}$$

holds for all non-negative integers  $\tau, \tau_1, \dots, \tau_{n-1}$  and  $p$  with  $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_r-1$  and  $0 \leq p \leq P_r-1$ .

PROOF: We proceed by induction on  $r$ . For  $r = 0$  the statement is proved in (9). Let  $r$  be an integer with  $0 \leq r \leq R-1$  for which

$$(12) \quad |F_{\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \{-\frac{1}{3}U\}$$

for  $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_r-1$  and  $0 \leq p \leq P_r-1$ . Since

$$F_{\tau_1 \dots \tau_{n-1}}(z) = \sum_{k_1} \dots \sum_{k_n} \sum_m \sum_v C_{k_1 \dots k_{nmv}} \eta_v^{n-1} \prod_{i=1}^{n-1} (\log \alpha_i)^{-\tau_i} \\ \times (z^m e^{-k_n z})^{(\tau)} \prod_{i=1}^{n-1} (\exp \{(k_i + k_n \gamma_i)(\log \alpha_i)z\})^{(\pi_i)}$$

we have for  $t = 0, 1, 2, \dots$

$$F_{\tau_1 \dots \tau_{n-1}}^{(t)}(z) = \sum_{\mu + \mu_1 + \dots + \mu_{n-1} = t} \frac{t!}{\mu! \mu_1! \dots \mu_{n-1}!} \\ \times \prod_{i=1}^{n-1} (\log \alpha_i)^{\mu_i} \times F_{\tau + \mu, \tau_1 + \mu_1, \dots, \tau_{n-1} + \mu_{n-1}}(z).$$

Together with (12) we obtain

$$(13) \quad |F_{\tau\tau_1 \dots \tau_{n-1}}^{(t)}(p)| \leq \exp \{-\tfrac{1}{4}U\}$$

for  $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_{r+1} - 1$ ,  $0 \leq t \leq T_{r+1} - 1$  and  $0 \leq p \leq P_r - 1$ .

From (7) we know that

$$(14) \quad \max_{|z| \leq 6P_{r+1}} |F_{\tau\tau_1 \dots \tau_{n-1}}(z)| \leq \exp \{c_3 N^{n+(n^2+n+nr+r)\delta} S(\log S)^{2+(2n+r+4)\delta}\}$$

for  $\tau + \tau_1 + \dots + \tau_{n-1} \leq T_{r+1} - 1$ . We apply Lemma 7 of [4] to  $F_{\tau\tau_1 \dots \tau_{n-1}}$  with  $R = P_{r+1}$ ,  $A = 6$ ,  $T = T_{r+1}$  and  $P = P_r$ . From (13), (14) and the inequality

$$N^{n+1} \log S \leq \exp \{N^\delta (\log S)^\delta\}$$

we then obtain

$$\max_{|z| \leq P_{r+1}} |F_{\tau\tau_1 \dots \tau_{n-1}}(z)| \leq \exp \{-2^{-(r+3)} N^{n+1+(n^2+n+nr+r)\delta} S(\log S)^{2+(2n+r+5)\delta}\}.$$

In particular,

$$|F_{\tau\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \{-2^{-(r+3)} N^{n+1+(n^2+n+nr+r)\delta} S(\log S)^{2+(2n+r+5)\delta}\}$$

for  $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_{r+1} - 1$  and  $0 \leq p \leq P_{r+1} - 1$ .

From (8) and (10) it follows that

$$(15) \quad |\Phi_{\tau\tau_1 \dots \tau_{n-1}}(p)| \leq \exp \{-2^{-(r+4)} N^{n+1+(n^2+n+nr+r)\delta} S(\log S)^{2+(2n+r+5)\delta}\}$$

for  $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_{r+1} - 1$  and  $0 \leq p \leq P_{r+1} - 1$ . But for these values of  $\tau, \tau_1, \dots, \tau_{n-1}$  and  $p$ , we can consider  $\Phi_{\tau\tau_1 \dots \tau_{n-1}}(p)$  as a polynomial in  $\eta, \alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_{n-1}$ , of degree less than  $T_{r+1} + N$  in  $\eta, KP_{r+1}$  in  $\alpha_1, \dots, \alpha_n$  and  $T_{r+1}$  in  $\gamma_1, \dots, \gamma_{n-1}$ . If  $B$  denotes the sum of the absolute values of the coefficients, then we have

$$B \leq \exp \{2N^{n+(n^2+n)\delta} S(\log S)^{2+(2n+4)\delta}\}.$$

According to Lemma 3 of [4] we thus have either  $\Phi_{\tau\tau_1 \dots \tau_{n-1}}(p) = 0$  or

$$(16) \quad |\Phi_{\tau\tau_1 \dots \tau_{n-1}}(p)| \geq \exp \{-c_4 N^{n+1+(n^2+n+nr+r)\delta} S(\log S)^{2+(2n+r+4)\delta}\}$$

for  $0 \leq \tau + \tau_1 + \dots + \tau_{n-1} \leq T_{r+1} - 1$  and  $0 \leq p \leq P_{r+1} - 1$ . Hence,  $\Phi_{\tau\tau_1 \dots \tau_{n-1}}(p) = 0$  for these  $\tau, \tau_1, \dots, \tau_{n-1}$  and  $p$ . Again from (8) and (10) we obtain



$$|F_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for  $0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T_{r+1} - 1$  and  $0 \leq p \leq P_{r+1} - 1$ . The lemma has been proved.

From (11) with  $r = R$  we get

$$(17) \quad |F_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for  $0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T_R - 1$  and  $0 \leq p \leq P_R - 1$ . From their definitions we have  $T_R = T'$ . Since

$$R \geq \frac{n-1}{\delta} + 2n^2 + n - 1$$

we see that

$$P_R \geq \frac{1}{2} N^{n^2-1+(2n^3+3n^2-1)\delta} S(\log S)^{n+(2n^2+n+1)\delta} \geq P'.$$

Hence,

$$(18) \quad |F_{\tau\tau_1\cdots\tau_{n-1}}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for  $0 \leq \tau + \tau_1 + \cdots + \tau_{n-1} \leq T' - 1$  and  $0 \leq p \leq P' - 1$ . From (6) it now follows that

$$(19) \quad |F^{(t)}(p)| \leq \exp \left\{ -\frac{1}{4}U \right\}$$

for  $t = 0, 1, \cdots, T' - 1$  and  $p = 0, 1, \cdots, P' - 1$ .

We apply Lemma 8 of [4] to  $F$  with  $K$  replaced by  $K^n$ . Let  $\Omega$  and  $\omega$  have the same meaning as in this lemma. Since the exponents of  $F$  have the form

$$k_1 \log \alpha_1 + \cdots + k_n \log \alpha_n + k_n(\sigma - \eta),$$

we see that

$$(20) \quad \Omega \leq c_5 N^{n+(n^2-1)\delta} (\log S)^{1+(2n+1)\delta}.$$

With this, the condition

$$T'P' \geq 2K^n M + 13\Omega P'$$

is easily checked. Further, we know from Lemma 1, applied with  $\varepsilon = \delta$ , that

$$(21) \quad |k_1 \log \alpha_1 + \cdots + k_n \log \alpha_n| \geq \exp \left\{ -(\log K)^{1+2\delta} \right\} \\ \geq \exp \left\{ -N^\delta (\log S)^\delta \right\}$$

for all integers  $k_1, \cdots, k_n$ , not all zero, with  $|k_1| \leq K-1, \cdots, |k_n| \leq K-1$ . From (21) and (5) it follows that

$$(22) \quad \omega \geq \exp \left\{ -N^\delta (\log S)^\delta \right\} - \exp \left\{ -\frac{1}{2}U \right\} \geq \exp \left\{ -2N^\delta (\log S)^\delta \right\}.$$

From (20) we have

$$(23) \quad \Omega \leq \exp \{N^\delta (\log S)^\delta\}.$$

From lemma 8 of [4], with (19), (22) and (23) we obtain

$$(24) \quad \left| \sum_{v=0}^{N-1} C_{k_1 \dots k_n m v} \eta^v \right| \leq \exp \{c_6 N^{n^2+n+(n^3+n^2+1)\delta} S(\log S)^{n+1+(2n^2+3n+4)\delta} - \frac{1}{4}U\} \\ \leq \exp \{-\frac{1}{5}U\}$$

for  $k_1, \dots, k_n = 0, 1, \dots, K-1$  and  $m = 0, 1, \dots, M-1$ .

But according to Lemma 3 of [4] we have either

$$\sum_{v=0}^{N-1} C_{k_1 \dots k_n m v} \eta^v = 0$$

or

$$(25) \quad \left| \sum_{v=0}^{N-1} C_{k_1 \dots k_n m v} \eta^v \right| \geq \exp \{-2N^{n+1+(n^2+n)\delta} S(\log S)^{2+(2n+4)\delta}\}$$

for the same values of  $k_1, \dots, k_n$  and  $m$ . Hence,

$$\sum_{v=0}^{N-1} C_{k_1 \dots k_n m v} \eta^v = 0 \text{ for } k_1, \dots, k_n = 0, 1, \dots, K-1$$

and  $m = 0, 1, \dots, M-1$ . Since  $\eta$  has the degree  $N$ , it follows that all integers  $C_{k_1 \dots k_n m v}$  are zero, in contradiction to their choice. This contradiction proves Theorem 1.

We have the following

**COROLLARY:** *Under the conditions of Theorem 1, there exists an effectively computable, number  $C_3 = C_3(\varepsilon, \log \alpha_1, \dots, \log \alpha_n, \gamma_1, \dots, \gamma_{n-1}) > 0$  such that*

$$(26) \quad |\gamma_1 \log \alpha_1 + \dots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta| \\ > \exp \{-C_3 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon}\}$$

for all algebraic numbers  $\eta$  of degree at most  $N$  and size at most  $S$ .

**PROOF:** There are only finitely many algebraic numbers  $\eta$  of size  $s(\eta) < S_1$ . Choose  $C_3 \geq 1$  such that (26) holds for these finitely many numbers.

**THEOREM 2:** *Let, for  $n \geq 2$ ,  $\alpha_1, \dots, \alpha_n$  and  $\beta_1, \dots, \beta_n$  be non-zero algebraic numbers such that, for any fixed values of the logarithms,  $\log \alpha_1, \dots, \log \alpha_n$  are linearly independent over  $\mathcal{Q}$ . Let  $\varepsilon$  be a positive number. Then there exists an effectively computable positive number  $C_4 = C_4(\varepsilon, \log \alpha_1, \dots, \log \alpha_n, \beta_1, \dots, \beta_n)$  such that*

$$(27) \quad |\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \xi| \\ > \exp \{ -C_4 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon} \}$$

for all algebraic numbers  $\xi$  of degree  $N$  and size  $S$ .

PROOF: Put  $\gamma_i = -\beta_i/\beta_n$  ( $i = 1, \dots, n-1$ ) and  $\eta = -\xi/\beta_n$ . Then

$$(-1/\beta_n)(\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \xi) \\ = \gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta.$$

We have  $d(\eta) \leq c_7 N$  with  $c_7 = d(\beta_n)$  and, by Lemma 2,  $s(\eta) \leq c_8 S$  with  $c_8 = 3d(\beta_n) + s(\beta_n)$ . From (26) we now obtain

$$|\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n - \xi| \\ = |\beta_n| |\gamma_1 \log \alpha_1 + \cdots + \gamma_{n-1} \log \alpha_{n-1} - \log \alpha_n - \eta| > \\ > |\beta_n| \exp \{ -C_3 (c_1 N)^{n^2+n+\varepsilon} c_2 S(1+\log c_2 S)^{n+1+\varepsilon} \} \\ \geq \exp \{ -C_4 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon} \}$$

for some effectively computable positive number  $C_4$ .

THEOREM 3: Under the conditions of Theorem 2, there exists an effectively computable number  $C_1 = C_1(\varepsilon, \log \alpha_1, \dots, \log \alpha_n, \beta_1, \dots, \beta_n) > 0$ , such that

$$\exp \{ -C_1 N^{n^2+n+\varepsilon} S(1+\log S)^{n+1+\varepsilon} \}$$

is a transcendence measure of  $\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n$ .

PROOF: Apply Lemma 9 of [4] to the result of Theorem 2 and put  $C_1 = 6C_4(1+\log 2)^{n+1+\varepsilon}$ .

#### 4. The case $e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$

THEOREM 4: Let  $n$  be a positive integer. Let  $\beta_0$  be algebraic and let  $\alpha_1, \dots, \alpha_n$  be non-zero algebraic numbers such that, for any fixed values of the logarithms,  $\log \alpha_1, \dots, \log \alpha_n$  are linearly independent over  $\mathbb{Q}$ . Let  $\beta_1, \dots, \beta_n$  be algebraic numbers, not all rational. Put

$$\alpha_i^{\beta_i} = e^{\beta_i \log \alpha_i} \text{ for } i = 1, \dots, n.$$

Let  $\varepsilon$  be a positive number. Then there exists an effectively computable number  $S_2 = S_2(\varepsilon, \log \alpha_1, \dots, \log \alpha_n, \beta_0, \beta_1, \dots, \beta_n) > 0$  such that

$$(28) \quad |e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n} - \xi| > \exp \{ -N^{n^2+2n+2+\varepsilon} S^{n+1+\varepsilon} \}$$

for all algebraic  $\xi$  of degree  $N$  and size  $S \geq S_2$ .

PROOF: Put  $\delta = (2n^3 + 8n^2 + 10n + 4)^{-1} \varepsilon$ . For the sake of brevity, put

$$\sigma = e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$$

and

$$U = N^{n^2+2n+2+(2n^3+8n^2+10n+4)\delta} S^{n+1+(2n^2+7n+11)\delta}.$$

It is sufficient to prove that

$$|\sigma - \xi| > \exp \{-U\}$$

if  $S \geq S_2$ ; in this proof we may assume that  $\delta$  is rather small. By  $c_1, c_2, \dots$  we shall denote positive numbers which depend on  $n, \log \alpha_1, \dots, \log \alpha_n, \beta_0, \beta_1, \dots, \beta_n$  only.

Suppose that

$$(29) \quad |\sigma - \xi| \leq \exp \{-U\}$$

for some algebraic number  $\xi$  of degree  $N$  and size  $S$ . We prove that this is impossible if  $S$  is sufficiently large.

Choose the following integers:

$$\begin{aligned} K &= [N^{n+(n^2+n)\delta} S^{1+(2n+3)\delta}], \\ L &= [N^{n+1+(n^2+n)\delta} S^{(2n+3)\delta}], \\ M &= [N^{n+1+(n^2+2n+1)\delta} S^{1+(2n+5)\delta}], \\ C &= 2[\tfrac{1}{2} \exp \{N^{n+1+(n^2+2n+1)\delta} S^{1+(2n+6)\delta}\}], \\ T &= [N^{n+1+(n^2+2n+1)\delta} S^{1+(2n+5)\delta}] (= M), \\ P &= [NS^{2\delta}], \\ R &= \left[ \frac{n}{\delta} + 2n^2 + 5n + 3 \right], \\ T' &= [2^{-R}T], \text{ and} \\ P' &= [\tfrac{1}{2}N^{n^2+n+1+(n^3+2n^2+n)\delta} S^{n+(2n^2+5n+4)\delta}]. \end{aligned}$$

Put

$$\begin{aligned} F(z) &= \sum_{k_1=0}^{K-1} \cdots \sum_{k_n=0}^{K-1} \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} \sum_{v=0}^{N-1} C_{k_1 \dots k_n l m v} \xi^v z^m \\ &\quad \times \exp \{ \ell \beta_0 z + \sum_{i=1}^n (k_i + \ell \beta_i) (\log \alpha_i) z \}, \end{aligned}$$

where the numbers  $C_{k_1 \dots k_n l m v}$  are integers of absolute values at most  $C$ ; they will be specified later.

For  $t = 0, 1, 2, \dots$  we have

$$(30) \quad F^{(t)}(z) = \sum_{\tau+\tau_1+\dots+\tau_n=t} \frac{t!}{\tau! \tau_1! \cdots \tau_n!} \prod_{i=1}^n (\log \alpha_i)^{\tau_i} F_{\tau \tau_1 \dots \tau_n}(z)$$

where

$$(31) \quad F_{\tau\tau_1 \dots \tau_n}(z) = \sum_{k_1} \dots \sum_{k_n} \sum_l \sum_m \sum_v C_{k_1 \dots k_n l m v} \zeta^v \sum_{\kappa=0}^m \left(\frac{\tau}{\kappa}\right) \frac{m!}{(m-\kappa)!} \\ \times z^{m-\kappa} \ell^{\tau-\kappa} \beta_0^{\tau-\kappa} \prod_{i=1}^n (k_i + \ell \beta_i)^{\tau_i} \exp \left\{ \sum_{i=1}^n k_i (\log \alpha_i) z \right\} \sigma^{lz}.$$

Define  $\Phi_{\tau\tau_1 \dots \tau_n}$  by

$$\Phi_{\tau\tau_1 \dots \tau_n}(z) = \sum_{k_1} \dots \sum_{k_n} \sum_l \sum_m \sum_v C_{k_1 \dots k_n l m v} \zeta^v \sum_{\kappa=0}^m \left(\frac{\tau}{\kappa}\right) \frac{m!}{(m-\kappa)!} \\ \times z^{m-\kappa} \ell^{\tau-\kappa} \beta_0^{\tau-\kappa} \prod_{i=1}^n (k_i + \ell \beta_i)^{\tau_i} \exp \left\{ \sum_{i=1}^n k_i (\log \alpha_i) z \right\} \xi^{lz}.$$

For  $\ell = 0, 1, \dots, L-1$  and

$$p = 0, 1, \dots, [N^{n^2+n+1+(2n^3+7n^2+8n+3)\delta} S^{n+(2n^2+5n+5)\delta}]$$

one has

$$|\sigma^{lp} - \xi^{lp}| \leq lp |\sigma - \xi| (|\sigma| + 1)^{lp} \leq \exp \{-\frac{1}{2}U\}.$$

Hence,

$$(32) \quad |F_{\tau\tau_1 \dots \tau_n}(p) - \Phi_{\tau\tau_1 \dots \tau_n}(p)| \leq \exp \{-\frac{1}{3}U\}$$

for  $\tau, \tau_1, \dots, \tau_n = 0, 1, \dots, T-1$  and

$$p = 0, 1, \dots, [N^{n^2+n+1+(2n^3+7n^2+8n+3)\delta} S^{n+(2n^2+5n+5)\delta}].$$

We apply Lemma 6 of [4] to the polynomials

$$P_{\tau\tau_1 \dots \tau_n p k_1 \dots k_n l m}(\tau, \tau_1, \dots, \tau_n = 0, 1, \dots, T-1; p = 0, 1, \dots, P-1; \\ k_1, \dots, k_n = 0, 1, \dots, K-1; \ell = 0, 1, \dots, L-1 \text{ and } m = 0, 1, \dots, M-1),$$

chosen in the appropriate way such that

$$\Phi_{\tau\tau_1 \dots \tau_n}(p) = \sum_{k_1} \dots \sum_{k_n} \sum_l \sum_m \sum_v C_{k_1 \dots k_n l m v} \zeta^v \\ \times P_{\tau\tau_1 \dots \tau_n p k_1 \dots k_n l m}(\xi, \alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n).$$

If  $r, s$  and  $B$  denote the same numbers as in Lemma 6 of [4], we have

$$r = T^{n+1}P \leq N^{n^2+2n+2+(n^3+3n^2+3n+1)\delta} S^{n+1+(2n^2+7n+7)\delta}, \\ s = K^n L M \geq \frac{1}{2} N^{n^2+2n+2+(n^3+3n^2+3n+1)\delta} S^{n+1+(2n^2+7n+8)\delta}$$

and

$$B \leq \exp \{c_1 N^{n+1+(n^2+2n+1)\delta} S^{1+(2n+5)\delta} \log S\}.$$

From these inequalities it is easy to check the conditions of this lemma. Hence, we can fix the numbers  $C_{k_1 \dots k_n l m v}$  as integers, not all zero, of

absolute values at most  $C$ , such that  $\Phi_{\tau\tau_1\ldots\tau_n p} = 0$  for  $\tau, \tau_1, \ldots, \tau_n = 0, 1, \ldots, T-1$  and  $p = 0, 1, \ldots, P-1$ . With (32) this implies

$$(33) \quad |F_{\tau\tau_1\ldots\tau_n}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for  $0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T-1$  and  $0 \leq p \leq P-1$ .

Define  $T_r$  and  $P_r$  for  $r = 0, 1, \ldots, R$  by

$$T_r = [2^{-r}T]$$

and

$$P_r = [(N^{n+1}S)^{\delta}P].$$

Observe that

$$(34) \quad P_R \leq N^{n^2+n+1+(2n^3+7n^2+8n+3)\delta} S^{n+(2n^2+5n+5)\delta}.$$

LEMMA: For  $r = 0, 1, \ldots, R$  the inequality

$$(35) \quad |F_{\tau\tau_1\ldots\tau_n}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

holds for all non-negative integers  $\tau, \tau_1, \ldots, \tau_n$  and  $p$  with  $0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T_r - 1$  and  $0 \leq p \leq P_r - 1$ .

PROOF: We use induction on  $r$ . For  $r = 0$  the inequality has already been proved in (33). Let  $r$  be an integer with  $0 \leq r \leq R-1$  for which

$$(36) \quad |F_{\tau\tau_1\ldots\tau_n}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for  $0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T_r - 1$  and  $0 \leq p \leq P_r - 1$ . Since

$$\begin{aligned} F_{\tau\tau_1\ldots\tau_n}(p) &= \sum_{k_1} \cdots \sum_{k_n} \sum_l \sum_m \sum_v C_{k_1 \cdots k_n l m v} \xi^v \\ &\quad \times \prod_{i=1}^n (\log \alpha_i)^{-\tau_i} \times (z^m e^{l\beta_0 z})^{(\tau)} \times \prod_{i=1}^n (e^{(k_i + l\beta_i)(\log \alpha_i)(\tau_i)})^{(\tau_i)} \end{aligned}$$

it follows that for  $t = 0, 1, 2, \ldots$

$$\begin{aligned} F_{\tau\tau_1\ldots\tau_n}^{(t)}(z) &= \sum_{\mu + \mu_1 + \cdots + \mu_n = t} \frac{t!}{\mu! \mu_1! \cdots \mu_n!} \\ &\quad \times \prod_{i=1}^n (\log \alpha_i)^{\mu_i} F_{\tau + \mu, \tau_1 + \mu_1, \ldots, \tau_n + \mu_n}(z). \end{aligned}$$

Hence, (36) implies

$$(37) \quad |F_{\tau\tau_1\ldots\tau_n}^{(t)}(p)| \leq \exp \left\{ -\frac{1}{3}U \right\}$$

for  $0 \leq \tau + \tau_1 + \cdots + \tau_n \leq T_{r+1} - 1$ ,  $0 \leq t \leq T_{r+1} - 1$  and  $0 \leq p \leq P_r - 1$ .

For the same values of  $\tau, \tau_1, \ldots, \tau_n$  we obtain from (31)

$$(38) \quad \max_{|z| \leq 6P_{r+1}} |F_{\tau\tau_1 \dots \tau_n}(z)| \leq \exp \{c_2 N^{n+1+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+6)\delta}\}.$$

We apply Lemma 7 of [4] to  $F_{\tau\tau_1 \dots \tau_n}$  with  $R = P_{r+1}$ ,  $A = 6$ ,  $T = T_{r+1}$  and  $P = P_r$ . From (37), (38) and (34) we then obtain

$$\max_{|z| \leq P_{r+1}} |F_{\tau\tau_1 \dots \tau_n}(z)| \leq \exp \{-2^{-(r+3)} N^{n+2+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+7)\delta}\}.$$

Consequently,

$$|F_{\tau\tau_1 \dots \tau_n}(p)| \leq \exp \{-2^{-(r+3)} N^{n+2+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+7)\delta}\}$$

for  $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_{r+1} - 1$  and  $0 \leq p \leq P_{r+1} - 1$ . From (32) and (34), it follows that

$$(39) \quad |\Phi_{\tau\tau_1 \dots \tau_n}(p)| \leq \exp \{-2^{-(r+4)} N^{n+2+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+7)\delta}\}$$

for  $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_{r+1} - 1$  and  $0 \leq p \leq P_{r+1} - 1$ .

However, for these values of  $\tau, \tau_1, \dots, \tau_n$  and  $p$ , we can consider  $\Phi_{\tau\tau_1 \dots \tau_n}(p)$  as a polynomial in  $\xi, \alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n$ , of degree less than  $LP_{r+1} + N$  in  $\xi, KP_{r+1}$  in  $\alpha_1, \dots, \alpha_n$  and  $T_{r+1}$  in  $\beta_0, \beta_1, \dots, \beta_n$ . The sum of the absolute values of its coefficients is at most

$$\exp \{2N^{n+1+(n^2+2n+1)\delta} S^{1+(2n+6)\delta}\}.$$

According to Lemma 3 of [4] we have either  $\Phi_{\tau\tau_1 \dots \tau_n}(p) = 0$  or

$$(40) \quad |\Phi_{\tau\tau_1 \dots \tau_n}(p)| \geq \exp \{-c_3 N^{n+2+(n^2+nr+2n+r+1)\delta} S^{1+(2n+r+6)\delta}\}.$$

Hence,

$$\Phi_{\tau\tau_1 \dots \tau_n}(p) = 0 \text{ for } 0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_{r+1} - 1 \text{ and } 0 \leq p \leq P_{r+1} - 1.$$

From (32) and (34) we see

$$|F_{\tau\tau_1 \dots \tau_n}(p)| \leq \exp \{-\frac{1}{3}U\}$$

for  $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_{r+1} - 1$  and  $0 \leq p \leq P_{r+1} - 1$ , which proves the lemma.

From (35) with  $r = R$  we get

$$|F_{\tau\tau_1 \dots \tau_n}(p)| \leq \exp \{-\frac{1}{3}U\}$$

for  $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T_R - 1$  and  $0 \leq p \leq P_R - 1$ . We have  $T_R = T'$ . From  $R \geq n/\delta + 2n^2 + 5n + 2$  we see

$$P_R \geq [\frac{1}{2}N^{n^2+n+1+(2n^3+7n^2+7n+2)\delta}S^{n+(2n^2+5n+4)\delta}] \geq P'$$

Thus,

$$|F_{\tau_1 \dots \tau_n}(p)| \leq \exp \{-\frac{1}{3}U\}$$

for  $0 \leq \tau + \tau_1 + \dots + \tau_n \leq T' - 1$  and  $0 \leq p \leq P' - 1$ . From (30) we now obtain

$$(41) \quad |F^{(t)}(p)| \leq \exp \{-\frac{1}{4}U\}$$

for  $t = 0, 1, \dots, T' - 1$  and  $p = 0, 1, \dots, P' - 1$ .

The exponents of  $F$  have the form

$$\ell\beta_0 + (k_1 + \ell\beta_1) \log \alpha_1 + \dots + (k_n + \ell\beta_n) \log \alpha_n.$$

Let  $\Omega$  and  $\omega$  have the same meaning as in Lemma 8 of [4]. Then

$$(42) \quad \Omega \leq c_4 N^{n+(n^2+n)\delta} S^{1+(2n+3)\delta},$$

from which the condition

$$T'P' \geq 2K^rLM + 13\Omega P'$$

follows by direct computation.

The difference of two exponents of  $F$  is of the form

$$\ell\beta_0 + (k_1 + \ell\beta_1) \log \alpha_1 + \dots + (k_n + \ell\beta_n) \log \alpha_n$$

with integral  $k_1, \dots, k_n, \ell$ , not all zero, and  $|k_i| \leq K - 1$  for  $i = 1, \dots, n$  and  $|\ell| \leq L - 1$ . Moreover, at least one of the numbers  $k_i + \ell\beta_i$  ( $i = 1, \dots, n$ ) is non-zero, since  $\beta_1, \dots, \beta_n$  are not all rational. The degrees of  $\ell\beta_0, k_1 + \ell\beta_1, \dots, k_n + \ell\beta_n$  are constants. We estimate their heights by means of Lemma 3; we then see that these heights do not exceed

$$c_5(2KL)^{c_6} \leq S^{c_7}$$

in which  $c_5$  and  $c_6$  are upper bounds for the heights and degrees resp. of  $\beta_0, \beta_1, \dots, \beta_n$ . From Lemma 1 with  $\varepsilon = \delta$  it follows that

$$|\ell\beta_0 + (k_1 + \ell\beta_1) \log \alpha_1 + \dots + (k_n + \ell\beta_n) \log \alpha_n| > \exp \{-(\log S)^{1+2\delta}\}.$$

Hence, the exponents of  $F$  are distinct and

$$(43) \quad \omega > \exp \{-(\log S)^{1+2\delta}\} > \exp \{-S^\delta\}.$$

From Lemma 8 of [4], using (41), (42) and (43) we obtain the inequality

$$(44) \quad \left| \sum_{v=0}^{N-1} C_{k_1 \dots k_n l m v} \xi^v \right| \leq \exp \{-\frac{1}{5}U\}$$

for  $k_1, \dots, k_n = 0, 1, \dots, K - 1$ ;  $\ell = 0, 1, \dots, L - 1$  and  $m = 0, 1, \dots, M - 1$ .



According to Lemma 3 of [4] we have either

$$(45) \quad \sum_{v=0}^{N-1} C_{k_1 \dots k_n l m v} \xi^v = 0 \text{ or } \left| \sum_{v=0}^{N-1} C_{k_1 \dots k_n l m v} \xi^v \right| > \exp \{ -2N^{n+2+(n^2+2n+1)\delta} S^{1+(2n+6)\delta} \}$$

for the same values of  $k_1, \dots, k_n, \ell$  and  $m$ . It follows that

$$\sum_{v=0}^{N-1} C_{k_1 \dots k_n l m v} \xi^v = 0$$

for all of these values. Since  $\xi$  has the degree  $N$ , this implies that all integers  $C_{k_1 \dots k_n l m v}$  are zero, in contradiction to their choice. The theorem has been proved.

Using the fact, that there are only finitely many algebraic numbers  $\xi$  of size  $S < S_2$ , and using Lemma 9 of [4], one immediately obtains the following theorem:

**THEOREM 5:** *Under the conditions of Theorem 4, there exists an effectively computable, number  $C_2 = C_2(\varepsilon, \log \alpha_1, \dots, \log \alpha_n, \beta_0, \beta_1, \dots, \beta_n) > 0$  such that*

$$\exp \{ -C_2 N^{n^2+2n+2+\varepsilon} S^{n+1+\varepsilon} \}$$

*is a transcendence measure of  $e^{\beta_0} \alpha_1^{\beta_1} \dots \alpha_n^{\beta_n}$ .*

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