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## DISCRETE SERIES AND THE UNIPOTENT SUBGROUP

G. I. Lehrer

### Introduction

In this paper we give an explicit decomposition of the restriction of any irreducible discrete series complex representation of  $GL(n, q)$  to the unipotent subgroup consisting of upper unitriangular matrices. The decomposition is as a tensor product of representations which are shown to be multiplicity free, and whose components are exhibited explicitly as representations induced from radical subgroups of a special type. The subgroups occurring correspond precisely to root subgroups, and it is hoped that the results presented here may lead to generalizations for other groups of Lie type.

A by-product is the result that there are certain representations of high degree in the restriction, and it is shown in the final section that these representations have maximal degree among the irreducible complex representations of the unipotent group. This leads in particular to the result that this maximal degree is a power  $q_1^{h(n)}$  of  $q$  which depends only on  $n$ . Also it is shown that the representations of maximal degree are in some sense 'dense'.

The decomposition shall take place in three parts. Firstly, some general propositions are proved which relate to representations of semi-direct products. Then the restriction of the discrete series characters is defined by means of its values, and an inductive property is used to achieve the tensor product decomposition. The factors are then analysed, using the results of the first part, and tensor products are calculated using Mackey's theorem.

### 1. Representations of semi-direct products

**PROPOSITION 1.1:** *Let  $G$  be a finite group expressible as a semi-direct product  $G = A \rtimes B$ , and let  $\rho$  be a complex representation of  $B$ . Then if  $\theta$  is the permutation representation of  $G$  on cosets of  $B$ , we have*

$$\rho^G = \theta \otimes \rho^*$$

where  $\rho^*$  denotes the lift (or 'pullback') of  $\rho$  from  $B$  to  $G$ .

PROOF: Let  $V_\rho$  be a  $CB$  module for the representation  $\rho$ . Then  $\rho^G$  has  $CG$  module  $W = \bigoplus_{a \in A} aV_\rho$  (see Serre [7]) where  $G$  acts as follows: if  $g \in G$ ,  $G = ab(a \in A, b \in B)$  then  $g(a_0v_0) = ab(a_0v_0) = (a \cdot ba_0b^{-1}) \cdot (bv_0) = a_1v_1(a_0, a_1 \in A; v_0, v_1 \in V_\rho)$  where  $a_1 = a \cdot ba_0b^{-1}$  and  $v_1 = bv_0$  ( $b$  acting by means of the representation  $\rho$  of  $B$  on  $V_\rho$ ). Thus  $g = ab$  sends  $a_0v_0$  to  $a_1v_1$  where  $a_1$  is the unique element of  $A$  in  $ga_0B$  and  $v_1 = bv_0$ . Now the elements of  $A$  are left coset representatives for  $B$  in  $G$  and the permutation  $g : a_0 \mapsto a_1$  defined above is the permutation  $g$  defines on the left  $B$ -cosets in  $G$ . Moreover  $V_\rho$  may be regarded as a  $CG$  module for the representation  $\rho^*$ , where if  $v \in V_\rho$ ,  $g = ab(a \in A, b \in B)$  then  $gv = bv$ . But any element of  $W$  has a unique expression as a sum  $\sum_{a \in A} a \cdot v_a(v_a \in V_\rho)$  and thus as  $C$ -vector space  $W$  is the tensor product of a space realising  $\theta$  (viz the set of  $C$ -linear combinations of elements of  $a$ ) and a space realising  $\rho^*$  (viz  $V_\rho$ ). Finally the description of the  $G$ -action on  $W$  above shows that  $\rho^G = \theta \otimes \rho^*$ , where equality here denotes equivalence of representations.

The object of the next proposition is to show that the processes of lifting and of induction commute with each other.

PROPOSITION 1.2: *Let  $G$  be as in proposition 1.1 and suppose  $C$  is a subgroup of  $B$ . Let  $\lambda$  be a complex character of  $C$ ,  $\lambda^*$  its lift to  $C^* = A \cdot C = A \rtimes C$ . Then  $\lambda^{*G} = (\lambda^{B^*})$ , where  $(\lambda^{B^*})$  denotes the lift of  $\lambda^B$  from  $B$  to  $G$ .*

PROOF: We can choose a common set of representatives  $b_1, \dots, b_n$  ( $b_i \in B$ ) for cosets of  $C$  in  $B$  and for cosets of  $C^*$  in  $B^* = G$ . Then if  $g \in G$ ,  $g = ab(a \in A, b \in B)$  we have

$$(\lambda^B)^*(g) = \lambda^B(b) = \sum_{i=1}^n \hat{\lambda}(b_i b b_i^{-1})$$

where  $\hat{\lambda}(x) = \lambda(x)$  if  $x \in C$  and  $\hat{\lambda}(x) = 0$  otherwise.

Also

$$(\lambda_C^*)^G(g) = \sum_{i=1}^n \hat{\lambda}^*(b_i a b b_i^{-1})$$

where  $\hat{\lambda}^*(y) = \lambda^*(y)$  if  $y \in C^*$  and  $\hat{\lambda}^*(y) = 0$  otherwise. But  $b_i a b b_i^{-1} = b_i a b_i^{-1} \cdot b_i b b_i^{-1} \in C^* \Leftrightarrow b_i a b_i^{-1} \in C$ . Hence  $\hat{\lambda}^*(b_i a b b_i^{-1}) = \hat{\lambda}(b_i b b_i^{-1})$  and the result follows.

The final proposition of this section describes the irreducible representations of the group  $G$  of proposition 1.1 in case  $A$  is abelian.

PROPOSITION 1.3:

(i) *With  $G$  as in proposition 1.1, let  $A$  be abelian. Then each irreducible complex representation of  $G$  is of the form  $(\chi\phi)^G$  where  $\chi$  is an irreducible*

representation of  $A$  and  $\phi$  is an irreducible representation of the centralizer  $B_\chi$  of  $\chi$  in  $B$ .

(ii) In (i) above,  $\chi$  may be replaced by any representation conjugate to it under the action of  $B$ .

PROOF: (i) is a theorem of Mackey ([6]) and (ii) is a simple consequence (see [3]).

## 2. Representations of the unipotent group

In this section we recall briefly (cf. [3]) the consequences of applying proposition 1.3 to the unipotent group. Let  $G = U_n$  be the group of upper unitriangular matrices of size  $n$  with coefficients in  $GF(q)$ . Let  $A$  be the subgroup of  $G$  consisting of matrices all of whose non-diagonal elements are zero except for the last column, and take for  $B$  the set of matrices whose non-diagonal entries in the last column are zero. Then  $G = A \rtimes B$ ,  $A \cong (GF(q)^+)^{n-1}$  is abelian and may be regarded as an  $(n-1)$ -dimensional vector space over  $GF(q)$  on which  $B$  acts:

$$\begin{bmatrix} X & | & 0 \\ \hline 0 & | & 1 \end{bmatrix} \begin{bmatrix} I & | & v \\ \hline & | & 1 \end{bmatrix} \begin{bmatrix} X^{-1} & | & 0 \\ \hline 0 & | & 1 \end{bmatrix} = \begin{bmatrix} I & | & Xv \\ \hline 0 & | & 1 \end{bmatrix}$$

In fact  $B \cong U_{n-1}$  and by an obvious identification  $B$  acts as  $U_{n-1}$  on column vectors of length  $n-1$  which are regarded as the elements of  $A$ .

An irreducible character  $\chi$  of  $A$  is given by a set of  $n-1$  characters  $\chi_i$  of  $GF(q)^+$ , where

$$\chi \begin{bmatrix} v_1 \\ \vdots \\ v_{n-1} \end{bmatrix} = \prod_{i=1}^{n-1} \chi_i(v_i),$$

and if  $b \in B$ ,  $\chi^b(v) = \chi(bvb^{-1})$ . Henceforth let  $\chi_0$  be a fixed non-trivial irreducible character of  $GF(q)^+$ ; then any irreducible character of  $GF(q)^+$  is of the form  $a\chi_0$ , where  $a\chi_0(f) = \chi_0(af)$  ( $a, f \in GF(q)$ ), and thus corresponds to an element of  $GF(q)$ . Hence characters  $\chi = \chi_1 \cdots \chi_{n-1}$  also can be regarded as row vectors over  $GF(q)$ .

With these identifications it is clear that if  $b \in B$  and  $\chi = (\chi_1, \cdots, \chi_{n-1})$  then  $\chi^b = (\chi_1, \cdots, \chi_{n-1})(b_{ij})$  where  $(b_{ij})$  is the leading  $(n-1) \times (n-1)$  part of  $b$ .

Let  $k$  be the least index such that  $\chi_k \neq 0$ , if  $\chi \neq (0, 0, \cdots, 0)$ ; then the  $B$ -orbit of  $\chi$  contains precisely all characters of the form  $(0, 0, \cdots, 0, \chi_k, \mu_{k+1}, \cdots, \mu_{n-1})$  where the  $\mu_i$  are arbitrary. In particular this orbit contains the character  $\chi_c = (0, 0, \cdots, 0, \chi_k, 0, \cdots, 0)$  which will be referred to as the canonical character in the orbit of  $\chi$ . The  $B$ -orbits in the

character group  $\hat{A}$  of  $A$  are thus represented by the  $(n-1)(q-1)$  canonical characters  $\chi_c$ , together with the zero (identity) character.

The centralizer  $B_{\chi_c}$  consists of matrices in  $B$  whose non-diagonal entries in the  $k^{\text{th}}$  row are zero. It depends, therefore, only on the index  $k$  which is referred to as the *type* of  $\chi$ . The identity is canonical of type 0.

DEFINITION: If the irreducible character  $\chi$  of  $A \triangleleft U_n$  is of type  $k$ , we write  $Z_{k,n-1} = B_{\chi_c}$  and  $U_{k,n} = A \rtimes Z_{k,n-1}$

Proposition 1.3 applied here gives

LEMMA 2.1: *Any irreducible representation  $\nu$  of  $U_n$  is of the form  $\nu = (\chi\phi)_{U_{k,n}}^{U_n}$  where  $\chi$  is a canonical character of type  $k$  and  $\phi$  is an irreducible representation of  $B_{\chi}$ .*

To conclude this section we introduce some notation. For each integer  $m$ ,  $1 \leq m \leq n$  we regard the group  $U_m$  as the specific subgroup of  $U_n$  consisting of matrices in  $U_n$  with all non-diagonal entries zero, except those in the leading  $m \times m$  square. Then  $U_m$  has an obvious normal complement  $C_m$  and  $U_n = C_m \rtimes U_m$ .

DEFINITION:

- (i) If  $H$  is a subgroup of  $U_m$  we write  $H^*$  for  $H^* = C_m \rtimes H$ .
- (ii)  $A_m = C_{m-1} \cap U_m$  is the normal complement of  $U_{m-1}$  in  $U_m$  (e.g.  $A_n$  is the  $A$  of the discussion above).

### 3. The discrete series

There is a family of distinguished irreducible representations of  $GL(n, q)$  which have degree  $(q-1)(q^2-1) \cdots (q^{n-1}-1)$  and whose importance is outlined in [8]. An explicit description of the values of the characters of these representations is given in Green's famous paper [2], and can also be found in [4]. It transpires that all discrete series representations have the same restriction to  $U_n$  since the character values are the same on  $U_n$ .

DEFINITION: For  $u \in U_n$ , define the rational integer  $\delta_n(u)$  by  $\delta_n(u) = (-1)^{n-1}(1-q)(1-q^2) \cdots (1-q^{s(u)})$  where  $s(u) = (n-1) - r(u)$  and  $r(u) = \text{rank of the matrix } 1-u$ .

Then  $\delta_n$  is the character of the restriction to  $U_n$  of any irreducible discrete series representation of  $GL(n, q)$ ; we denote this representation of  $U_n$  by  $A_n$  and it is with its decomposition that we are concerned. The first step is the remark

PROPOSITION 3.1: *We have  $\delta_n = \delta_{n-1}^{U_n} - \delta_{n-1}^*$ , where  $\delta_{n-1}$  is the discrete series character of  $U_{n-1}$  as defined in the previous section.*

PROOF: This is simple to verify, using the formula for calculating induced characters; the explicit calculation may be found in [5]. This fact has been remarked on by several authors, including Ennola [1].

Proposition 1.1 enables us to exploit this inductive property of  $\Delta_n$ . We define the representation  $\rho_n$  as follows: Let  $\Gamma_n$  be the set of cosets of  $U_{n-1}$  in  $U_n$  and let  $\mathcal{F}_n$  be the set of functions  $f: \Gamma_n \rightarrow \mathbb{C}$ . Then  $U_n$  acts naturally on the  $q^{n-1}$ -dimensional space  $\mathcal{F}_n$ , and this is simply the permutation representation  $\theta_n$  of  $U_n$  on the cosets of  $U_{n-1}$ . Let

$$\overline{\mathcal{F}}_n = \{f \in \mathcal{F}_n \mid \sum_{\gamma \in \Gamma_n} f(\gamma) = 0\}.$$

Then  $U_n$  stabilizes  $\overline{\mathcal{F}}_n$ , and we define  $\rho_n$  to be the  $(q^{n-1} - 1)$ -dimensional representation of  $U_n$  on  $\overline{\mathcal{F}}_n$ .

PROPOSITION 3.2: *We have  $\Delta_n = \rho_n \otimes \Delta_{n-1}^*$*

PROOF: Let  $f_0 \in \mathcal{F}_n$  be the function taking the value 1 at each element of  $\Gamma_n$ , and let  $\langle f_0 \rangle$  be the 1-dimensional space spanned by  $f_0$ . Then  $\mathcal{F}_n = \overline{\mathcal{F}}_n \oplus \langle f_0 \rangle$  and both components are stable under  $U_n$ . But the representation of  $U_n$  on  $\langle f_0 \rangle$  is the identity, and so we have  $\theta_n = \rho_n \oplus 1$ .

By proposition 1.1  $\Delta_{n-1}^{U_n} = \theta_n \otimes \Delta_{n-1}^*$  and this by the above is equal to  $(\rho_n \oplus 1) \otimes \Delta_{n-1}^*$ . Hence  $\Delta_{n-1}^{U_n} = \rho_n \otimes \Delta_{n-1}^* \oplus \Delta_{n-1}^*$ . The result is now immediate from proposition 3.1.

An immediate corollary is

THEOREM 3.3: *Let  $\rho_i$  be the  $(q^{i-1} - 1)$ -dimensional representation of  $U_i$  defined as above. Then*

$$\Delta_n = \rho_n \otimes \rho_{n-1} \otimes \dots \otimes \rho_2$$

where  $\rho_i$  is identified with its lift from  $U_i$  to  $U_n$ .

We now turn our attention to the  $\rho_i$ .

#### 4. The representation $\rho_n$

LEMMA 4.1: *The restriction of  $\rho_n$  to  $A_n$  is the sum of all the irreducible representations (characters) of  $A_n$  except for the identity representation, each one occurring with multiplicity one.*

PROOF: The action of  $A_n$  on  $\Gamma_n$  is the permutation action of  $A_n$  on its own elements by left translation since the elements of  $A_n$  form a set of coset representatives. Hence the representation of  $A_n$  on  $\mathcal{F}_n$  is the regular representation of  $A_n$ , which is the sum of all the irreducible representations of  $A_n$ , since  $A_n$  is abelian. But the representation of  $A_n$  on  $\langle f_0 \rangle$  is the identity representation, and so the representation of  $A_n$  on  $\overline{\mathcal{F}}_n$  (which is  $\rho_n|_{A_n}$ ) is the sum of the non-identity representations of  $A_n$ .

**COROLLARY 4.1:**  $\rho_n$  is a multiplicity free representation of  $U_n$ .

This is clear, since its restriction to  $A_n$  is multiplicity free.

**LEMMA 4.2:** *There is a 1-1 correspondence  $\chi \rightarrow \alpha(\chi)$  between canonical non-trivial characters  $\chi$  of  $A_n$  and irreducible constituents  $\alpha(\chi)$  of  $\rho_n$  such that if  $\chi$  is of type  $k$ , degree  $(\alpha(\chi)) = q^{n-k-1}$  ( $k = 1, 2, \dots, n-1$ ).*

**PROOF:** Let  $\alpha$  be an irreducible constituent of  $\rho_n$ . By Clifford's theorem together with lemma 4.1, the restriction  $\alpha|_{A_n}$  is the sum of the characters of  $A_n$  in a  $U_{n-1}$ -orbit of non-trivial characters. By the discussion in Section 2, these orbits are represented by canonical characters. Let  $\chi$  be a canonical character of  $A_n$ , of type  $k$ . Then if  $\alpha$  is the unique constituent of  $\rho_n$  containing  $\chi$  in its restriction to  $A_n$ , we have that the degree of  $\alpha$  is the number of characters of  $A_n$  in the orbit of  $\chi$  under  $U_{n-1}$ . By the discussion preceding lemma 2.1 this is  $q^{n-k-1}$ .

In view of this correspondence we introduce the following notation.

**DEFINITION:**

(i) Let  $\chi$  be a non-trivial character of  $GF(q)^+$ . Denote by  $\chi^{(k)}$  the type  $k$  canonical character  $(0, \dots, 0, \chi, 0, \dots, 0)$  of  $A_n$ .

(ii) Write  $\alpha_{kn}(\chi)$  for the irreducible constituent of  $\rho_n$  containing  $\chi^{(k)}$  in its restriction to  $A_n$ .

**COROLLARY 4.2:** *We have  $\alpha_{kn}(\chi) = (\chi^{(k)} \cdot \phi)_{U_{kn}}^{U_n}$  where  $\phi$  is a representation of degree one of  $Z_{k,n-1}$ .*

**PROOF:** By lemma 2.1  $\alpha_{kn}(\chi)$  is of the form stated;  $\phi$  has degree one by Lemma 4.2.

We next show that  $\phi$  can in fact be taken as the identity representation of  $Z_{k,n-1}$  in each case.

**LEMMA 4.3:** *The restriction  $\rho_n|^{U_{kn}}$  contains the one-dimensional representations  $\chi^{(k)} \cdot 1$  for each non-trivial character  $\chi$  of  $GF(q)^+$*

**PROOF:** We construct a subspace  $\overline{\mathcal{F}}_n(\chi^{(k)})$  of  $\overline{\mathcal{F}}_n$  which realizes the representation  $\chi^{(k)} \cdot 1$  of  $U_{k,n}$ .

We consider the elements of  $\Gamma_n$  as identified with  $A_n$ , and let  $f$  be the function in  $\overline{\mathcal{F}}_n$  such that

$$f(a) = \chi^{(k)}(a) \quad (a \in A_n)$$

and take  $\overline{\mathcal{F}}_n(\chi^{(k)})$  to be the 1-dimensional space spanned by  $f$ . Then  $A_n$  clearly acts on  $\overline{\mathcal{F}}_n(\chi^{(k)})$  according to  $\chi^{(k)}$ . Moreover  $Z_{k,n-1}$  acts on  $f$  according to

$$zf(a) = f(zaz^{-1}) = \chi^{(k)}(zaz^{-1}) = \chi^{(k)}(a) = f(a), \quad (z \in Z_{k,n-1}),$$

since  $Z_{k,n-1}$  centralizes the character  $\chi^{(k)}$  of  $A_n$ . Hence  $Z_{k,n-1}$  fixes the function  $f$  and  $\overline{\mathcal{F}}_n(\chi^{(k)})$  is a module for  $U_{k,n}$  realizing the representation  $\chi^{(k)} \cdot 1$ .

This gives us the following explicit form for  $\alpha_{k,n}(\chi)$ .

**COROLLARY 4.3:** *We have  $\alpha_{k,n}(\chi) = (\chi^{(k)} \cdot 1)_{U_{k,n}}^{U_n}$*

**PROOF:** The right hand side is an irreducible representation of  $U_n$  which occurs in  $\rho_n$  by lemma 4.3 and Frobenius reciprocity. Moreover it (the R.H.S.) contains  $\chi^{(k)}$  in its restriction to  $A_n$ . Hence the result.

Collecting together the last three results we have

**THEOREM 4.4:** *The representation  $\rho_n$  has additive decomposition*

$$\rho_n = \bigoplus_{\substack{k=1 \\ \chi}}^{n-1} \alpha_{k,n}(\chi)$$

where  $\chi$  runs over the non-trivial characters of  $GF(q)^+$  and  $\alpha_{k,n}(\chi)$  is induced from the 1-dimensional representation  $\chi^{(k)} \cdot 1$  of  $U_{k,n}$ .

Theorem 4.4 of course immediately yields the decomposition of  $\rho_i$  for each  $i$  (see theorem 3.3), but using proposition 1.2 we can obtain the constituents of  $\rho_i$  directly as induced representations.

**PROPOSITION 4.5:** *Let  $\lambda_{ki}(\chi)$  be the character of  $U_{ki}^*$  (for notation see Section 2) given by  $\lambda_{ki}(\chi)(u) = \chi(u_{ki})$  where  $u_{ki}$  is the  $(k, i)$  matrix coefficient of  $u(u \in U_{ki}^*)$ . Then if  $\alpha_{ki}(\chi)$  is the representation induced from  $\lambda_{ki}(\chi)$  we have*

$$\rho_i = \bigoplus_{\substack{k=1 \\ \chi}}^{i-1} \alpha_{ki}(\chi)$$

where  $\chi$  runs over the non-trivial characters of  $GR(q)^+$ , and  $\rho_i$  is as in 3.3.

**PROOF:** This follows from Theorem 4.4 and proposition 1.2, which says that lifting the analogues of the  $\alpha_{kn}(\chi)$  from  $U_i$  to  $U_n$  gives the same result as first lifting the 1-dimensional representation of  $U_{ki}$  and then inducing.

In summary, we have the following decomposition of  $\Delta_n$ :

**THEOREM 4.6:** *We have  $\Delta_n = \rho_n \otimes \rho_{n-1} \otimes \cdots \otimes \rho_2$  where*

$$\rho_i = \bigoplus_{\substack{k=1 \\ \chi}}^{i-1} \alpha_{ki}(\chi)$$

and the  $\alpha_{ki}(\chi)$  are induced from the 1-dimensional representations  $\lambda_{ki}(\chi)$  of  $U_{ki}^*$ , so that  $\text{degree}(\alpha_{ki}(\chi)) = q^{i-k-1}$ .

### 5. Tensor products

Theorem 4.6 shows that to obtain an additive decomposition of  $\Delta_n$ , it is necessary to work out tensor products of the form

$$\alpha_{k_2, 2}(\chi_2) \otimes \alpha_{k_3, 3}(\chi_3) \otimes \cdots \otimes \alpha_{k_n, n}(\chi_n).$$

The object of the present section is to show that many such tensor products are irreducible.

**LEMMA 5.1:** *Suppose  $i < j$  and  $k > l$ ; then  $\alpha_{ki}(\chi) \otimes \alpha_{lj}(\chi')$  is irreducible.*

**PROOF:** The tensor product above is the lift of a corresponding tensor product from  $U_j$  to  $U_n$ . Hence we may assume that  $j = n$  and  $i < n$ .

We have  $\alpha_{ki}(\chi) \otimes \lambda_{ln}(\chi') = \lambda_{ki}(\chi)^{U_n} \otimes \lambda_{ln}(\chi')^{U_n}$  where  $\lambda_{ki}(\chi)$  and  $\lambda_{ln}(\chi')$  are one-dimensional representations of  $U_{ki}^*$  and  $U_{ln}$  respectively. By Mackey's formula for tensor products, we have, since  $U_{ki}^* \cdot U_{ln} = U_n$  that

$$\alpha_{ki}(\chi) \otimes \alpha_{ln}(\chi') = (\lambda_{ki}(\chi) \cdot \lambda_{ln}(\chi')^{U_{*_{ki} \cap U_{ln}}})^{U_n}.$$

Let  $P = U_{i-1} \cap U_{ln}$ ,  $Q = A_i \cap U_{ln}$  and  $R = PQ < Z_{l, n-1}$ . Then  $R = U_i \cap U_{ln} = Q \rtimes P$  with  $Q$  abelian, and proposition 1.3 applies.

Now we have

$$(\lambda_{ki}(\chi) \cdot \lambda_{ln}(\chi')^{U_{*_{ki} \cap U_{ln}}})^{U_n} = \{[(\lambda_{ki}(\chi) \cdot 1)^R_{U_{*_{ki} \cap U_{ln}}}]\}_{Z_{l, n-1}}^* \lambda_{ln}(\chi')^{U_{ln}}$$

where  $*$  denotes the lift to  $Z_{l, n-1}$ . Moreover since  $k > l$  an easy calculation shows that  $Z_{k, i-1} \cap U_{ln} < P$  is the full centralizer of the character  $\lambda_{ki}(\chi)$  of  $Q$ . Hence the first factor above is an irreducible representation of  $Z_{l, n-1}$  by proposition 1.3, and by another application of 1.3, so is the representation  $\alpha_{ki}(\chi) \otimes \alpha_{ln}(\chi')$ .

We have almost as a corollary

**THEOREM 5.2:** *Suppose  $i_1 < i_2 < \cdots < i_r$  and  $k_1 > k_2 > \cdots > k_r$  ( $1 < r < n$ ). Then*

$$\alpha_{k_1, i_1}(\chi_1) \otimes \alpha_{k_2, i_2}(\chi_2) \otimes \cdots \otimes \alpha_{k_r, i_r}(\chi_r)$$

*is an irreducible representation of  $U_n$ .*

**PROOF:** Here we have

$$U_{k_j+1, i_j+1}^* \cdot (U_{k_j, i_j}^* \cap \cdots \cap U_{k_1, i_1}^*) = U_n$$

since the  $k_j$  are distinct. We may also assume as in 5.1 that  $i_r = n$ . Then Mackey's theorem shows that

$$\alpha_{k_1, i_1}(\chi_1) \otimes \cdots \otimes \alpha_{k_r, i_r}(\chi_r) = \lambda_{k_1, i_1}(\chi_1) \cdots \lambda_{k_r, i_r}(\chi_r)_{U_{k_1, i_1} \cap \cdots \cap U_{k_r, i_r}}^{U_n}$$

The proof now proceeds inductively; the induced representation on the right is expressed as a representation induced in stages through subgroups of the form  $R$  of 5.1, and the irreducibility of the result of each step has the same proof as lemma 5.1. The details are left to the reader.

**COROLLARY 5.2':** *Let  $\mu(n) = (n-2) + (n-4) + \dots$ . Then there are irreducible components of  $\Delta_n$  which have degree  $q^c$  for each integer  $c$  such that  $0 \leq c \leq \mu(n)$ .*

**PROOF:** The degree of  $\alpha_{k_i}(\chi)$  is  $q^{i-k-1}$ . Hence

$$\text{degree} (\alpha_{k_1, i_1}(\chi_1) \otimes \dots \otimes \alpha_{k_r, i_r}(\chi_r))$$

is  $q^c$  where

$$c = \sum_{j=1}^r (i_j - k_j - 1).$$

Thus the corollary amounts to finding sequences  $(k_1, k_2, \dots)$  such that

$$0 < k_i < i, k_{i+1} < k, \text{ and } \sum_{i=1}^n (1 - k_i - 1)$$

is specified between 0 and  $\mu(n)$ . Taking  $k_n = 1, k_{n-1} = 2, \dots, k_{\lfloor n/2 \rfloor + 1} = \lfloor n/2 \rfloor - 1, k_i = i - 1$  for  $i < \lfloor n/2 \rfloor + 1$  we obtain corresponding degree  $q^{\mu(n)}$ , since if  $k_i = i - 1$  then  $\alpha_{k_i, i}(\chi)$  has degree one. Taking  $k_i = i - 1$  for each  $i$  we obtain corresponding degree  $q^0$ . It is clear that by perturbing these choices for  $k_i$  we can obtain irreducible representations of degree  $q^c$  for each  $c$  such that  $0 \leq c \leq \mu(n)$ .

**COROLLARY 5.2'':** *The group  $U_n$  has at least  $(q-1)^{\lfloor n/2 \rfloor}$  non-isomorphic irreducible representations of degree  $q^{\mu(n)}$ , and the sum of the squares of their degrees is an integer polynomial in  $q$  with leading term  $q^{\frac{1}{2}n(n-1)}$ .*

**PROOF:** It is easy to show by induction on  $n$  that the  $(q-1)^{\lfloor n/2 \rfloor}$  irreducible representations

$$\alpha_{1, n}(\chi_1) \otimes \alpha_{2, n-1}(\chi_2) \otimes \dots \otimes \alpha_{\lfloor n/2 \rfloor, \lfloor (n+1)/2 \rfloor + 1}(\chi_{\lfloor n/2 \rfloor})$$

(where the  $\chi_i$  range over all combinations of non-trivial characters of  $GF(q)^+$ ) are mutually non-isomorphic, which explicitly constructs the required number of irreducible representations of degree  $q^{\mu(n)}$ . The conclusion about the sum of squares of the degrees follows from the integer identity  $2\mu(n) + \lfloor n/2 \rfloor = \frac{1}{2}n(n-1)$  which may be directly verified.

### 6. Representations of maximal degree

**PROPOSITION 6.1:** *Let  $G = A \rtimes B$  be a semi-direct product with  $A$  abelian. Then the degree of any irreducible complex representation of  $G$  is not greater than  $|B|$ .*

PROOF: Let  $\psi$  be an irreducible representation of  $G$ . By proposition 1.3,  $\psi$  is of the form  $\psi = (\chi\phi)_{A \rtimes B_\chi}^G$  where  $\chi$  is a character of  $A$ , and  $B_\chi$  is its centralizer in  $B$ . Then  $\text{degree}(\psi) = (\text{degree} \phi) \cdot [B : B_\chi]$ . But  $\phi$  is an irreducible representation of  $B_\chi$  and so has  $\text{degree} \leq |B_\chi|$ . The result follows.

**THEOREM 6.2:** *The degree of any irreducible complex representation of  $U_n$  of maximal degree is  $q^{\mu(n)}$ .*

PROOF: By corollary 5.2'' it remains to show only that any irreducible representation has  $\text{degree} \leq q^{\mu(n)}$ . For this let  $A$  be the abelian normal subgroup of  $U_n$  given by matrices whose non-diagonal entries are zero except for those in the upper right  $[n/2] \times [(n+1)/2]$  rectangle, and let  $B$  be its natural complement  $U_{[n/2]} \times U_{[n+1/2]}$ . Then  $|A| = q^l$  where  $l = [n/2] \cdot [(n+1)/2]$  and  $U_n = A \rtimes B$ . By proposition 6.1 any irreducible representation  $\chi$  of  $U_n$  has  $\text{degree} \leq |B|$ . But  $|B| = q^{N-l}$  where  $N = \frac{1}{2} n(n-1)$ , and we complete the proof by observing that  $N-l = \mu(n)$  which is easily verified using the relations  $2\mu(n) + [n/2] = N$  and  $2l - [n/2] = N$ .

We conclude with a conjecture on the representations of  $U_n$ , related to that made at the ICM in Moscow by Professor J. G. Thompson.

**CONJECTURE 6.3:**

- (i)  $U_n$  has irreducible complex representations only of degree  $q^c$  for  $0 \leq c \leq \mu(n)$ .
- (ii) The number of irreducible representations of degree  $q^c$  is an integer polynomial in  $q$ .

Part (i) in fact follows from theorem 6.2 and corollary 5.2' if we assume a result recently communicated privately to Professor Thompson by E. Goutkin of Moscow. The result states that all irreducible complex representations of  $U_n$  have degree a power of  $q$ .

If we assume that  $q$  is prime, then we obtain

**COROLLARY 6.4:** *For  $q$  prime the degrees of the irreducible complex representations of  $U_n$  are polynomials in  $q$ , depending only on  $n$ .*

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It has also come to the attention of the author that the results in Section 2 have been independently obtained by P. V. Lambert and G. van Dijk [9].

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