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ON THE WEAK*-BASIS THEOREM

N. J. Kalton

Suppose (E, τ) is a locally convex space; then a sequence (x_n) is called a basis of E if for every $x \in E$ there is a unique sequence of scalars (a_n) with

$$x = \sum_{n=1}^{\infty} a_n x_n$$

If, furthermore the coefficients a_n are given by

$$a_n = f_n(x)$$

where each f_n is a τ -continuous linear functional, we say that (x_n) is a Schauder basis of E .

The weak basis theorem of Mazur (see [2]) states that if X is a Banach space, then a basis of X in the weak topology is a Schauder basis of X in the strong topology; in particular it is a Schauder basis. This theorem has been extended to various classes of locally convex spaces. In particular it is natural to ask whether a basis (f_n) of X^* in the weak* topology $\sigma(X^*, X)$ is necessarily a Schauder basis; this is equivalent (see [8] p. 155) to asking whether there exists a basis (x_n) of X with (f_n) the corresponding coefficient functionals. Unfortunately Singer shows by example ([8] p. 153 or see [7]) that a weak* basis need not be Schauder.

However it is trivial that a weak*-basis of the dual of a reflexive Banach space is Schauder; in this paper we give another important class of spaces for which this theorem is true.

Let τ be an $\langle X, X^* \rangle$ polar topology on X^* , and let (f_k) be a τ -basis of X^* ; suppose $(p_\lambda; \lambda \in \Lambda)$ is a collection of semi-norms defining the topology τ . We define

$$p_\lambda^*(x) = \sup_n p_\lambda \left(\sum_{k=1}^n a_k x_k \right)$$

where

$$x = \sum_{k=1}^{\infty} a_k x_k(\tau).$$

Then the collection of semi-norms $(p_\lambda^*; \lambda \in \Lambda)$ define a topology τ^* on X^* . We then have the following lemma (see McArthur [6] Lemma 2 or Bennett and Cooper [1] Lemma 1).

LEMMA 1: (X^*, τ^*) is complete and (f_k) is a Schauder basis of (X^*, τ^*) .

PROOF: This is proved by a method similar to [1] Lemma 1 or [6] Lemma 3. It is only necessary to assume that whenever $\sum a_k f_k$ is a τ -Cauchy series then it converges; this follows from the sequential completeness of (X^*, τ) .

LEMMA 2: τ^* is weaker than the norm topology on X^* .

PROOF: For

$$f = \sum_{k=1}^{\infty} a_k f_k (\tau),$$

the sequence

$$\sum_{k=1}^n a_k f_k$$

is τ -bounded and therefore norm bounded. Let

$$\|f\|^* = \sup_n \left\| \sum_{k=1}^n a_k f_k \right\|$$

Then the standard argument, used in [1] Lemma 1, shows that $(X^*, \|\cdot\|^*)$ is a Banach space. As the identity map $(X^*, \|\cdot\|^*) \rightarrow (X^*, \|\cdot\|)$ is continuous, we obtain, by the open mapping theorem, a constant $K > 0$ such that

$$\|f\|^* \leq K\|f\|$$

However as τ is weaker than the norm topology on X^* ; then for each $\lambda \in \mathcal{A}$ there exists K_λ with

$$p_\lambda(x) \leq K_\lambda \|f\| \quad (x \in E)$$

and so

$$\begin{aligned} p_\lambda^*(x) &\leq K_\lambda \|f\|^* \\ &\leq KK_\lambda \|f\| \end{aligned}$$

and τ^* is weaker than the norm topology.

THEOREM: Let μ be a (positive) measure on a set S ; suppose X is a closed subspace of $L_1(\mu)$ and that τ is an $\langle X, X^* \rangle$ polar topology on X^* . Then any basis of (X^*, τ) is a Schauder basis.

PROOF: Suppose $\{f_k\}$ is a basis of (X^*, τ) ; then $\{f_k\}$ is a Schauder basis of (X^*, τ^*) , and so it is sufficient to show that every τ^* -continuous linear functional is also τ -continuous.

Let $J: X \rightarrow L_1(\mu)$ denote the inclusion map, and let B and C be the closed unit balls of X^* and $[L_1(\mu)]^*$ respectively; then we have $J^*(C) = B$. Let I be the identity map on X^* . The map $IJ^*: [L_1(\mu)]^* \rightarrow (X^*, \tau^*)$

is continuous by Lemma 2; furthermore, by Lemma 1, (X^*, τ^*) is a separable complete locally convex space.

We use the well-known result that $[L_1(\mu)]^*$ is isometrically isomorphic with the space $C(S)$ of continuous functions on a compact Stone space. This follows, in the case of μ σ -finite, from the remarks of Grothendieck [3] p. 167. More generally we may use the results of Kakutani ([4], [5]) to show that the real space $[L_1(\mu)]^*$ is an abstract M -space with unit, and therefore lattice isomorphic and isometric with a space $C(S)$ where S is compact and Hausdorff; as $[L_1(\mu)]^*$ is also clearly order-complete it follows that S is a Stone space.

Now, by a result of Grothendieck [3], p. 168, $IJ^* : [L_1(\mu)]^* \rightarrow (X^*, \tau^*)$ is weakly compact. Let σ^* denote the weak topology associated with τ^* ; we have that $IJ^*(C) = B$ is σ^* -relatively compact. However B is τ -closed and therefore τ^* -closed; as B is convex we can deduce that B is σ^* -closed. Thus B is σ^* -compact; hence on B , σ^* coincides with the weaker Hausdorff topology $\sigma(X^*, X)$. If ϕ is a τ^* -continuous linear functional on X^* , then ϕ is σ^* -continuous and therefore $\sigma(X^*, X)$ continuous on B ; it follows that ϕ is $\sigma(X^*, X)$ -continuous and therefore τ -continuous. This completes the proof.

We conclude by remarking that if X satisfies the hypotheses of the theorem then X is weakly sequentially complete; conversely we may ask whether the theorem holds if X is weakly sequentially complete. This would seem very likely but we have been unable to prove it.

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