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Introduction

In this paper we consider matrix transformations on the set of entire sequences into itself. We call such methods entire. By adopting M. S. Macphail's technique of applying a theorem of K. Knopp and G. G. Lorentz we obtain necessary and sufficient conditions on the elements of a matrix in order that it be an entire method. After some examples and preliminary Lemmas we then prove a consistency type theorem for entire methods of summation.

1. Entire methods of summation

Let s represent the set of all sequences of complex numbers. A member of s, say $x = \{x_k\}$, $k = 0, 1, 2, \cdots$, is called an *entire sequence* if $\sum_{k=0}^{\infty} |x_k| p^k$ converges for every $p > 0$. Let $\mathscr{E}$ designate the set of entire sequences and let $A = (a_{nk})$ $(n, k = 0, 1, 2, \cdots)$ be an infinite matrix of complex numbers. The set of equations

$$(1) \qquad y_n = \sum_{k=0}^{\infty} a_{nk} x_k \qquad (n = 0, 1, \cdots)$$

defines an *entire method of summation* if each series in (1) converges and $y = \{y_n\} \in \mathscr{E}$ whenever $x \in \mathscr{E}$. If, in addition,

$$\sum_{n=0}^{\infty} y_n = \sum_{k=0}^{\infty} x_k$$

then A is called a *regular entire method.*

For each positive integer p, let $\mathscr{E}_p$ represent the set of sequences $\{x_k\}$ such that

$$\sum_{k=0}^{\infty} |x_k| p^k < \infty.$$

In [3; p. 389], M. S. Macphail designates this set by $l(p)$ and observes that the mapping

$$\{x_k\} \to \{x_k p^k\}$$

is a one-to-one correspondence between $\mathscr{E}_p$ and l (the set of absolutely convergent series). It was shown by K. Knopp and G. G. Lorentz [2] that a necessary and sufficient condition for a matrix $A = (a_{nk})$ to transform l into itself (that is, for A to be an $l-l$ method) is that there exists a constant M such that

$$\sum_{n=0}^{\infty} |a_{nk}| < M \qquad (k = 0, 1, 2, \cdots), \tag{2}$$

and a necessary and sufficient condition for A to be absolutely regular (that is, $\sum y_n = \sum x_k$ whenever $x \in l$) is that in addition to (2) the equations

$$\sum_{n=0}^{\infty} a_{nk} = 1 \qquad (k = 0, 1, 2, \cdots)$$

hold. Thus, the matrix (a_{nk}) maps $\mathscr{E}_p$ into l if and only if the matrix $(a_{nk}p^{-k})$ is an $l-l$ method. That is, (a_{nk}) maps $\mathscr{E}_p$ into l if and only if there exists a constant $M(p)$ such that

$$\sum_{n=0}^{\infty} |a_{nk}|p^{-k} < M(p) \qquad (k = 0, 1, 2, \cdots).$$

Similarly, for each positive integer q, a matrix (b_{nk}) maps l into $\mathscr{E}_q$ if and only if the matrix $(b_{nk}q^n)$ is an $l-l$ method, that is, if and only if there exists a constant $M(q)$ such that

$$\sum_{n=0}^{\infty} |b_{nk}|q^n < M(q) \qquad (k = 0, 1, 2, \cdots).$$

Now $\mathscr{E} = \cap \{\mathscr{E}_q : q = 1, 2, \cdots\}$; hence, a matrix $A = (a_{nk})$ is an entire method if and only if to each positive integer q, there corresponds a positive number $p = p(q) \geqq q$ such that A transforms $\mathscr{E}_p$ into $\mathscr{E}_q$. In other words, A is an entire method if and only if to each $q = 1, 2, \cdots$, there corresponds a $p = p(q) \geqq q$ such that the matrix $(a_{nk}q^n p^{-k})$ is an $l-l$ method. By taking $q = 1$ we obtain necessary and sufficient conditions for A to be a regular entire method. We summarize these remarks in the following theorem.

THEOREM 1. *A necessary and sufficient condition for A to be an entire method is that for each positive integer q there exist $p(q) \geqq q$ and a constant $M(p, q)$ such that*

$$\sum_{n=0}^{\infty} |a_{nk}|q^n p^{-k} < M(p, q) \qquad (k = 0, 1, 2, \cdots), \tag{3}$$

and a necessary and sufficient condition for A to be a regular entire method is that in addition to (3) *the equations*

$$\sum_{n=0}^{\infty} a_{nk} = 1 \qquad (k = 0, 1, 2, \cdots)$$

hold.

Remark. In order that A be an entire method it is necessary that each column of A be an entire sequence. Also, by taking $q = 1$ and $p = p(1)$, it is necessary that each row be analytic, that is, for each $n = 0, 1, 2, \cdots$, the sequence

$$\{|a_{n0}|, |a_{nk}|^{1/k} : k = 1, 2, \cdots\}$$

be bounded. However, one may easily show that these conditions are not sufficient. Indeed, the matrix defined by the set of equations

$$\begin{aligned} a_{nn} &= n!, && n = 0, 1, \cdots, \\ a_{nk} &= 0, && \text{otherwise,} \end{aligned}$$

has both entire rows and entire columns. However, the entire sequence $\{1/n!\}$ is transformed into the constant sequence $\{1\}$.

2. Examples

For each complex number t, the Euler-Knopp series-to-series method is defined by the set of equations

$$\begin{aligned} E_{nk}(t) &= \binom{n}{k} t^{k+1}(1-t)^{n-k}, && k \leqq n, \\ E_{nk}(t) &= 0, && k > n. \end{aligned}$$

The transformations $E(0)$ and $E(1)$ are, respectively, the zero matrix and the identity, both of which are entire methods. However, if t is any other complex number, then the k^{th} column of (E_{nk}) is not an entire sequence and so $E(t)$ cannot be an entire method. (See the Remark.) Contrary to this, the Taylor matrix [1] is always entire. For each complex number t, the Taylor matrix $T(t)$ is defined by the set of equations

$$\begin{aligned} T_{nk}(t) &= 0, && n > k, \\ T_{nk}(t) &= \binom{k}{n} (1-t)^{n+1} t^{k-n}, && n \leqq k. \end{aligned}$$

The trivial cases $T(0)$ (identity) and $T(1)$ (zero) are certainly entire methods. If t is any complex number other than 0 or 1, then

$$
(4)\qquad \begin{aligned}\sum_{n=0}^{k}\binom{k}{n}|1-t|^{n+1}|t|^{k-n}q^{n}p^{-k} &= |1-t|p^{-k}(q|1-t|+|t|)^{k}\\ &\leqq |1-t|(q+(q+1)|t|)^{k}/p^{k}\\ &\leqq (1+R)(q+(q+1)R)^{k}/p^{k},\end{aligned}
$$

where R is chosen to be so large that $|t| \leqq R$. We may now choose $p = 2(q+(q+1)R)$. Then (4) is dominated by $(1+R)(1/2)^k$, which shows that (3) is satisfied with $M = 1+R$. Thus, $T(t)$ is entire.

Notice also that

$$
\sum_{n=0}^{k}\binom{k}{n}(1-t)^{n+1}t^{k-n} = 1-t,
$$

so that $T(t)$ is regular if and only if $t = 0$, that is, if and only if T is the identity matrix.

3. Preliminary lemmas

It is well known that $\mathscr{E}$ is a locally convex FK space with its FK topology being given by the family of seminorms $\{h_n : n = 1, 2, \cdots\}$, where for each $x \in \mathscr{E}$,

$$
h_n(x) = \max_{|z|=n} |\sum_{i=0}^{\infty} x_i z^i|.
$$

Also, if we define an analytic sequence x to mean that the sequence $\{|x_0|, |x_k|^{1/k} : k = 1, 2, \cdots\}$ is a bounded sequence, then every continuous linear functional f on $\mathscr{E}$ has the representation

$$
f(x) = \sum_{n=0}^{\infty} t_n x_n,
$$

for some analytic sequence t. (For a discussion of $\mathscr{E}$, see, for example, C. Goffman and G. Pedrick, *First Course in Functional Analysis*, pp. 220—222, 224, Prentice-Hall, New Jersey.)

Now let A be an entire method of summation and let $\mathscr{E}_A$ represent its summability field, that is,

$$
\mathscr{E}_A = \{x \in s : Ax \in \mathscr{E}\}.
$$

An application of [4, Theorem 1, p. 226 and Theorem 5, p. 230] shows how $\mathscr{E}_A$ may inherit a locally convex FK topology given by the following family of seminorms:

$$p_n(x) = |x_n|, \qquad (n = 0, 1, 2, \cdots),$$

$$q_n(x) = \sup_m |\sum_{k=0}^{m} a_{nk}x_k|, \qquad (n = 0, 1, 2, \cdots),$$

$$h_n(x) = \max_{|z|=n} |\sum_{i=0}^{\infty} (\sum_{k=0}^{\infty} a_{nk}x_k)z^i|, \qquad (n = 1, 2, 3, \cdots).$$

Also, every $f \in \mathscr{E}'_A$ (the dual space of $\mathscr{E}_A$) may be evaluated as

$$(5) \qquad f(x) = \sum_{n=0}^{\infty} t_n \sum_{k=0}^{\infty} a_{nk}x_k + \sum_{k=0}^{\infty} \alpha_k x_k$$

for some analytic sequences t and α, and all $x \in \mathscr{E}_A$. (α is analytic because $\mathscr{E}_A \supseteqq \mathscr{E}$ and so $\sum \alpha_k x_k$ converges for every $x \in \mathscr{E}$.)

To each entire method A there corresponds the functional S_A given by

$$S_A(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{nk}x_k.$$

Since every matrix map between FK spaces is continuous [4; p. 204], it follows that $S_A \in \mathscr{E}'_A$.

Lemma 1. *If $f \in \mathscr{E}'_A$, then there exists an entire method B such that $\mathscr{E}_B \supseteqq \mathscr{E}_A$ and $S_B(x) = f(x)$ for every $x \in \mathscr{E}_A$.*

Proof. Given an entire method A define the matrix $B = (b_{nk})$ by the set of equations

$$b_{0k} = \alpha_k + t_0 a_{0k} \qquad (k = 0, 1, 2, \cdots)$$

$$b_{nk} = t_n a_{nk} \qquad (n = 1, 2, \cdots; k = 0, 1, 2, \cdots),$$

where t and α are the analytic sequences given by equation (5) in the representation of f.

Let N be the smallest integer greater than or equal to the number $\max(M(\alpha), M(t))$, where

$$M(\alpha) = \max\left(\sup_k (|\alpha_0|, |\alpha_k|^{1/k}), 1\right)$$

and

$$M(t) = \max\left(\sup_n (|t_0|, |t_n|^{1/n}), 1\right).$$

N depends only on f.

To show that B is an entire method we apply Theorem 1. Let q be any positive integer whatsoever. Choose $p \geqq N \cdot q$ so that

$$\sup_k \sum_{n=0}^{\infty} |a_{nk}|(N \cdot q)^n p^{-k} = M(p, q) < \infty.$$

(This is possible because A is an entire method.) For this choice of p observe that for each $k = 0, 1, 2, \cdots$,

$$|\alpha_k|^{1/k}/p \leqq 1.$$

Thus, for each $k = 0, 1, 2, \cdots$,

$$\begin{aligned}\sum_{n=0}^{\infty} |b_{nk}| q^n p^{-k} &\leqq \frac{|\alpha_k|}{p^k} + \sum_{n=0}^{\infty} |t_n a_{nk}| q^n p^{-k} \\ &= \left(\frac{|\alpha_k|^{1/k}}{p}\right)^k + \sum_{n=0}^{\infty} |a_{nk}| (|t_n|^{1/n} \cdot q)^n p^{-k} \\ &\leqq \left(\frac{|\alpha_k|^{1/k}}{p}\right)^k + \sum_{n=0}^{\infty} |a_{nk}| (N \cdot q)^n p^{-k}.\end{aligned}$$

It follows that

$$\sup_k \sum_{n=0}^{\infty} |b_{nk}| q^n p^{-k} < \infty;$$

hence, B is an entire method.

That $\mathscr{E}_B \supseteqq \mathscr{E}_A$ follows immediately from the construction of B.

Finally, if $x \in \mathscr{E}_A$, then

$$\begin{aligned}S_B(x) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_{nk} x_k \\ &= \sum_{k=0}^{\infty} (\alpha_k + t_0 a_{0k}) x_k + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} t_n a_{nk} x_k \\ &= f(x),\end{aligned}$$

which proves the Lemma.

Let now A and B be two entire methods. Define $C = (C_{nk})$ by the set of equations

$$C_{2n,k} = a_{nk} \qquad (n, k = 0, 1, 2, \cdots),$$
$$C_{2n+1,k} = -b_{nk} \qquad (n, k = 0, 1, 2, \cdots).$$

LEMMA 2. *C is an entire method such that $\mathscr{E}_C = \mathscr{E}_A \cap \mathscr{E}_B$ and $S_C(x) = S_A(x) - S_B(x)$ for every $x \in \mathscr{E}_C$.*

PROOF. Since A and B are entire methods, given any positive integer q we may choose $p \geqq q^2$ so that

$$\sup_k \sum_{n=0}^{\infty} |a_{nk}| (q^2)^n p^{-k} < \infty$$

and

$$q \cdot \sup_k \sum_{n=0}^{\infty} |b_{nk}| (q^2)^n p^{-k} < \infty.$$

Thus,

$$\sup_k \sum_{n=0}^{\infty} |C_{nk}| q^n p^{-k} \leqq \sup_k \sum_{n=0}^{\infty} |a_{nk}| q^{2n} p^{-k} + \sup_k \sum_{n=0}^{\infty} |b_{nk}| q^{2n+1} p^{-k} < \infty,$$

so that C is an entire method.

Next, $x \in \mathscr{E}_C$ if and only if for every $p > 0$,

$$\left|\sum_{k=0}^{\infty} a_{0k} x_k\right| p^0 + \left|\sum_{k=0}^{\infty} b_{0k} x_k\right| p^1 + \left|\sum_{k=0}^{\infty} a_{1k} x_k\right| p^2 + \left|\sum_{k=0}^{\infty} b_{1k} x_k\right| p^3 + \cdots < \infty. \tag{6}$$

Since this is a series of non-negative terms, it is satisfied for every $p > 0$ if and only if

$$\sum_{n=0}^{\infty} \left|\sum_{k=0}^{\infty} a_{nk} x_k\right| (p^2)^n + p \cdot \sum_{n=0}^{\infty} \left|\sum_{k=0}^{\infty} b_{nk} x_k\right| (p^2)^n < \infty$$

for every $p > 0$, that is, if and only if $x \in \mathscr{E}_A \cap \mathscr{E}_B$.

Finally, let $x \in \mathscr{E}_C$. Then

$$S_C(x) = \sum_{k=0}^{\infty} a_{0k} x_k - \sum_{k=0}^{\infty} b_{0k} x_k + \sum_{k=0}^{\infty} a_{1k} x_k - \sum_{k=0}^{\infty} b_{1k} x_k \pm \cdots.$$

Since this is an absolutely convergent series [take $p = 1$ in equation (6)], we may rearrange its terms to obtain

$$\begin{aligned} S_C(x) &= \sum_n \sum_k a_{nk} x_k - \sum_n \sum_k b_{nk} x_k \\ &= S_A(x) - S_B(x). \end{aligned}$$

4. Consistency of entire methods of summation

Two entire methods A and B will be called *consistent* (relative to the functionals S_A and S_B) if $S_A(x) = S_B(x)$ for every $x \in \mathscr{E}_A \cap \mathscr{E}_B$.

THEOREM 2. *In order that an entire method A be consistent with every entire method B whenever $S_A(x) = S_B(x)$ for $x \in \mathscr{E}$, it is necessary and sufficient that $\mathscr{E}$ be dense in $\mathscr{E}_A \cap \mathscr{E}_B$ whenever $S_B(x) = S_A(x)$ for $x \in \mathscr{E}$ (where the closure is taken in the FK topology of $\mathscr{E}_A \cap \mathscr{E}_B$).*

PROOF. Assume $\mathscr{E}$ is dense in $\mathscr{E}_A \cap \mathscr{E}_B$ and that $S_A(x) = S_B(x)$ for every $x \in \mathscr{E}$. Then $F(x) = S_A(x) - S_B(x)$ defines a continuous linear functional on $\mathscr{E}_A \cap \mathscr{E}_B$ which vanishes on $\mathscr{E}$; hence, it must vanish on $\mathscr{E}_A \cap \mathscr{E}_B$. Thus, A and B are consistent.

Conversely, assume that A is an entire method which is consistent with every entire method that agrees with A on $\mathscr{E}$. Suppose there exists an entire method B such that $S_B(x) = S_A(x)$ for $x \in \mathscr{E}$ and yet $\mathscr{E}$ is not dense in $\mathscr{E}_A \cap \mathscr{E}_B$. Then there exists $f \in \mathscr{E}'_C$ such that f vanishes on $\mathscr{E}$ and $f(y) \neq 0$ for some $y \in \mathscr{E}_C$, where C is the entire method constructed from A and B as in Lemma 2.

By Lemma 1, there exists an entire method D such that $\mathscr{E}_D \supseteqq \mathscr{E}_C$ and $S_D(x) = f(x)$ for $x \in \mathscr{E}_C$.

Define $E = (e_{nk})$ by the set of equations

$$e_{nk} = d_{nk} + a_{nk} \qquad (n, k = 0, 1, 2, \cdots).$$

Then E is an entire method because for every k,

$$\sum_{n=0}^{\infty} |e_{nk}| q^n p^{-k} \leqq \sum_{n=0}^{\infty} |d_{nk}| q^n p^{-k} + \sum_{n=0}^{\infty} |a_{nk}| q^n p^{-k}.$$

Since $\mathscr{E}_D \supseteqq \mathscr{E}_C$ we have $\mathscr{E}_E \supseteqq \mathscr{E}_C$. Moreover, for

$$x \in \mathscr{E},\ S_E(x) = S_D(x) + S_A(x) = S_A(x)$$

since f vanishes on $\mathscr{E}$. However, E is not consistent with A since $y \in \mathscr{E}_E \cap \mathscr{E}_A$, $f(y) \neq 0$, and $S_E(y) = f(y) + S_A(y)$. This contradicts our assumption, and the Theorem is proved.

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