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on Banach space**

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## A theorem on Riesz summability $(R, \omega, 2)$ on Banach space

by

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A series  $\sum_{i=0}^{i=\infty} x_i$  is said to be summable  $(R, \omega, 2)$ , to sum  $s$ , if

$$R_{\omega}^2 = \sum_{i \leq \omega} \left(1 - \frac{i}{\omega}\right)^2 x_i$$

tends to a limit  $s$ , as  $\omega \rightarrow \infty$ .  $R_{\omega}^2$  is called the Riesz mean of second order,  $(R, \omega, 2)$ , of  $\sum x_i$ . Similarly, we say that  $\sum x_i/i^p$  is summable  $x^*$ , if

$$R_{\omega}^2(p) = \sum_{i \leq \omega} \left(1 - \frac{i}{\omega}\right)^2 \frac{x_i}{i^p} \rightarrow x^*, \text{ as } \omega \rightarrow \infty.$$

On the other hand, Cèsaro summability,  $(C, 2)$ , of  $\sum x_i$  or  $\sum x_i/i^p$  is defined as follows:

$$\sigma_n^2 = \sum_{i=0}^n \left(1 - \frac{i}{n+1}\right) \left(1 - \frac{i}{n+2}\right) x_i$$

or

$$\sigma_n^2(p) = \sum_{i=1}^n \left(1 - \frac{i}{n+1}\right) \left(1 - \frac{i}{n+2}\right) \frac{x_i}{i^p}.$$

The relationship between these two summabilities in our case can be reduced to the following lemma:

**LEMMA.**

$$(1) \quad \sigma_n^2 = \frac{2}{(n+1)(n+2)} \left\{ 2\left(n + \frac{1}{4}\right)^2 R_{n+\frac{1}{4}}^2 - \frac{1}{2}\left(n + \frac{1}{2}\right)^2 R_{n+\frac{1}{2}}^2 + 4\left(n + \frac{3}{4}\right)^2 R_{n+\frac{3}{4}}^2 \right\},$$

$$(2) \quad R_{\omega}^2(p) = \frac{1}{2\omega^2} \left\{ (n+1)(n+2)\lambda^2 \sigma_n^2(p) + n(n+1)(1+2\lambda-2\lambda^2) \sigma_{n-1}^2(p) + (n-1)n(1-\lambda)^2 \sigma_{n-2}^2(p) \right\}$$

where  $\omega = n + \lambda$ ,  $0 \leq \lambda < 1$ . Since its truth is trivial, we omit the prove here.

A normed linear space becomes a metric space if the distance  $d(x, y)$  is defined as  $\|x - y\|$ , and it is called a Banach space if it is complete in this metric. Banach space is generally a complex space. However, it is assumed to be a real Banach space throughout this paper. But our conclusions can readily be generalized to complex case.

The purpose of this paper is to prove Theorem A.

**THEOREM A.** *Assume that  $0 \leq q < 1$  and  $0 < p < q$ . If there exists an element  $x \in X$ , where  $X$  is a Banach space, such that*

$$\|R_\omega^2(q) - x\| = O(\omega^{-q})$$

*then we can choose another element  $x^*$  from  $X$  such that*

$$\|R_\omega^2(-p) - x^*\| = O(\omega^{-q+p}).$$

**PROOF.** Our proof depends on the following relations. From (1) and hypothesis, we have

$$(3) \quad \|\sigma_n^2 - x\| = O(n^{-q}).$$

If, based on (1), we can choose  $x^* \in X$  such that

$$(4) \quad \|\sigma_n^2 - x^*\| = O(n^{-q+p}) = O(\omega^{-q+p}),$$

then,

$$\|R_\omega^2(-p) - x^*\| = O(n^{-q+p}) = O(\omega^{-q+p})$$

follows. The crucial point in the proof is how to deduce (4) from (3). If we suppose  $x = 0$ , (3) is

$$\|\sigma_n^2\| = O(n^{-q})$$

Now,

$$\begin{aligned} \sigma_k^2(-p) - \sigma_{k-1}^2(-p) &= \sum_{i=1}^k \left(1 - \frac{i}{k+1}\right) \left(1 - \frac{i}{k+2}\right) \frac{x_k}{i^{-p}} \\ &\quad - \sum_{i=1}^{k-1} \left(1 - \frac{i}{k+1}\right) \left(1 - \frac{i}{k+2}\right) \frac{x_k}{i^{-p}} \\ &= 2 \left[ \frac{\sum_{i=1}^k i^{1+p} x_i}{k(k+2)} - \frac{\sum_{i=1}^k i^{2+p} x_i}{k(k+1)(k+2)} \right]. \end{aligned}$$

Set  $m > n$ . Then,

$$\begin{aligned} \sigma_m^2(-p) - \sigma_n^2(-p) &= \sum_{k=n+1}^m [\sigma_k^2(-p) - \sigma_{k-1}^2(-p)] \\ &= 2 \left[ \sum_{k=n+1}^m \frac{\sum_{i=1}^k i^{1+p} x_i}{k(k+2)} - \sum_{k=n+1}^m \frac{\sum_{i=1}^k i^{2+p} x_i}{k(k+1)(k+2)} \right]. \end{aligned}$$

Set

$$Y_i = \sum_{j=1}^i x_j, \quad Y'_i = \sum_{j=1}^i Y_j, \quad \Delta_1(i) = i^{1+p} - (i+1)^{1+p},$$

$$\Delta_j(i) = \Delta_{j-1}(i) - \Delta_{j-1}(i+1).$$

Then,

$$Y'_i = \frac{1}{2}[(i+1)(i+2)\sigma_i^2 - i(i+1)\sigma_{i-1}^2].$$

By the "3rd Abel Transformation", we have

$$\sum_{i=1}^k i^{1+p} x_i = \frac{1}{2} \sum_{i=1}^k \Delta_3(i) \cdot (i+1)(i+2)\sigma_i^2 + \frac{1}{2} \Delta_2(k+1) \cdot (k+1)(k+2)\sigma_k^2$$

$$+ \frac{1}{2} \Delta_1(k+1) \{ (k+1)(k+2)\sigma_k^2 - k(k+1)\sigma_{k-1}^2 \} + (k+1)^{1+p} Y_k$$

$$= \sum'_{1k} + \sum'_{2k} + \sum'_{3k} + \sum'_{4k}.$$

From (5) and  $\Delta_j(i) = O(i^{p+1-j})$ ,

$$\left\| \sum_{k=n+1}^m \frac{\sum'_{1k}}{k(k+2)} \right\| = \sum_{k=n+1}^m O\left(\frac{1}{k^2}\right) \left[ \sum_{i=1}^k O(i^{p-2}) O(i^2) O\left(\frac{1}{i^q}\right) \right]$$

$$= \sum_{k=n+1}^m O\left(\frac{1}{k^{-p+q+1}}\right) = O(n^{p-q}),$$

$$\left\| \sum_{k=n+1}^m \frac{\sum'_{2k}}{k(k+2)} \right\| = \sum_{k=n+1}^m O\left(\frac{1}{k^2}\right) O(k^{p-1} \cdot k^2 \cdot k^{-q}) = O(n^{p-q}).$$

And

$$\sum_{k=n+1}^m \frac{\sum'_{3k}}{k(k+2)} = \frac{1}{2} \sum_{k=n+1}^m \left[ \frac{(k+1)^{1+p} - (k+2)^{1+p}}{k(k+2)} \right. \\ \left. - \frac{(k+2)^{1+p} - (k+3)^{1+p}}{(k+1)(k+3)} \right] (k+1)(k+2)\sigma_k^2$$

$$- \frac{1}{2} \frac{(n+2)^{1+p} - (n+3)^{1+p}}{(n+1)(n+3)} (n+1)(n+2)\sigma_n^2$$

$$+ \frac{1}{2} \frac{(m+2)^{1+p} - (m+3)^{1+p}}{(m+1)(m+3)} (m+1)(m+2)\sigma_m^2,$$

$$\frac{(k+1)^{1+p} - (k+2)^{1+p}}{k(k+2)} - \frac{(k+2)^{1+p} - (k+3)^{1+p}}{(k+1)(k+3)} = O(k^{p-3}).$$

Hence,

$$\left\| \sum_{k=n+1}^m \frac{\sum'_{3k}}{k(k+2)} \right\| = O\left(\sum_{k=n+1}^m k^{p-3} \cdot k^2 \cdot k^{-q}\right) + O(n^{p-q}) + O(m^{p-q})$$

$$= O(n^{p-q}).$$

Finally,

$$\begin{aligned} \sum_{k=n+1}^m \frac{\sum'_{4k}}{k(k+2)} &= \sum_{k=n+1}^m \frac{(k+1)^{1+p}}{k(k+2)} Y_k \\ &= \sum_{k=n+1}^m \left[ \frac{(k+1)^{1+p}}{k(k+2)} - \frac{(k+2)^{1+p}}{(k+1)(k+3)} \right] Y'_k \\ &\quad + \frac{(m+2)^{1+p}}{(m+1)(m+3)} Y'_m - \frac{(n+2)^{1+k}}{(n+1)(n+3)} Y'_n. \end{aligned}$$

Let  $\sum_{mn}$  denote the first term of the above equality. And we employ the first Abel Transformation for it.

$$\begin{aligned} \sum_{mn} &= \sum_{k=n+1}^m \left[ \frac{(k+1)^{1+p}}{k(k+2)} - \frac{2(k+2)^{1+p}}{(k+1)(k+3)} \right. \\ &\quad \left. + \frac{(k+3)^{1+p}}{(k+2)(k+4)} \right] (k+1)(k+2)\sigma_k^2 \\ &\quad + \left[ \frac{(m+2)^{1+p}}{(m+1)(m+3)} - \frac{(m+3)^{1+p}}{(m+2)(m+4)} \right] (m+1)(m+2)\sigma_m^2 \\ &\quad - \left[ \frac{(n+2)^{1+p}}{(n+1)(n+3)} - \frac{(n+3)^{1+p}}{(n+2)(n+4)} \right] (n+1)(n+2)\sigma_n^2. \end{aligned}$$

Hence,

$$\|\sum_{mn}\| = O\left(\sum_{k=n+1}^m k^{-p-3} \cdot k^2 \cdot k^{-q}\right) + O(n^{p-q}) + O(m^{-p-q}) = O(n^{p-q}).$$

From the previous results, we have

$$\sum_{k=n+1}^m \frac{\sum_{i=1}^k i^{1+p} x_i}{k(k+2)} = \sum'_{mn} + \frac{(m+2)^{1+p}}{(m+1)(m+3)} Y'_m - \frac{(n+2)^{1+p}}{(n+1)(n+3)} Y'_n$$

where

$$\|\sum'_{mn}\| = O(n^{p-q}).$$

Similarly, if we set

$$\Delta'_1(i) = i^{2+p} - (i+1)^{2+p}, \quad \Delta'_j(i) = \Delta'_{j-1}(i) - \Delta'_{j-1}(i+1),$$

then

$$\begin{aligned} \sum_{i=1}^k (i)^{2+p} x_i &= \frac{1}{2} \sum_{i=1}^k \Delta'_3(i)(i+1)(i+2)\sigma_i^2 + \frac{1}{2} \Delta'_2(k+1) \cdot (k+1)(k+2)\sigma_k^2 \\ &\quad + \frac{1}{2} \Delta'_1(k+1)\{(k+1)(k+2)\sigma_k^2 - k(k+1)\sigma_{k-1}^2\} + (k+1)^{(2+p)} Y_k \\ &= \sum_{1k}^2 + \sum_{2k}^2 + \sum_{3k}^2 + \sum_{4k}^2. \end{aligned}$$

It is ready to prove

$$\left\| \sum_{k=n+1}^m \frac{\sum_{jk}^2}{k(k+1)(k+2)} \right\| = O(n^{p-a}) \quad (j = 1, 2, 3)$$

and

$$\begin{aligned} \sum_{k=n+1}^m \frac{\sum_{jk}^2}{k(k+1)(k+2)} &= \sum_{mn}^* + \frac{(m+2)^{2+p}}{(m+1)(m+2)m+3} Y'_m \\ &\quad - \frac{(n+2)^{2+p}}{(n+1)(n+2)(n+3)} Y'_n \end{aligned}$$

where

$$\|\sum_{mn}^*\| = O(n^{p-a}).$$

Hence

$$\sum_{k=n+1}^m \frac{\sum_{i=1}^k i^{2+p} x_i}{k(k+1)(k+2)} = \sum_{mn}^2 + \frac{(m+2)^{1+p}}{(m+1)(m+3)} Y'_m - \frac{(n+1)^{1+p}}{(n+1)(n+3)} Y'_n,$$

where

$$\|\sum_{mn}^2\| = O(n^{p-a}).$$

Finally,

$$\|\sigma_m^2(-p) - \sigma_n^2(-p)\| = 2\|\sum_{mn}^1 - \sum_{mn}^2\| = O(n^{p-a}).$$

Since  $X$  is a complete space, there exists  $x^* \in X$  such that

$$\|\sigma_m^2(-p) - x^*\| \rightarrow 0,$$

as  $m \rightarrow \infty$ . We have completed our proof.

(Oblatum 18-4-64)