

# COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 15 (1962-1964), p. 210-226

[http://www.numdam.org/item?id=CM\\_1962-1964\\_\\_15\\_\\_210\\_0](http://www.numdam.org/item?id=CM_1962-1964__15__210_0)

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# Integral functions of two complex variables

by

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1. Let <sup>3)</sup>

$$f(z_1, z_2) = \sum_{m_1, m_2=0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2},$$

be a function of the two complex variables  $z_1$  and  $z_2$ , regular for  $|z_t| \leq r_t$ ,  $t = 1, 2$ . If  $r_1$  and  $r_2$  are arbitrarily large, then  $f(z_1, z_2)$  is an integral function of the two complex variables. We know that

$$M(r_1, r_2) = \max_{|z_t| \leq r_t} |f(z_1, z_2)|, \quad t = 1, 2,$$

is the maximum modulus of  $f(z_1, z_2)$  for  $|z_t| \leq r_t$ .

In this paper we have defined maximum term and the ranks of the maximum term, and have extended the method of systematic determination of these as in the case of one variable by Newton's polygon ((1), p. 28). Also we have obtained relations between these, and inequalities involving these and the maximum modulus. Further, we have defined order and have obtained necessary and sufficient conditions for the function to be of finite order, and also the same for functions of finite order and type  $\tau$ .

2. Let <sup>4)</sup>

$$f(z_1, z_2) = \sum_{m_1, m_2=0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}$$

be a function of the two complex variables  $z_1$  and  $z_2$ , regular in  $|z_1| \leq r_1$  and  $|z_2| \leq r_2$ . Writing  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$ ,

$$f(z_1, z_2) = U(r_1, r_2, \theta_1, \theta_2) + iV(r_1, r_2, \theta_1, \theta_2)$$

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<sup>1)</sup> I regret to announce the sudden and untimely death of Devendra Sharma on 18th June, 1957.

<sup>2)</sup> We are thankful to the referee for the valuable criticism.

<sup>3)</sup> We have considered only two variables for simplicity. The results can easily be extended to several variables.

<sup>4)</sup> If  $(z_1^0, z_2^0)$  is any given point, then by neighbourhood of this point we would mean a bicylinder  $|z_1 - z_1^0| < r_1, |z_2 - z_2^0| < r_2, r_1 > 0, r_2 > 0$ .

and

$$(2.1) \quad \begin{aligned} a_{m_1, m_2} &= \alpha_{m_1, m_2} + i\beta_{m_1, m_2}, \\ U(r_1, r_2, \theta_1, \theta_2) &= \sum_{m_1, m_2=0}^{\infty} r_1^{m_1} r_2^{m_2} \{ \alpha_{m_1, m_2} \cos(m_1 \theta_1 + m_2 \theta_2) \\ &\quad - \beta_{m_1, m_2} \sin(m_1 \theta_1 + m_2 \theta_2) \}. \end{aligned}$$

This series is convergent for all values of  $\theta_1$  and  $\theta_2$  because  $\sum_{m_1, m_2=0}^{\infty} |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2}$  is uniformly convergent, by hypothesis. We, therefore, multiply both sides by  $\cos(m_1 \theta_1 + m_2 \theta_2)$  or  $\sin(m_1 \theta_1 + m_2 \theta_2)$  and integrate term by term between the limits zero and  $2\pi$ . We thus have

$$\frac{(2\pi)^2}{2} r_1^{m_1} r_2^{m_2} \alpha_{m_1, m_2} = \int_0^{2\pi} \int_0^{2\pi} U \cos(m_1 \theta_1 + m_2 \theta_2) d\theta_1 d\theta_2$$

and

$$\frac{(2\pi)^2}{2} r_1^{m_1} r_2^{m_2} \beta_{m_1, m_2} = - \int_0^{2\pi} \int_0^{2\pi} U \sin(m_1 \theta_1 + m_2 \theta_2) d\theta_1 d\theta_2.$$

Multiplying the second by  $i$  and adding, we get

$$\frac{(2\pi)^2}{2} a_{m_1, m_2} r_1^{m_1} r_2^{m_2} = \int_0^{2\pi} \int_0^{2\pi} U e^{-i(m_1 \theta_1 + m_2 \theta_2)} d\theta_1 d\theta_2.$$

Hence

$$(2.2) \quad \frac{(2\pi)^2}{2} |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2} \leq \int_0^{2\pi} \int_0^{2\pi} |U| d\theta_1 d\theta_2.$$

Again, if we integrate (2.1) with respect to  $\theta_1$  and  $\theta_2$  in the range zero to  $2\pi$ , we get

$$(2.3) \quad (2\pi)^2 \alpha_{0,0} = \int_0^{2\pi} \int_0^{2\pi} U d\theta_1 d\theta_2.$$

From (2.2) and (2.3) follows

$$\frac{(2\pi)^2}{2} \{ |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2} + 2\alpha_{0,0} \} \leq \int_0^{2\pi} \int_0^{2\pi} \{ |U| + U \} d\theta_1 d\theta_2.$$

Now, if  $U$  is positive, then the integrand is equal to  $2U$  and if  $U$  is negative or zero, then the right hand side is zero. Hence we have

**THEOREM 1.** *If the function  $f(z_1, z_2)$  is regular for  $|z_1| \leq r_1$  and  $|z_2| \leq r_2$  and if  $A(r_1, r_2)$  is its maximum real part for  $|z_1| \leq r_1$  and  $|z_2| \leq r_2$ , then for all positive values of  $m_1$  and  $m_2$ , the number  $|a_{m_1, m_2}| r_1^{m_1} r_2^{m_2}$  is less than or equal to the greatest of the two numbers  $-2\alpha_{0,0}$  and  $4A(r_1, r_2) - 2\alpha_{0,0}$ .*

COROLLARY. If  $f(z_1, z_2)$  is an integral function, then

$$(2.4) \quad |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2} \leq 4A(r_1, r_2) - 2\alpha_{0,0},$$

for all positive values of  $m_1$  and  $m_2$ , and for all  $r_1 > r_1^0$ ,  $r_2 > r_2^0$ .

Similar results can be obtained for minimum of  $U(r_1, r_2, \theta_1, \theta_2)$ . Also for the maximum and minimum of  $V(r_1, r_2, \theta_1, \theta_2)$ .

THEOREM II. If  $f(z_1, z_2)$  is an integral function, and  $q_1$  and  $q_2$  are two fixed finite positive numbers such that  $|f(z_1, z_2)|$  is algebraically less than  $Kr_1^{q_1} r_2^{q_2}$  for  $|z_1| = r_1$  and  $|z_2| = r_2$ , where  $r_1$  and  $r_2$  are arbitrary large numbers and  $K$  is a constant, then  $f(z_1, z_2)$  is a polynomial of degree not greater than  $q_1 + q_2$ .

PROOF: Since  $|f(z_1, z_2)| \leq Kr_1^{q_1} r_2^{q_2}$  for  $|z_1| = r_1$  and  $|z_2| = r_2$ , therefore, from (2.4) follows

$$|a_{m_1, m_2}| \leq 4Kr_1^{q_1 - m_1} r_2^{q_2 - m_2} - 2\alpha_{0,0} r_1^{-m_1} r_2^{-m_2}.$$

If  $m_1 > q_1$  or  $m_2 > q_2$ , then the right hand side vanishes as  $r_1$  or  $r_2$  respectively tends to infinity, and so  $a_{m_1, m_2}$  is zero for  $m_1 > q_1$  or  $m_2 > q_2$ .

THEOREM III. If  $f(z_1, z_2)$  is an integral function of the two complex variables  $z_1$  and  $z_2$ , then

$$M(r_1, r_2) < \frac{R_1 R_2}{(R_1 - r_1)(R_2 - r_2)} [4A(R_1, R_2) + 3|a_{0,0}|],$$

for all  $R_1 > r_1 > r_1^0$  and  $R_2 > r_2 > r_2^0$ , where

$$M(r_1, r_2) = \max |f(z_1, z_2)|,$$

for  $|z_1| \leq r_1$  and  $|z_2| \leq r_2$ .

PROOF: We can write (2.4) as

$$|a_{m_1, m_2}| r_1^{m_1} r_2^{m_2} \leq [4A(R_1, R_2) - 2\alpha_{0,0}] \left(\frac{r_1}{R_1}\right)^{m_1} \left(\frac{r_2}{R_2}\right)^{m_2}$$

and taking  $r_1 < R_1$  and  $r_2 < R_2$ , it follows that

$$\begin{aligned} M(r_1, r_2) &\leq \sum_{m_1, m_2=0}^{\infty} |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2} \\ &\leq |a_{0,0}| + [4A(R_1, R_2) - 2\alpha_{0,0}] \sum'_{m_1, m_2} \left(\frac{r_1}{R_1}\right)^{m_1} \left(\frac{r_2}{R_2}\right)^{m_2} \\ &= |a_{0,0}| + [4A(R_1, R_2) - 2\alpha_{0,0}] \frac{r_1}{R_1 - r_1} \cdot \frac{r_2}{R_2 - r_2} \\ &< \frac{R_1 R_2}{(R_1 - r_1)(R_2 - r_2)} [4A(R_1, R_2) + 3|a_{0,0}|]. \end{aligned}$$

COROLLARY: If  $f(z_1, z_2)$  is zero at  $z_1 = 0, z_2 = 0$ , then

$$M(r_1, r_2) \leq \frac{4r_1 r_2}{(R_1 - r_1)(R_2 - r_2)} A(R_1, R_2)$$

for  $R_1 > r_1, R_2 > r_2$ .

3. Consider the moduli of the terms of the double series in the expansion of the integral function<sup>1)</sup>

$$(3.1) \quad f(z_1, z_2) = \sum_{m_1, m_2=0}^{\infty} c_{m_1, m_2} z_1^{m_1} z_2^{m_2},$$

$$(3.2) \quad \begin{matrix} C_{0,0} & C_{0,1} r_2 & C_{0,2} r_2^2 & \dots & C_{0,m_2} r_2^{m_2} & \dots \\ C_{1,0} r_1 & C_{1,1} r_1 r_2 & C_{1,2} r_1 r_2^2 & \dots & C_{1,m_2} r_1 r_2^{m_2} & \dots \\ C_{2,0} r_1^2 & C_{2,1} r_1^2 r_2 & C_{2,2} r_1^2 r_2^2 & \dots & C_{2,m_2} r_1^2 r_2^{m_2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ C_{m_1,0} r_1^{m_1} & C_{m_1,1} r_1^{m_1} r_2 & C_{m_1,2} r_1^{m_1} r_2^2 & \dots & C_{m_1,m_2} r_1^{m_1} r_2^{m_2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{matrix}$$

where

$$C_{m_1, m_2} = |c_{m_1, m_2}|.$$

If we consider any column or row, then the sequence thus obtained tends to zero for all values of  $r_1$  or  $r_2$ . Hence for every value of  $r_1$ , keeping  $m_2$  and  $r_2$  fixed, there is, therefore, one term of the sequence thus obtained which is greater than or equal to all the rest. This term will be the maximum term in that column and will be denoted by  $\mu(m_2; r_1, r_2)$ . If there are more than one such term, then the term of the highest rank will be regarded as the maximum term of this column and the rank will be denoted by  $\nu_1(m_2; r_1)$ . We next give different values to  $m_2$ , i.e. consider different columns and suppose the greatest term occurs in the  $\nu_2$ th column, then the term  $\mu(\nu_2; r_1, r_2)$  will be the maximum term with respect to columns and the rank of this term will be denoted by  $\nu_1(r_1)$ . If there are more than one column containing such term, then the term of the highest rank with regard to column will be regarded as maximum term, i.e. the column of highest rank which contains the maximum term is  $\nu_2$ th. Similarly, if we consider  $m_1$ th row, keeping  $r_1, r_2$  and  $m_1$  fixed, then we shall have

<sup>1)</sup> Suffix 1 will indicate row, for example,  $m_1, p_1, \nu_1$  etc. and suffix 2 will indicate column,  $m_2, p_2, \nu_2$  etc.

$\mu(m_1; r_1, r_2)$  as the maximum term in that row and the rank as  $\nu_2(m_1; r_2)$ . Further, for different values of  $m_1$ , suppose the greatest term occurs in the  $\nu_1$ th row, then the maximum term will be denoted by  $\mu(\nu_1; r_1, r_2)$  and the rank of this term will be denoted by  $\nu_2(r_2)$ . If there are more than one such term, then the same convention as for columns is adopted. Hence we shall denote the *maximum term* for given values of  $r_1$  and  $r_2$  by  $\mu(r_1, r_2)$  and the rank of this term will be denoted by  $\nu(r_1, r_2)$ .

For a systematic study of finding the maximum term we shall extend the method of Valiron ((1), p. 28) for one variable.

Let  $\log C_{m_1, m_2} = -g_{m_1, m_2}$ , then

$$(3.3) \quad \lim_{m_1 \rightarrow \infty} \frac{g_{m_1, m_2}}{m_1} = +\infty$$

and

$$\lim_{m_2 \rightarrow \infty} \frac{g_{m_1, m_2}}{m_2} = +\infty.$$

Since  $\sqrt[m_1]{C_{m_1, m_2}}$  and  $\sqrt[m_2]{C_{m_1, m_2}}$  tend to zero as  $m_1$  and  $m_2$  respectively tend to infinity.

Taking  $OX, OY, OZ$  as the axes of coordinates, if we plot the points  $A_{m_1, m_2}$  of coordinates  $(m_1, m_2, g_{m_1, m_2})$ , then, from (3.3), it follows that we can construct a surface with plane faces and every section of this surface by planes parallel to the  $XZ$ -plane and  $YZ$ -plane form a Newton's polygon, having certain of the points  $A_{m_1, m_2}$ , lying in this vertical plane. Out of these some of them coincide with the vertices of the polygon, whilst the remainder lie either on or above it. Let us denote this surface by  $S(f)$  and call it *Newton's polyhedron*.

If  $\nu_1$  and  $\nu_2$  be the rank of a maximum term as defined above and  $m_1 \neq \nu_1$  or  $m_2 \neq \nu_2$ , then, it easily follows:

$$g_{m_1, m_2} - m_1 \log r_1 - m_2 \log r_2 \geq g_{\nu_1, \nu_2} - \nu_1 \log r_1 - \nu_2 \log r_2.$$

Now, let us consider the geometrical interpretation of this inequality. Let  $D_{r_1, r_2}$  denote a tangent plane, having direction cosines proportional to  $-\log r_1, -\log r_2$  and 1, passing through the point  $A_{\nu_1, \nu_2}$  or  $A_{m_1, \nu_2}$ . If we now draw the plane parallel to the  $XZ$ -plane through  $A_{\nu_1, m_2}$ , then those points  $A_{m_1, m_2}$  which lie in the plane do not lie below the line  $L_{r_1}$ , the line of intersection with the plane  $D_{r_1, r_2}$ , of slope  $\log r_1$ . Similarly, if we consider the plane parallel to the  $YZ$ -plane through  $A_{m_1, \nu_2}$ , those points  $A_{m_1, m_2}$  which lie in that plane do not lie below the line  $L_{r_2}$ , the line of

intersection with the plane  $D_{r_1 r_2}$ , of slope  $\log r_2$ . The point  $A_{\nu_1, m_2}$  or  $A_{m_1, \nu_2}$  is therefore a point of the polyhedron  $S(f)$  and the plane  $D_{r_1 r_2}$  is a tangent to this polyhedron.

Now, we take a vertical plane parallel to  $XZ$ -plane through  $A_{\nu_1, m_2}$ , then  $m_2$  is fixed, i.e. in (3.2) we are considering a column. The intersection of the plane with the surface will include a Newton's polygon and the tangent line  $L_{r_1}$  of slope  $\log r_1$  will pass through  $A_{\nu_1, m_2}$ . The point  $A_{\nu_1, m_2}$  is uniquely determined in the plane, when  $\log r_1$  is not equal to the slope of one of the sides of  $\pi_1(f)$ , and for such values of  $r_1$ , there is only one term in the sequence (column of (3.2) under consideration) equal to  $\mu(m_2; r_1, r_2)$ . When  $\log r_1$  is equal to the slope of a side of  $\pi_1(f)$ , there are several such terms and their number is equal to the number of the points  $A_{\nu_1, m_2}$  which lie on this side of this polygon. When more than one term are equal to  $\mu(m_2; r_1, r_2)$ , we shall take the term of highest rank amongst them as the maximum term, with respect to  $m_2$  and  $r_1$ .

Thus having obtained the maximum term or terms with respect to  $m_2$  and  $r_1$ , we must draw a plane parallel to  $YZ$ -plane through  $A_{m_1, \nu_2}$ , then  $m_1$  is fixed, i.e. in (3.2) we are considering  $m_1$ th row. Again the intersection will include a Newton's polygon and the tangent line  $L_{r_2}$  of slope  $\log r_2$  will pass through  $A_{m_1, \nu_2}$ . Here again the point  $A_{m_1, \nu_2}$  is uniquely determined in the plane, when  $\log r_2$  is not equal to the slope of one of the sides of  $\pi_2(f)$ , and for such values of  $r_2$  there is only one term in the sequence of terms in the  $m_1$ th row equal to  $\mu(m_1; r_1, r_2)$ . When  $\log r_2$  is equal to the slope of a side of  $\pi_2(f)$ , there are several terms and their number is equal to the number of points  $A_{m_1, \nu_2}$  which lie on the side of the polygon. As in the case of  $m_2$  and  $r_1$ , we shall take the term of highest rank amongst them as the maximum term, with respect to  $m_1$  and  $r_2$ . Finally the greater of the two terms obtained above shall be denoted by  $\mu(r_1, r_2)$ .

We have thus, with this convention, obtained one term as maximum with respect to  $m_1$ ,  $m_2$ ,  $r_1$  and  $r_2$ . Thus  $\nu_1(m_2; r_1, r_2)$  or  $\nu_1(r_1, r_2)$ ,  $r_2$  fixed,  $\nu_2(m_1; r_1, r_2)$  or  $\nu_2(r_1, r_2)$ ,  $r_1$  fixed, and  $\nu(r_1, r_2)$  will be used to denote ranks of the maximum term of the double series.  $\nu_1(m_2; r_1, r_2)$ ,  $\nu_2(m_1; r_1, r_2)$  and  $\nu(r_1, r_2)$  are unbounded non-decreasing functions of  $r_1$  and  $r_2$ . Further,  $\nu_1(m_2; r_1, r_2)$  and  $\nu_2(m_1; r_1, r_2)$  have left hand discontinuity wherever  $r_1$  and  $r_2$  respectively pass through a value such that  $\log r_1$  and  $\log r_2$  respectively equal the slope of one of the sides of the polygons  $\pi_1(f)$  and  $\pi_2(f)$ . Hence  $\nu(r_1, r_2)$  has also discontinuity for such values of  $r_1$  and  $r_2$ .

4. Two functions  $f(z_1, z_2)$  and  $g(z_1, z_2)$  having the same polyhedron will have the same maximum term and the rank.

Let us consider the function

$$(4.1) \quad W(r_1, r_2) = \sum_{m_1, m_2=0}^{\infty} e^{-G_{m_1, m_2}} r_1^{m_1} r_2^{m_2},$$

where  $G_{m_1, m_2}$  is the  $Z$ -coordinate of the point, whose  $X$  and  $Y$  coordinates are  $m_1$  and  $m_2$ .

The function  $W(r_1, r_2)$  is a dominant function for  $f(z_1, z_2)$  and has the same maximum term. Also it is the simplest function corresponding to the polyhedron  $S(f)$ . The ratio

$$R_{m_1}^{m_2} = e^{G_{m_1, m_2} - G_{m_1-1, m_2}}, \quad S_{m_2}^{m_1} = e^{G_{m_1, m_2} - G_{m_1, m_2-1}}$$

and

$$R_{m_1, m_2} = e^{G_{m_1, m_2} - G_{m_1-1, m_2-1}}.$$

of the coefficients in  $W(r_1, r_2)$  corresponds to the ratio of  $a_{m_1, m_2}$  and  $a_{m_1-1, m_2}$ ,  $a_{m_1, m_2}$  and  $a_{m_1, m_2-1}$ , and  $a_{m_1, m_2}$  and  $a_{m_1-1, m_2-1}$ . We shall call these as rectified ratios. The logarithm of  $R_{m_1}^{m_2}$  is equal to the slope of the Newton's polygon obtained by plotting the points  $(m_1, G_{m_1, m_2})$  in a plane parallel to  $XZ$ -plane at a distance  $m_2$ , and is therefore a non-decreasing function of  $m_1$  tending to infinity. Similarly the logarithm of  $S_{m_2}^{m_1}$  and  $R_{m_1, m_2}$  are non-decreasing functions of  $m_1$ , and  $m_1$  and  $m_2$  tending to infinity.

Suppose for simplicity that  $G_{0,0} = 0$ . Then we have

$$(4.2) \quad \mu(\nu_1; r_1, r_2) = \frac{r_1^{\nu_1} r_2^{\nu_2}}{(S_{\nu_2}^{\nu_1} S_{\nu_2-1}^{\nu_1} \dots S_1^{\nu_1})(R_{\nu_1}^0 R_{\nu_1-1}^0 \dots R_1^0)}$$

and since

$$\int_{R_i^0}^{R_{i+1}^0} \nu_1(0; x_1) \frac{dx_1}{x_1} = i(\log R_{i+1}^0 - \log R_i^0)$$

and

$$\int_{S_j^{\nu_1}}^{S_{j+1}^{\nu_1}} \nu_2(\nu_1; x_2) \frac{dx_2}{x_2} = j(\log S_{j+1}^{\nu_1} - \log S_j^{\nu_1}),$$

therefore,

$$(4.3) \quad \log \mu(\nu_1; r_1, r_2) = \int_0^{\nu_1} \nu_1(0; x_1) \frac{dx_1}{x_1} + \int_0^{\nu_2} \nu_2(\nu_1; x_2) \frac{dx_2}{x_2}.$$

We may also put (4.2) as

$$\mu(\nu_2; r_1, r_2) = \frac{r_1^{\nu_1} r_2^{\nu_2}}{(R_{\nu_1}^{\nu_2} R_{\nu_1-1}^{\nu_2} \dots R_1^{\nu_2})(S_{\nu_2}^0 S_{\nu_2-1}^0 \dots S_1^0)}$$

and hence

$$(4.4) \quad \log \mu(\nu_2; r_1, r_2) = \int_0^{r_2} \nu_2(0, x_2) \frac{dx_2}{x_2} + \int_0^{r_1} \nu_1(\nu_2; x_1) \frac{dx_1}{x_1}.$$

Since  $\mu(r_1, r_2)$  is greater of  $\mu(\nu_1; r_1, r_2)$  and  $\mu(\nu_2; r_1, r_2)$ , therefore,  $\log \mu(r_1, r_2)$  will be given by either (4.3) or (4.4) or both.

We are now in a position to find a relation between  $\mu(r_1, r_2)$  and  $\nu_1(r_1, r_2)$ ,  $\nu_2(r_1, r_2)$  and  $\nu(r_1, r_2)$ . In the first place

$$\mu(r_1, r_2) < M(r_1, r_2).$$

Also it is obvious that  $M(r_1, r_2)$  does not exceed the value of the function  $W(r_1, r_2)$ . Suppose that  $p_1$  and  $p_2$  are integers greater than  $\nu_1 = \nu_1(r_1)$  and  $\nu_2 = \nu_2(r_2)$  and such that the rectified ratio  $R_{p_1}^{\nu_1-1} > r_1$  and  $S_{p_2}^{\nu_2-1} > r_2$ . Then, for  $q_1 \geq p_1$  and  $q_2 \geq p_2$ ,

$$\begin{aligned} e^{-G_{q_1, q_2}} r_1^{q_1} r_2^{q_2} &= e^{-G_{p_1-1, p_2-1}} r_1^{p_1-1} r_2^{p_2-1} \\ &\times \frac{r_1^{q_1-p_1+1} r_2^{q_2-p_2+1}}{R_{q_1}^{q_2} R_{q_1-1}^{q_2} \dots R_{p_1}^{q_2} S_{q_2}^{p_1-1} S_{q_2-1}^{p_1-1} \dots S_{p_2}^{p_1-1}} \\ &< \mu(r_1, r_2) \left(\frac{r_1}{R_{p_1}^{q_2}}\right)^{q_1-p_1+1} \left(\frac{r_2}{S_{p_2}^{p_1-1}}\right)^{q_2-p_2+1} \end{aligned}$$

Hence

$$\begin{aligned} W(r_1, r_2) &= \sum_{m_1, m_2=0}^{p_1-1, p_2-1} e^{-G_{m_1, m_2}} r_1^{m_1} r_2^{m_2} + \sum_{m_1=p_1, m_2=p_2}^{\infty} e^{-G_{m_1, m_2}} r_1^{m_1} r_2^{m_2} \\ &+ \sum_{m_1=0}^{p_1-1} \sum_{m_2=p_2}^{\infty} e^{-G_{m_1, m_2}} r_1^{m_1} r_2^{m_2} + \sum_{m_2=0}^{p_2-1} \sum_{m_1=p_1}^{\infty} e^{-G_{m_1, m_2}} r_1^{m_1} r_2^{m_2} \\ &< \mu(r_1, r_2) \left[ p_1 p_2 + \sum_{q_2=p_2}^{\infty} \sum_{q_1=p_1}^{\infty} \left(\frac{r_1}{R_{p_1}^{q_2}}\right)^{q_1-p_1+1} \left(\frac{r_2}{S_{p_2}^{p_1-1}}\right)^{q_2-p_2+1} \right. \\ &\quad \left. + p_1 \sum_{q_2=p_2}^{\infty} \left(\frac{r_2}{S_{p_2}^{p_1-1}}\right)^{q_2-p_2+1} + p_2 \sum_{q_1=p_1}^{\infty} \left(\frac{r_1}{R_{p_1}^{p_2-1}}\right)^{q_1-p_1+1} \right]. \\ &\leq \mu(r_1, r_2) \left[ p_1 p_2 + \sum_{q_2=p_2}^{\infty} \left(\frac{r_2}{S_{p_2}^{p_1-1}}\right)^{q_2-p_2+1} \frac{r_1}{R_{p_1}^{q_2}-r_1} \right. \\ &\quad \left. + p_1 \sum_{q_2=p_2}^{\infty} \left(\frac{r_2}{S_{p_2}^{p_1-1}}\right)^{q_2-p_2+1} + p_2 \sum_{q_1=p_1}^{\infty} \left(\frac{r_1}{R_{p_1}^{p_2-1}}\right)^{q_1-p_1+1} \right] \\ &\leq \mu(r_1, r_2) \left[ p_1 p_2 + \frac{r_1}{R_{p_1}^{p_2-1}-r_1} \sum_{q_2=p_2}^{\infty} \left(\frac{r_2}{S_{p_2}^{p_1-1}}\right)^{q_2-p_2+1} \right. \\ (4.5) \quad &\quad \left. + \frac{p_1 r_2}{S_{p_2}^{p_1-1}-r_2} + \frac{p_2 r_1}{R_{p_1}^{p_2-1}-r_1} \right] \\ &\leq \mu(r_1, r_2) \left[ p_1 p_2 + \frac{r_1}{R_{p_1}^{p_2-1}-r_1} \cdot \frac{r_2}{S_{p_2}^{p_1-1}-r_2} \right. \\ &\quad \left. + \frac{p_1 r_2}{S_{p_2}^{p_1-1}-r_2} + \frac{p_2 r_1}{R_{p_1}^{p_2-1}-r_1} \right]. \end{aligned}$$

In order that the terms in the bracket may be substantially equivalent, we choose

$$p_1 = \nu_1\left(r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)}\right) + 1$$

and

$$p_2 = \nu_2\left(r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)}\right) + 1,$$

which implies that

$$R_{p_1}^{p_1-1} > r_1 + \frac{r_1}{\nu_1(r_1, r_2)}$$

and

$$S_{p_2}^{p_2-1} > r_2 + \frac{r_2}{\nu_2(r_1, r_2)}.$$

Hence we have the following result:

$$\begin{aligned} \mu(r_1, r_2) &< M(r_1, r_2) \\ &< \mu(r_1, r_2) \left[ 3\nu_1\left(r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)}\right) \right. \\ (4.6) \quad &\times \left. \nu_2\left(r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)}\right) + 3 \right] \\ &\leq \mu(r_1, r_2) \left[ 3\nu\left(r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)}\right) + 3 \right]. \end{aligned}$$

5. We shall now define *order*. Here we shall restrict to integral functions of *finite order*.

Let

$$\overline{\lim}_{r_1 \rightarrow \infty} \overline{\lim}_{r_2 \rightarrow \infty} \frac{\log \nu_1(r_1, r_2)}{\log r_1} = \rho_1, \quad \overline{\lim}_{r_1 \rightarrow \infty} \overline{\lim}_{r_2 \rightarrow \infty} \frac{\log \nu_2(r_1, r_2)}{\log r_2} = \rho_2,$$

and let

$$\rho = \max. (\rho_1, \rho_2), \quad 0 \leq \rho < \infty,$$

then

$$(5.1) \quad \overline{\lim}_{r_1, r_2 \rightarrow \infty} \frac{\log \nu(r_1, r_2)}{\log (r_1 r_2)} = \rho,$$

and  $f(z_1, z_2)$  is said to be an integral function of *finite order*  $\rho$ . Further,  $\rho_1(r_2)$  and  $\rho_2(r_1)$  be defined as *proximate orders*.

Now, for those functions which satisfy (5.1), the relation (4.6) appears in an especially simple form, and it may be written

$$\log M(r_1, r_2) = \left[ 1 + \theta \frac{\log \left\{ 3\nu \left( r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)} \right) + 3 \right\}}{\log \mu(r_1, r_2)} \right] \log \mu(r_1, r_2),$$

$0 < \theta < 1.$

Hence

$$\overline{\text{Lt}}_{r_1, r_2 \rightarrow \infty} \frac{\log M(r_1, r_2)}{\log \mu(r_1, r_2)} = 1.$$

We may also define order as

$$(5.2) \quad \overline{\text{Lt}}_{r_1, r_2 \rightarrow \infty} \frac{\log \log M(r_1, r_2)}{\log(r_1 r_2)} = \rho.$$

We will now prove the equivalence of the two definitions.

Let us start with the definition (5.1). From (4.3) and (4.4) and the definitions of  $\rho_1, \rho_2$  and  $\rho$ , we have

$$\log \mu(r_1, r_2) < \log \mu(r_{\varepsilon_1}, r_{\varepsilon_2}) + k_1 r_1^{\rho+\varepsilon} \int_{r_{\varepsilon_1}}^{r_1} x_2^{\rho-1+\varepsilon} dx_2,$$

or

$$\log \mu(r_1, r_2) < \log \mu(r_{\varepsilon_1}, r_{\varepsilon_2}) + k_2 r_2^{\rho+\varepsilon} \int_{r_{\varepsilon_2}}^{r_2} x_1^{\rho-1+\varepsilon} dx_1,$$

where  $\varepsilon = \max.(\varepsilon_1, \varepsilon_2)$  and  $k_1 > 1, k_2 > 1$ , and so

$$\log \mu(r_1, r_2) < \frac{k}{\rho + \varepsilon} (r_1 r_2)^{\rho+\varepsilon} + K, \quad k = \max(k_1, k_2),$$

whence, in virtue of (4.6)

$$(5.3) \quad \overline{\text{Lt}}_{r_1, r_2 \rightarrow \infty} \frac{\log \log M(r_1, r_2)}{\log(r_1 r_2)} = \overline{\text{Lt}}_{r_1, r_2 \rightarrow \infty} \frac{\log \log \mu(r_1, r_2)}{\log(r_1 r_2)} \leq \rho.$$

Next, if we suppose that

$$\log M(r_1, r_2) < (r_1 r_2)^{\rho+\varepsilon}, \quad \text{for } r_1 > r_\varepsilon, r_2 > r_\varepsilon,$$

then

$$\begin{aligned} \{\nu_2(0; r_2) + \nu_1(\nu_2; r_1, r_2)\} \log 2 &< \int_{r_2}^{2r_2} \frac{\nu_2(0; x_2)}{x_2} dx_2 + \int_{r_1}^{2r_1} \frac{\nu_1(\nu_2; x_1)}{x_1} dx_1, \\ &\leq \log \mu(2r_1, 2r_2) \\ &< (r_1 r_2)^{\rho+\varepsilon}, \end{aligned}$$

or

$$\{\nu_1(0; r_1) + \nu_2(\nu_1; r_1, r_2)\} \log 2 < \log \mu(2r_1, 2r_2) < (r_1 r_2)^{\rho+\varepsilon}.$$

Hence

$$\begin{aligned} \overline{Lt}_{r_1, r_2 \rightarrow \infty} \frac{\log \nu(r_1, r_2)}{\log(r_1 r_2)} &\leq \overline{Lt}_{r_1, r_2 \rightarrow \infty} \frac{\log \log \mu(r_1, r_2)}{\log(r_1 r_2)} \\ (5.4) \qquad \qquad \qquad &= \overline{Lt}_{r_1, r_2 \rightarrow \infty} \frac{\log \log M(r_1, r_2)}{\log(r_1 r_2)}. \end{aligned}$$

Thus, from (5.3) and (5.4), the equivalence of the two definitions follows.

6. We shall now come to the necessary and sufficient condition for an integral function  $f(z_1, z_2)$  to be of finite order  $\rho$ . The result is as follows:

**THEOREM IV.** *The integral function*

$$f(z_1, z_2) = \sum_{m_1, m_2=0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}$$

*is of finite order, if and only if*

$$(6.1) \qquad \mu = \overline{Lt}_{m_1, m_2 \rightarrow \infty} \frac{\log(m_1^{m_1} m_2^{m_2})}{\log(1/|a_{m_1, m_2}|)}$$

*is finite; and then the order  $\rho$  of  $f(z_1, z_2)$  is equal to  $\mu$ .*

**PROOF:** We first prove that  $\rho \geq \mu$ . We may note that in case  $\mu = \infty$  the above statement is to be interpreted as meaning that the order is infinite, or else  $f(z_1, z_2)$  is not an integral function.

We know that

$$\begin{aligned} |a_{m_1, m_2}| &= \frac{1}{m_1! m_2!} \left| \frac{\partial^{m_1+m_2} f}{\partial z_1^{m_1} \partial z_2^{m_2}} \right| \\ (6.2) \qquad \qquad &= \left| \left( \frac{1}{2\pi i} \right)^2 \int_{|z_1|=r_1} \int_{|z_2|=r_2} \frac{f(z_1, z_2)}{z_1^{m_1+1} z_2^{m_2+1}} dz_1 dz_2 \right| \\ &\leq \frac{M(r_1, r_2)}{r_1^{m_1} r_2^{m_2}}. \end{aligned}$$

(i) If  $\mu = 0$ ,  $\rho \geq \mu$ , since  $\rho$  is not negative. Let us suppose that  $0 < \varepsilon < \rho < \infty$ . Then, from (6.1), we have

$$(\mu - \varepsilon) \log(1/|a_{m_1, m_2}|) \leq \log(m_1^{m_1} m_2^{m_2})$$

i.e.,

$$(6.3) \qquad \log |a_{m_1, m_2}^{\dagger}| \geq -(\mu - \varepsilon)^{-1}(m_1 \log m_1 + m_2 \log m_2),$$

for an infinite sequence of values of  $m_1$  and  $m_2$ .

Also (6.2) may be written as

$$\begin{aligned} \log M(r_1, r_2) &\geq \log |a_{m_1, m_2}| + \log(r_1^{m_1} r_2^{m_2}) \\ &\geq -(\mu - \varepsilon)^{-1}(m_1 \log m_1 + m_2 \log m_2) \\ &\quad + m_1 \log r_1 + m_2 \log r_2 \\ &= m_1 \left( \log r_1 - \frac{1}{\mu - \varepsilon} \log m_1 \right) + m_2 \left( \log r_2 - \frac{1}{\mu - \varepsilon} \log m_2 \right). \end{aligned}$$

Let  $r_1 = (em_1)^{1/\mu - \varepsilon}$  and  $r_2 = (em_2)^{1/\mu - \varepsilon}$ . Then

$$\log M(r_1, r_2) \geq \frac{m_1}{\mu - \varepsilon} + \frac{m_2}{\mu - \varepsilon} = \frac{r_1^{\mu - \varepsilon} + r_2^{\mu - \varepsilon}}{e(\mu - \varepsilon)}.$$

Since  $\mu - \varepsilon$  is independent of  $r_1$  and  $r_2$ , therefore,

$$\rho = \overline{\lim}_{r_1, r_2 \rightarrow \infty} \frac{\log \log M(r_1, r_2)}{\log(r_1 r_2)} \geq \mu - \varepsilon.$$

Further,  $\varepsilon$  is arbitrary and so  $\rho \geq \mu$ .

(ii) Next we prove that  $\rho \leq \mu$ . We note that if  $\mu = \infty$ , the result is obvious. So we suppose that  $\mu < \infty$ . Let  $\varepsilon > 0$ . Then from (6.1) follows

$$0 \leq \frac{\log(m_1^{m_1} m_2^{m_2})}{\log(1/|a_{m_1, m_2}|)} \leq \mu + \varepsilon,$$

for  $m_1 > m_1^0$  and  $m_2 > m_2^0$ ,

i.e.

$$\log m_1^{-\frac{m_1}{\mu + \varepsilon}} + \log m_2^{-\frac{m_2}{\mu + \varepsilon}} \geq \log |a_{m_1, m_2}|,$$

or

$$|a_{m_1, m_2}| \leq m_1^{-\frac{m_1}{\mu + \varepsilon}} m_2^{-\frac{m_2}{\mu + \varepsilon}}.$$

Hence

$$\begin{aligned} M(r_1, r_2) &< Ar_1^{m_1^0} r_2^{m_2^0} + \sum_{m_1=m_1^0+1, m_2=m_2^0+1}^{\infty} |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2} \\ &+ \sum_{m_1=0}^{m_1^0} \sum_{m_2=m_2^0+1}^{\infty} |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2} + \sum_{m_2=0}^{m_2^0} \sum_{m_1=m_1^0+1}^{\infty} |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2} \\ (6.4) \quad &\leq Ar_1^{m_1^0} r_2^{m_2^0} + \sum_{m_1, m_2}^{\infty} m_1^{-\frac{m_1}{\mu + \varepsilon}} m_2^{-\frac{m_2}{\mu + \varepsilon}} r_1^{m_1} r_2^{m_2} \\ &+ Br_1^{m_1^0} \sum_{m_2=m_2^0+1}^{\infty} m_2^{-\frac{m_2}{\mu + \varepsilon}} r_2^{m_2} + Cr_2^{m_2^0} \sum_{m_1=m_1^0+1}^{\infty} m_1^{-\frac{m_1}{\mu + \varepsilon}} r_1^{m_1}. \end{aligned}$$

Let  $\Sigma_1$  be the part of the double series in (6.4) for which  $m_1 < (2r_1)^{\mu+\varepsilon} < m_2 < (2r_2)^{\mu+\varepsilon}$ . We estimate  $\Sigma_1$  by taking the largest value of  $r_1^{m_1} r_2^{m_2}$ . Then

$$\begin{aligned}
 \Sigma_1 &= \sum_{m_1 < (2r_1)^{\mu+\varepsilon}} \sum_{m_2 < (2r_2)^{\mu+\varepsilon}} m_1^{-\frac{m_1}{\mu+\varepsilon}} m_2^{-\frac{m_2}{\mu+\varepsilon}} r_1^{m_1} r_2^{m_2} \\
 (6.5) \quad &\leq r_1^{(2r_1)^{\mu+\varepsilon}} r_2^{(2r_2)^{\mu+\varepsilon}} \sum_{m_1, m_2} m_1^{-\frac{m_1}{\mu+\varepsilon}} m_2^{-\frac{m_2}{\mu+\varepsilon}} \\
 &= O\{e^{(2r_1)^{\mu+2\varepsilon} + (2r_2)^{\mu+2\varepsilon}}\},
 \end{aligned}$$

since the series in (6.5) is convergent and is independent of  $r_1$  and  $r_2$ .

Let  $\Sigma_2$  contain the terms for which  $m_1 \geq (2r_1)^{\mu+\varepsilon}$  and  $m_2 \geq (2r_2)^{\mu+\varepsilon}$  and so in  $\Sigma_2$ , we have  $r_1 m_1^{-1/\mu+\varepsilon} \leq \frac{1}{2}$  and  $r_2 m_2^{-1/\mu+\varepsilon} \leq \frac{1}{2}$ , and hence

$$\begin{aligned}
 \Sigma_2 &= \sum_{m_1 \geq (2r_1)^{\mu+\varepsilon}} \sum_{m_2 \geq (2r_2)^{\mu+\varepsilon}} m_1^{-\frac{m_1}{\mu+\varepsilon}} m_2^{-\frac{m_2}{\mu+\varepsilon}} r_1^{m_1} r_2^{m_2} \\
 &\leq \sum_{m_1, m_2} \left(\frac{1}{2}\right)^{m_1} \left(\frac{1}{2}\right)^{m_2} \leq 1.
 \end{aligned}$$

Let  $\Sigma_3$  be the part of the series for which  $m_1 < (2r_1)^{\mu+\varepsilon}$  and  $m_2 \geq (2r_2)^{\mu+\varepsilon}$ , then

$$\Sigma_3 = \sum_{m_1 < (2r_1)^{\mu+\varepsilon}} m_1^{-\frac{m_1}{\mu+\varepsilon}} r_1^{m_1} \sum_{m_2 \geq (2r_2)^{\mu+\varepsilon}} m_2^{-\frac{m_2}{\mu+\varepsilon}} r_2^{m_2}.$$

Since

$$\sum_{m_2 \geq (2r_2)^{\mu+\varepsilon}} m_2^{-\frac{m_2}{\mu+\varepsilon}} r_2^{m_2} \leq 1$$

and

$$\sum_{m_1 < (2r_1)^{\mu+\varepsilon}} m_1^{-\frac{m_1}{\mu+\varepsilon}} r_1^{m_1} \leq O\{e^{(2r_1)^{\mu+2\varepsilon}}\},$$

therefore,

$$\Sigma_3 \leq O\{e^{(2r_1)^{\mu+2\varepsilon}}\}.$$

Let the remaining part of the double series in (6.4) be denoted by  $\Sigma_4$  i.e. for  $m_1 \geq (2r_1)^{\mu+\varepsilon}$  and  $m_2 < (2r_2)^{\mu+\varepsilon}$ , then

$$\Sigma_4 \leq O\{e^{(2r_2)^{\mu+2\varepsilon}}\}.$$

Further,

$$B r_1^{m_1^0} \sum_{m_2} m_2^{-\frac{m_2}{\mu+\varepsilon}} r_2^{m_2} \leq O\{e^{(2r_2)^{\mu+2\varepsilon}}\}$$

$$C r_2^{m_2^0} \sum_{m_1} m_1^{-\frac{m_1}{\mu+\varepsilon}} r_1^{m_1} \leq O\{e^{(2r_1)^{\mu+2\varepsilon}}\}.$$

Hence, substituting these values in (6.4), we get

$$\begin{aligned} M(r_1, r_2) &< \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + 0\{e^{(2r_1)^\mu + 2\varepsilon}\} + 0\{e^{(2r_2)^\mu + 2\varepsilon}\} \\ &\leq 0\{e^{(2r_1)^\mu + 2\varepsilon + (2r_2)^\mu + 2\varepsilon}\} \\ &\leq 0\{e^{(4r_1 r_2)^\mu + 2\varepsilon}\}, \end{aligned}$$

and hence

$$(6.6) \quad \overline{Lt}_{r_1, r_2 \rightarrow \infty} \frac{\log \log M(r_1, r_2)}{\log(r_1 r_2)} \leq \mu,$$

since  $\varepsilon$  is arbitrary and independent of  $r_1$  and  $r_2$ .

7. Suppose  $0 < \rho < \infty$  and let us define

$$(7.1) \quad \alpha = \overline{Lt}_{m_1, m_2 \rightarrow \infty} \{m_1^{m_1} m_2^{m_2} |a_{m_1, m_2}|^\rho\}^{\frac{1}{m_1 + m_2}}$$

and

$$\tau = \overline{Lt}_{r_1, r_2 \rightarrow \infty} \frac{\log M(r_1, r_2)}{r_1^\rho + r_2^\rho}.$$

The functions which satisfy the latter equality are said to be functions of exponential type  $\tau$ .

If  $\alpha = e\tau\rho$  and using the Sterling's formula

$$m_1! \sim m_1^{m_1} e^{-m_1} (2\pi m_1)^{\frac{1}{2}} e^{\delta/(12m_1)},$$

(7.1) takes the form

$$\begin{aligned} \tau\rho &= \overline{Lt}_{m_1, m_2 \rightarrow \infty} \frac{1}{e} \{m_1^{m_1} m_2^{m_2} |a_{m_1, m_2}|^\rho\}^{\frac{1}{m_1 + m_2}} \\ &= \overline{Lt}_{m_1, m_2 \rightarrow \infty} \frac{1}{e} \left\{ (m_1^{m_1} m_2^{m_2})^{\frac{1}{m_1 + m_2}} \left( \frac{1}{m_1! m_2!} \left| \frac{\partial^{m_1 + m_2}}{\partial z_1^{m_1} \partial z_2^{m_2}} f(z_1, z_2) \right| \right)^{\frac{\rho}{m_1 + m_2}} \right\}, \\ (7.2) \quad &= \overline{Lt}_{m_1, m_2 \rightarrow \infty} \left\{ \left( \frac{m_1^{m_1} m_2^{m_2}}{e^{m_1 + m_2}} \right)^{\frac{1-\rho}{m_1 + m_2}} \left| \frac{\partial^{m_1 + m_2}}{\partial z_1^{m_1} \partial z_2^{m_2}} f(z_1, z_2) \right|^{\frac{\rho}{m_1 + m_2}} \right\}, \end{aligned}$$

where  $z_1, z_2$  are any (fixed) complex numbers.

If in this we put  $\rho = 1$ , then

$$\tau = \overline{Lt}_{m_1, m_2 \rightarrow \infty} \left\{ \left| \frac{\partial^{m_1 + m_2}}{\partial z_1^{m_1} \partial z_2^{m_2}} f(z_1, z_2) \right|^{\frac{1}{m_1 + m_2}} \right\}.$$

We now deduce the following result:

**THEOREM V.** *If  $0 < \alpha < \infty$ , the function*

$$f(z_1, z_2) = \sum_{m_1, m_2=0}^{\infty} a_{m_1, m_2} z_1^{m_1} z_2^{m_2}$$

is of order  $\rho$  and type  $\tau$ , if and only if  $\alpha = \epsilon\tau\rho$ .

PROOF: From (7.1), we have

$$(7.3) \quad |a_{m_1, m_2}| \leq (\alpha + \epsilon)^{\frac{m_1 + m_2}{\rho}} m_1^{-\frac{m_1}{\rho}} m_2^{-\frac{m_2}{\rho}},$$

for  $\epsilon > 0$  and  $m_1, m_2$  large.

We shall first prove that  $\tau \leq \alpha/\epsilon\rho$ . Since we may add a polynomial to  $f(z_1, z_2)$  without affecting its type, we may suppose that (7.3) holds for all  $m_1$  and  $m_2$ , interpreting its right hand side as 1 for  $m_1 = 0, m_2 = 0$ . Then

$$\begin{aligned} |f(z_1, z_2)| &\leq \sum_{m_1, m_2=0}^{\infty} |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2} \\ &\leq \sum_{m_1, m_2=0}^{\infty} (\alpha + \epsilon)^{\frac{m_1 + m_2}{\rho}} m_1^{-\frac{m_1}{\rho}} m_2^{-\frac{m_2}{\rho}} r_1^{m_1} r_2^{m_2}. \end{aligned}$$

The general term of the right hand side does not exceed its maximum.

Let

$$\begin{aligned} \phi(m_1, m_2) &= \frac{m_1 + m_2}{\rho} \log(\alpha + \epsilon) - \frac{m_1}{\rho} \log m_1 + m_1 \log r_1 \\ &\quad - \frac{m_2}{\rho} \log m_2 + m_2 \log r_2. \end{aligned}$$

Then for  $\phi$  to be maximum,

$$\frac{\partial \phi}{\partial m_1} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial m_2} = 0,$$

i.e.

$$m_1 = \frac{(\alpha + \epsilon)r_1^\rho}{e} \quad \text{and} \quad m_2 = \frac{(\alpha + \epsilon)r_2^\rho}{e}.$$

Therefore,

$$\begin{aligned} \text{max. term} &= \exp \left\{ \frac{m_1 + m_2}{\rho} \log(\alpha + \epsilon) - \frac{m_1}{\rho} \log m_1 \right. \\ &\quad \left. - \frac{m_2}{\rho} \log m_2 + m_1 \log r_1 + m_2 \log r_2 \right\} \\ &= \exp \left\{ \frac{\alpha + \epsilon}{e\rho} (r_1^\rho + r_2^\rho) \right\}. \end{aligned}$$

Thus the maximum term is  $\exp \{(r_1^\rho + r_2^\rho)(\alpha + \epsilon)/e\rho\}$ , attained for

$m_1 = r_1^\rho (\alpha + \varepsilon)/e$  and  $m_2 = r_2^\rho (\alpha + \varepsilon)/e$ . Let  $\Sigma_1$  denote the part of the series for which  $m_1 \leq (\alpha + 2\varepsilon)r_1^\rho$  and  $m_2 \leq (\alpha + 2\varepsilon)r_2^\rho$ . Then

$$\begin{aligned} \Sigma_1 &\leq (\alpha + 2\varepsilon)^2 (r_1 r_2)^\rho \exp \left\{ \frac{\alpha + \varepsilon}{\rho e} (r_1^\rho + r_2^\rho) \right\} \\ &= 0 \left[ \exp \left\{ \frac{\alpha + \varepsilon}{e \rho} (r_1^\rho + r_2^\rho) \right\} \right]. \end{aligned}$$

Let  $\Sigma_2$  denote the part of the series for which  $r_1^\rho < m_1/(\alpha + 2\varepsilon)$  and  $r_2^\rho < m_2/(\alpha + 2\varepsilon)$ . Then

$$\begin{aligned} \Sigma_2 &< \sum_{m_1 > (\alpha + 2\varepsilon)r_1^\rho} \sum_{m_2 > (\alpha + 2\varepsilon)r_2^\rho} (\alpha + \varepsilon)^{\frac{m_1 + m_2}{\rho}} m_1^{-\frac{m_1}{\rho}} m_2^{-\frac{m_2}{\rho}} \\ &\quad \times \left( \frac{m_1}{\alpha + 2\varepsilon} \right)^{\frac{m_1}{\rho}} \left( \frac{m_2}{\alpha + 2\varepsilon} \right)^{\frac{m_2}{\rho}} \\ &= \sum_{m_1 > (\alpha + 2\varepsilon)r_1^\rho} \left( \frac{\alpha + \varepsilon}{\alpha + 2\varepsilon} \right)^{\frac{m_1}{\rho}} \sum_{m_2 > (\alpha + 2\varepsilon)r_2^\rho} \left( \frac{\alpha + \varepsilon}{\alpha + 2\varepsilon} \right)^{\frac{m_2}{\rho}} \\ &= 0(1). \end{aligned}$$

Let  $\Sigma_3$  denote the part of the series for which  $m_1 \leq (\alpha + 2\varepsilon)r_1^\rho$  and  $m_2 > (\alpha + 2\varepsilon)r_2^\rho$ . Then

$$\begin{aligned} \Sigma_3 &= \sum_{m_1 \leq (\alpha + 2\varepsilon)r_1^\rho} (\alpha + \varepsilon)^{\frac{m_1}{\rho}} m_1^{-\frac{m_1}{\rho}} r_1^{m_1} \sum_{m_2 > (\alpha + 2\varepsilon)r_2^\rho} (\alpha + \varepsilon)^{\frac{m_2}{\rho}} m_2^{-\frac{m_2}{\rho}} r_2^{m_2} \\ &\leq K \sum_{m_1 \leq (\alpha + 2\varepsilon)r_1^\rho} (\alpha + \varepsilon)^{\frac{m_1}{\rho}} m_1^{-\frac{m_1}{\rho}} r_1^{m_1}, \end{aligned}$$

since

$$\sum_{m_2 > (\alpha + 2\varepsilon)r_2^\rho} = 0(1).$$

Therefore

$$\Sigma_3 \leq 0 \left\{ e^{\frac{\alpha + \varepsilon}{e \rho} r_1^\rho} \right\}.$$

Let the remaining part of the series be denoted by  $\Sigma_4$ , i.e. for  $m_1 > (\alpha + 2\varepsilon)r_1^\rho$  and  $m_2 \leq (\alpha + 2\varepsilon)r_2^\rho$ . Then, as in  $\Sigma_3$ , we obtain

$$\Sigma_4 \leq 0 \left\{ e^{\frac{\alpha + \varepsilon}{e \rho} r_2^\rho} \right\}.$$

Thus

$$\begin{aligned} |f(z_1, z_2)| &\leq \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 \\ &\leq 0 \left\{ e^{\frac{\alpha + \varepsilon}{e \rho} (r_1^\rho + r_2^\rho)} \right\}. \end{aligned}$$

Hence

$$\tau \leq \frac{\alpha}{e\rho},$$

since  $f(z_1, z_2)$  is of exponential type  $\tau$ .

Next, to show that  $\tau \geq \alpha/e\rho$ , we have again from (7.1), for an infinite sequence of values of  $m_1$  and  $m_2$ ,

$$|a_{m_1, m_2}| \geq (\alpha - \varepsilon)^{-\frac{m_1+m_2}{\rho}} m_1^{-\frac{m_1}{\rho}} m_2^{-\frac{m_2}{\rho}}, \quad 0 < \varepsilon < \alpha.$$

If we take  $r_1$  and  $r_2$  such that

$$r_1^\rho = \frac{m_1 e}{\alpha - \varepsilon} \quad \text{and} \quad r_2^\rho = \frac{m_2 e}{\alpha - \varepsilon},$$

for these values of  $m_1$  and  $m_2$ , we have

$$M(r_1, r_2) \geq |a_{m_1, m_2}| r_1^{m_1} r_2^{m_2} \geq \exp\left\{\frac{\alpha - \varepsilon}{e\rho} (r_1^\rho + r_2^\rho)\right\}$$

for a sequence of values of  $r_1$  and  $r_2$ , tending to infinity. Hence

$$\tau \geq \frac{\alpha}{e\rho}.$$

It remains now to show that  $f(z_1, z_2)$  is of order  $\rho$  at most if  $\alpha < \infty$ , and is of order  $\rho$  at least if  $\alpha > 0$ .

For large  $m_1, m_2$ , we have from (7.1)

$$\alpha + \varepsilon \geq \{m_1^{m_1} m_2^{m_2} |a_{m_1, m_2}|^\rho\}^{\frac{1}{m_1+m_2}}, \quad \varepsilon > 0,$$

and hence

$$\frac{\log(m_1^{m_1} m_2^{m_2})}{\log(1/|a_{m_1, m_2}|)} \leq \frac{\rho}{1 - \frac{(m_1+m_2) \log(\alpha + \varepsilon)}{m_1 \log m_1 + m_2 \log m_2}}.$$

By (6.1), the order of  $f(z_1, z_2)$  is  $\rho$  at most. Similarly if  $\alpha > 0$  the order of  $f(z_1, z_2)$  is at least  $\rho$ .

#### REFERENCE

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(Oblatum 8-6-60).