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On ε -nets in a complex

by

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§ 1. Let F be a compact metric space or a closed subset of such a space. A finite subset A of F is called an ε -net in F if $\varrho(x, A) < \varepsilon$ for any point x of F .¹⁾

The point x of F is said to be of order λ with respect to the net A if there are exactly λ points $x_1, x_2, \dots, x_\lambda$ of A for which $\varrho(x, x_i) = \varrho(x, A)$, ($i = 1, 2, \dots, \lambda$).¹⁾

Alexandroff has then stated the problem²⁾:

Is it possible, for any λ -dimensional closed set F and for any ε , to find an ε -net so that no point of F has an order $> \lambda + 1$ with respect to the net?

The object of this paper is to prove that to any n -dimensional complex K there is a homeomorphic metric space K' for which the answer to Alexandroff's question is in the affirmative.

Let K be a finite, connected, n -dimensional complex which we imagine to be topologically immersed in the Euclidean R_{2n+1} so that the metric in K may be taken as the metric of R_{2n+1} i.e. the distance between two points of K is their distance in R_{2n+1} .

Consider the infinite sequence of complexes K_0, K_1, K_2, \dots in which K_0 is the complex K and K_{i+1} is a regular subdivision³⁾ of K_i such that the new vertices introduced are centres⁴⁾ of simplexes of K_i . Let the vertices of K_i be $x_{i,1}, x_{i,2}, \dots, x_{i,\alpha_i}$ and those of K_{i+1} be $x_{i+1,1}, x_{i+1,2}, \dots, x_{i+1,\alpha_i}, x_{i+1,\alpha_i+1}, \dots, x_{i+1,\alpha_{i+1}}$ where $x_{i+1,j} = x_{i,j}$ ($j = 1, 2, \dots, \alpha_i$). The set $\{x_{i,j}\}$ of all vertices $x_{i,j}$ ($i = 0, 1, 2, \dots; j = 1, 2, \dots, \alpha_i$) is dense in K .

¹⁾ P. ALEXANDROFF, Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension [Annals of Math. (2) 30 (1928), 123].

²⁾ P. ALEXANDROFF l.c.¹⁾ 125.

³⁾ O. VEBLEN, Colloquium Lectures on Analysis Situs (1922).

⁴⁾ By the centre of a simplex is understood that point whose barycentric coordinates with respect to the vertices are all equal.

§ 2. Introduction of the new metric in K . By a path in K_i joining vertices x and y of K_i is understood a 1-chain $x_{i,1}x_{i,2} + x_{i,2}x_{i,3} + \dots + x_{i,l}x_{i,l+1}$ where $x_{i,1} = x$ and $x_{i,l+1} = y$; then (2.1) the length of this path is defined to be $\frac{l}{2^i}$. Of the finite number of paths in K_i which join vertices $x_{i,s}$ and $x_{i,t}$ of K_i there are one or more whose lengths as above defined have the minimum possible value — such a path is called a minimum path in K_i . We now define:

(2.2) The distance $\varrho(x_{i,s}, x_{i,t})$ is the length of a minimum path in K_i joining $x_{i,s}$ and $x_{i,t}$, ($i = 0, 1, 2, \dots$). We then have

(2.3) $\varrho(x_{i,r}, x_{i,s}) + \varrho(x_{i,s}, x_{i,t}) \geq \varrho(x_{i,r}, x_{i,t})$, ($i = 0, 1, 2, \dots$); for otherwise a path in K_i from $x_{i,r}$ to $x_{i,t}$ via $x_{i,s}$ would have a length $< \varrho(x_{i,r}, x_{i,t})$ contrary to the definition (2.2).

Let a minimum path L_i in K_i joining $x_{i,s}$ and $x_{i,t}$ consist of l_i 1-cells of K_i , ($i = 0, 1, 2, \dots$).

(2.4) No two 1-cells of L_i belong to the same simplex E of K_i , ($i = 0, 1, \dots$); for otherwise two or more 1-cells of L_i could be replaced by a single 1-cell of E , thus replacing L_i by a shorter path in K_i contrary to hypothesis.

Let the upper index α indicate that the vertex x^α of K_{i+1} is the centre of an α -simplex of K_i . We then have:

(2.5) A minimum path L_{i+1} of K_{i+1} joining $x_{i,s} = x_{i+1,s}$ and $x_{i,t} = x_{i+1,t}$ has the form

$$x_1^{\alpha_1} x_2^{\alpha_2} + x_2^{\alpha_2} x_3^{\alpha_3} + \dots + x_{l_{i+1}}^{\alpha_{l_{i+1}}} x_{l_{i+1}+1}^{\alpha_{l_{i+1}+1}},$$

where $\alpha_1 = \alpha_{l_{i+1}+1} = 0$, $\alpha_{2m-1} < \alpha_{2m} > \alpha_{2m+1}$, $l_{i+1} = 2h$ and $x_{2m-1}^{\alpha_{2m-1}} x_{2m}^{\alpha_{2m}} + x_{2m}^{\alpha_{2m}} x_{2m+1}^{\alpha_{2m+1}}$ is in the subdivision of a simplex E_m of K_i , ($m = 1, 2, \dots, h$).

(a) $\alpha_1 = \alpha_{l_{i+1}+1} = 0$, since $x_1^{\alpha_1}$ and $x_{l_{i+1}+1}^{\alpha_{l_{i+1}+1}}$ are the vertices $x_{i,s}$ and $x_{i,t}$ of K_i .

(b) Assume $\alpha_{2m-1} < \alpha_{2m}$; then $x_{2m-1}^{\alpha_{2m-1}} x_{2m}^{\alpha_{2m}}$ is in the subdivision of an α_{2m} -simplex E_m of K_i of centre $x_{2m}^{\alpha_{2m}}$; if $\alpha_{2m} < \alpha_{2m+1}$, then $x_{2m+1}^{\alpha_{2m+1}}$ would be the centre of an α_{2m+1} -simplex of K_i having E_m in its boundary and $x_{2m-1}^{\alpha_{2m-1}} x_{2m}^{\alpha_{2m}} x_{2m+1}^{\alpha_{2m+1}}$ would be a 2-simplex of K_{i+1} contrary to (2.4), hence $\alpha_{2m} > \alpha_{2m+1}$, hence $x_{2m+1}^{\alpha_{2m+1}}$ is the centre of a face of E_m and $x_{2m-1}^{\alpha_{2m-1}} x_{2m}^{\alpha_{2m}} + x_{2m}^{\alpha_{2m}} x_{2m+1}^{\alpha_{2m+1}}$ is in the sub-

division of E_m . Similarly if $\alpha_{2m+1} > \alpha_{2m+2}$, $x_{2m}^{\alpha_{2m}} x_{2m+1}^{\alpha_{2m+1}} x_{2m+2}^{\alpha_{2m+2}}$ would be a 2-simplex of K_{i+1} contrary to (2.4), hence $\alpha_{2m+1} < \alpha_{2m+2}$ i.e. $\alpha_{2(m+1)-1} < \alpha_{2(m+1)}$. From (a) and (b), (2.5) follows by induction (to prove that $l_{i+1} = 2h$ we merely note that when $\alpha_j > \alpha_{j+1}$, j is even, and since $\alpha_{l_{i+1}} > \alpha_{l_{i+1}+1} = 0$, $\alpha_{l_{i+1}}$ is even).

From (2.5) $x_{i,s}$ and $x_{i,t}$ can be joined by the path $L'_i = E'_1 + E'_2 + \dots + E'_h$ in K_i where E'_m is a 1-simplex of the simplex E_m of K_i and $h = \frac{1}{2}l_{i+1}$.

Since L_i is a minimum path in K_i joining $x_{i,s}$ and $x_{i,t}$ we have (c) length $L_i \leq \text{length } L'_i = \frac{1}{2}l_{i+1} = \frac{l_{i+1}}{2^{i+1}} = \text{length } L_{i+1}$. But by a regular subdivision of L_i we obtain a path L'_{i+1} in K_{i+1} joining $x_{i+1,s} = x_{i,s}$ and $x_{i+1,t} = x_{i,t}$ and composed of $2l_i$ 1-simplexes of K_{i+1} ; hence length $L'_{i+1} = \frac{2l_i}{2^{i+1}} = \frac{l_i}{2^i} = \text{length } L_i$. Since L_{i+1} is a minimum path in K_{i+1} , length $L'_{i+1} \geq \text{length } L_{i+1}$, hence (d) length $L_i \geq \text{length } L_{i+1}$. From (c) and (d) we have length $L_i = \text{length } L_{i+1}$, hence

$$(2.6) \quad \varrho(x_{i,s}, x_{i,t}) = \varrho(x_{i+1,s}, x_{i+1,t}), \quad (i = 0, 1, 2, \dots)$$

Let $x_{i,r}$, $x_{j,s}$ and $x_{k,t}$ be any three vertices of $\{x_{i,j}\}$ ($i = 0, 1, 2, \dots$; $j = 1, 2, \dots, \alpha_i$), then

$$(2.7) \quad \varrho(x_{i,r}, x_{j,s}) + \varrho(x_{j,s}, x_{k,t}) \geq \varrho(x_{i,r}, x_{k,t});$$

for let m be an integer greater than i, j and k , such that $x_{i,r} = x_{m,r}$, $x_{j,s} = x_{m,s}$ and $x_{k,t} = x_{m,t}$, then by (2.3) we have

$$\varrho(x_{m,r}, x_{m,s}) + \varrho(x_{m,s}, x_{m,t}) \geq \varrho(x_{m,r}, x_{m,t})$$

from which, using (2.6), we obtain (2.7).

Let now x and y be any points of K and x_{i,r_i} and x_{i,s_i} vertices of K_i such that the sequences $x_{1,r_1}, x_{2,r_2}, \dots$ and $x_{1,s_1}, x_{2,s_2}, \dots$ converge to x and y respectively in R_{2n+1} ; we then make the definition

$$(2.8) \quad \varrho(x, y) = \lim_{i \rightarrow \infty} \varrho(x_{i,r_i}, x_{i,s_i}).$$

(2.9) From (2.2) and (2.7) it follows that the metric thus introduced satisfies the usual axioms

$$\begin{cases} \varrho(x, x) = 0, \\ \varrho(x, y) = \varrho(y, x), \\ \varrho(x, y) + \varrho(y, z) \geq \varrho(x, z). \end{cases}$$

The points of K with the new metric thus constitute a metric space K' .

§ 3. (3.1) The distance in K' from a vertex of a simplex of K_i to a point of the opposite face is $\leq \frac{1}{2^i}$.

Let x_0 be a vertex of an h -dimensional simplex

$$x_0E_0 = x_0x_1 \dots x_h \text{ of } K_i, \quad (h = 1, 2, \dots, n), \quad E_0$$

being the face opposite x_0 ; let y be any point of E_0, E_{m+1} that simplex of K_{i+m+1} in the subdivision of E_m which contains y , ($m = 0, 1, \dots$), and x_{m+1} the centre of x_mE_m ; then

$$\varrho(x_0, y) \leq \sum_{m=0,1,\dots,\infty} \varrho(x_m, x_{m+1}) = \sum_{j=1,2,\dots,\infty} \frac{1}{2^{i+j}} = \frac{1}{2^i}.$$

A similar proof gives:

(3.2) The distance in K' from a vertex of a simplex of K_i to any point of the simplex or its boundary is $\leq \frac{1}{2^i}$.

(3.3) The distance in K' from a vertex of a simplex of K_i to a point of the opposite face is $\frac{1}{2^i}$.

Let x_1 and x_2 be vertices of a simplex E of K_{i-1} , x_1E' and x_2E' simplexes of K_i in the subdivision of E having a common face E' , and y any point of E' ; then by (3.1)

$$\varrho(x_1, y) \leq \frac{1}{2^i} \text{ and } \varrho(x_2, y) \leq \frac{1}{2^i},$$

hence if $\varrho(x_1, y) < \frac{1}{2^i}$ we should have

$$\varrho(x_1, x_2) \leq \varrho(x_1, y) + \varrho(y, x_2) < \frac{1}{2^{i-1}};$$

but by (2.1), $\varrho(x_1, x_2) = \frac{1}{2^{i-1}}$ since x_1 and x_2 are vertices of the simplex E of K_{i-1} ; from this contradiction we have $\varrho(x_1, y) = \frac{1}{2^i}$; the theorem (3.3) is thus true for the vertex x_1 and face E' of x_1E' ; but from definitions (2.1) and (2.2) the distance from a vertex of a simplex of K_i to a point of the opposite face is a function of i only ($i = 0, 1, \dots$), so that (3.3) holds for all simplexes of K_i .

From (3.3) we have:

(3.4) The distance in K' from the centre of a simplex star of K_i to a point of its boundary ⁵⁾ is $\frac{1}{2^i}$, ($i = 0, 1, \dots$).

From (2.2), (2.8) and (3.4) we have:

- (3.5) The distance in K' from the centre of a simplex star of K_i to a point of K_i neither in the interior ⁵⁾ nor boundary of the star is $> \frac{1}{2^i}$, ($i = 0, 1, \dots$).

From (3.2) we have:

- (3.6) The vertices of K_i constitute a $\frac{1}{2^{i+1}}$ -net in K' ,
($i = 0, 1, \dots$).

§ 4. Let $E = x_1 x_2 \dots x_{h+1}$ be any simplex of K_i of centre c , then from (3.4)

$$\varrho(c, x_j) = \frac{1}{2^{i+1}}, \quad (j = 1, 2, \dots, h+1; i = 0, 1, \dots),$$

hence:

- (4.1) The order of c with respect to the net of the vertices of E is $h+1$.

Let x be an inner point of E other than c , then there are vertices x_j and x_k of E such that x is in the interior or boundary of the star of K_{i+1} of centre x_j but neither in the interior nor boundary of the star of K_{i+1} of centre x_k ; hence from (3.2) and (3.5) we have:

$$(4.2) \quad \varrho(x, x_j) \leq \frac{1}{2^{i+1}}, \quad \varrho(x, x_k) > \frac{1}{2^{i+1}}, \quad \text{thus } \varrho(x, x_j) < \varrho(x, x_k);$$

hence:

- (4.3) The order of x with respect to the net of the vertices of E is $< h+1$.
- (4.4) The order of an inner point x of E with respect to the net of all vertices of K_i is equal to its order with respect to the net of vertices of E .

For let y be any vertex of K_i other than a vertex of E ; if $yx_1 x_2 \dots x_{h+1}$ is a simplex of K_i then by (3.1) $\varrho(x, y) = \frac{1}{2^i}$; if $yx_1 x_2 \dots x_{h+1}$ is not a simplex of K_i then $\varrho(x, y) > \frac{1}{2^i}$, thus in all cases $\varrho(x, y) \geq \frac{1}{2^i}$; but by (3.4) if $x = c$ or by (4.2) if $x \neq c$,

⁵⁾ Those simplexes of K_i having a common vertex constitute a simplex star whose centre is this vertex; the points of those simplexes of the star of which the centre is not a vertex constitute the boundary of the star and the remaining points of the star constitute its interior.

there is a vertex x_j of E such that $\varrho(x, x_j) \leq \frac{1}{2^{i+1}} < \varrho(x, y)$, so that the order of x with respect to the net $y, x_1, x_2, \dots, x_{h+1}$ is equal to its order with respect to the net x_1, x_2, \dots, x_{h+1} .

From (4.1), (4.3) and (4.4):

- (4.5) An inner point of an h -dimensional simplex E of K_i , ($h = 1, 2, \dots, n; i = 0, 1, \dots$), is of order $h + 1$ or $< h + 1$ with respect to the vertices of K_i according as it is or is not, respectively, the centre of E .

An immediate consequence is:

- (4.6) The order of any point of K' with respect to the net of vertices of K_i is $\leq n + 1$.

§ 5. Let D_i ($i = 0, 1, \dots$) be the maximum of the diameters, in K , of the simplexes of K_i , then:

- (5.1) $D_i \leq \left(\frac{n}{n+1}\right)^i D_0$, hence $\lim_{i \rightarrow \infty} D_i = 0$.

From (3.3) we have:

- (5.2) The diameter in K' of a simplex star Δ_i of K_i is $\frac{2}{2^i}$, thus $\lim_{i \rightarrow \infty} (\text{diam. } \Delta_i \text{ in } K') = 0$.

- (5.3) If x be an arbitrary point of K there is a sequence Δ_i , ($i = 0, 1, \dots$), of simplex stars such that Δ_i is a star of K_i containing x in its interior, $\Delta_{i+1} \subset \Delta_i$ and, K being closed and compact, $x = \prod_{i=0,1,\dots,\infty} \Delta_i$.

The proof is sufficiently obvious to be omitted.

- (5.4) Let $\bar{\Delta}_i$ represent Δ_i together with those simplexes of K_i having vertices in common with Δ_i , then:

- (a) $\text{Diam. } \bar{\Delta}_i \text{ in } K' = \frac{4}{2^i}$, by (3.3) and (5.2);
 (b) $\text{Diam. } \bar{\Delta}_i \text{ in } K \leq 4\left(\frac{n}{n+1}\right)^i D_0$, by (5.1);
 (c) $\bar{\Delta}_i \supset \Delta_i \supset x$ and
 (d) $\varrho(x, K - \bar{\Delta}_i) \geq \frac{1}{2^i}$.

Let $S(x, r)$ be the spherical region of R_{2n+1} of centre x and radius r , then:

- (5.5) We can choose r so small that for arbitrary i $S(x, r) \subset \Delta_i$; further, from (5.4b) and (5.4c), for arbitrary r we can choose i so great that $S(x, r) \supset \bar{\Delta}_i$.

Let x_1, x_2, \dots converge to x in K ; then each $S(x, r)$ contains almost all the x_j , hence by (5.5) each Δ_i contains almost all the x_j , hence by (5.2) and (5.3), x_1, x_2, \dots converges to x in K' .

Let x_1, x_2, \dots converge to x in K' ; then by (5.4d) each $\bar{\Delta}_i$ contains almost all the x_j , hence by (5.5) each $S(x, r)$ contains almost all the x_j , hence x_1, x_2, \dots converges to x in K .

Thus K and K' are continuous images of each other and since each point corresponds to itself we have:

(5.6) K and K' are homeomorphic.

From (4.6) and (5.6) we have:

(5.7) To any finite n -dimensional complex K there can be constructed a homeomorphic metric space K' in which, for arbitrary ε , an ε -net can be constructed such that the order of any point of K' with respect to the net is $\leq n + 1$.

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