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On the argument functions of simple closed curves and simple arcs

by

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I. In his papers on oriented line elements [1]), Ostrowski proved certain theorems about directions of tangents to plane curves. In an additional paper [2]), Hopf showed that a better insight into the matter treated can be obtained by emphasizing the directions of secants. The purpose of this note is to point out the great advantage that can be obtained by considering secants only, discarding, together with the tangents, the restrictive condition that the curves considered are differentiable. After all, any theorem on the direction of tangents can be obtained by a direct limiting process from a theorem on the direction of secants. A good example of the simplification that is thus sometimes obtained is Ostrowski's theorem 1 (p. 178), since this theorem does not state more than the existence of the function $\varphi(s, t)$ of III below (i.e., of the function $t(s_1, s_2)$, Hopf, l.c., p. 55).

In II we prove the „Umlaufsatz" for arbitrary simple closed curves in essentially the same way as Hopf (p. 51). However, slightly more care is necessary because no differentiability condition is used. It should be noted that by fixing the orientation of the curve in advance we eliminate the ambiguity in sign usually occurring in the statement of this theorem.

In III we prove a theorem on argument functions of simple arcs. The Rolle-Ostrowski theorem in its generalized form for non-differentiable curves is now derived very easily in IV. In V we state a modified version of Ostrowski's theorem 2 (l.c., p. 178) and show how this may be used to give a second proof of the theorem in III.

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[1]) Three notes. Compositio Mathematica 2 (1935), 26—49, and 177—200.

[2]) Compositio Mathematica 2 (1935), 50—62. See these papers for further references.

II. In order to determine a positive direction on a simple closed curve C, we construct three half lines l_i, $i = 1, 2, 3$, each l_i having in common with C only its endpoint P_i, and such that no two l_i have a point in common. The half lines are numbered in such a way that the orientation of the plane [3]) determined by the order l_1, l_2, l_3 is positive. We then determine the positive orientation of C by the order P_1, P_2, P_3.

The curve C can be determined by a continuous funtion $P(s)$, $-\infty < s < +\infty$, whose value corresponds to a point in the plane. We suppose that $P(s) = P(t)$, if and only if $s \equiv t \pmod 1$, and that the orientation of C determined by increasing s is the positive orientation. Then we can find numbers t_i, $i = 1, 2, 3$, such that $P_i = P(t_i)$ and $t_1 < t_2 < t_3 < t_1 + 1$.

It is easily seen [4]) that we can find a continuous function $\varphi(s, t)$ defined for $s < t < s + 1$ and representing one of the possible values for the argument of the oriented segment $P(s)P(t)$. We prove now.

$$\varphi(t, s+1) - \varphi(s, t) = \pi. \tag{1}$$

The right hand side of (1) is obviously an odd multiple of π and is independent of s and t, so that we only need to prove (1) for fixed values of s and t. Replacing s and t in (1) by t and $s+1$ and adding we find $\varphi(s+1, t+1) = \varphi(s, t) + 2\pi$. This formula gives the „Umlaufsatz" without any differentiability condition for C.

Another consequence of (1) is that the orientation of C is independent of the particular half lines l_i used in its definition.

In order to prove (1), we remark that the difference $\varphi(t_1, t_3) - \varphi(t_1, t_2)$ can be determined by letting the endpoint P of a segment P_1P move from P_2 to P_3 along any path in the plane that does not meet the halfline l_1. Similar remarks hold for $\varphi(t_2, t_3) - \varphi(t_1, t_3)$ and $\varphi(t_2, t_1+1) - \varphi(t_2, t_3)$. The three paths may be chosen as polygons, with a total of not more than four sides. Thus it is necessary to prove (1) only in case C is a quadrilateral [5]). The passage from a quadrilateral to a triangle is trivial, so that the rest of the proof of (1) is evident.

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[3]) It is clear that the ordered triple of half lines determines the same orientation on every circle which has the three endpoints in its interior.

[4]) A detailed proof is given bij Hopf in „Vorbemerkungen", l.c. p. 51.

[5]) It is also possible to reduce (1) to the case where C is one of the circles mentioned in [3]).

A continuation of this argument leads to a proof of the Jordan curve theorem, closely related to the one given by E. Schmidt[6]), the main difference being that the curve is oriented a priori, so that, for instance, the index of an interior point with respect to the curve is $+1$ instead of ± 1. Now if the orientation of the curve is known and the curve contains a line segment, it is possible to decide which side of the line segment is interior and which side is exterior to the curve. This will be used in III, c.

III. A simple arc D can be determined by a continuous function P_s, $0 \leqq s \leqq 1$, of which the values correspond to points in the plane and such that $P_s \neq P_t$ whenever $s \neq t$.

We can find[4]) a continuous function $\varphi(s, t)$, $0 \leqq s < t \leqq 1$, representing for given s and t a value of the argument of the oriented segment $P_s P_t$. The following theorem (2) remains true if the word „maximum" is replaced by „minimum". It immediately implies the Rolle-Ostrowski theorem, as shown in IV.

(2) *If D is a simple arc determined by the continuous function P_s, $0 \leqq s \leqq 1$, and if $\varphi(s, t)$, $0 \leqq s < t \leqq 1$, is its argument function, finally if $\varphi(0, 1)$ is a maximum for both $\varphi(0, s)$ and $\varphi(s, 1)$, then D is the line segment $P_0 P_1$.*

a. We assume that D is not the segment $P_0 P_1$, that $\varphi(0, 1) = 0$ and that $\varphi(0, s)$ and $\varphi(s, 1)$ never take a positive value. We must then prove a contradiction.

The line $P_0 P_1$ and the open segment $P_0 P_1$ we call m and l respectively. The open half line determined on m by P_0 and not containing P_1 we call l_0; the half line l_1 is similarly defined. The half plane containing all points S such that the argument of the oriented segment RS, R on m, is between 0 and π we call α, the other half plane determined by m we call β.

b. If a point P_t of D is not on m and the arc $t \leqq s \leqq 1$ of D does not meet l_0, then $-\pi < \varphi(0, t) < 0$, so that P_t is in β and $\varphi(t, 1)$ is congruent mod 2π to a number between 0 and π. Since $\varphi(t, 1) < 0$, it follows that $\varphi(t, 1) < -\pi < \varphi(0, t)$. Similarly we may prove that $\varphi(t, 1) > -\pi > \varphi(0, t)$ for certain values of t sufficiently near to 0. Since both $\varphi(s, 1)$ and $\varphi(0, s)$ are continuous, it follows that $\varphi(0, s) = \varphi(s, 1)$ for certain values of s.

Now if $\varphi(0, s) = \varphi(s, 1)$ then P_s is a common point of l and D. On the other hand $\varphi(0, s) = \varphi(s, 1) = 0$ is impossible, since if

[6]) Sitz. Ber. Preuss. Akad. 1923, 318—329.

this equality is true for P_s and s is varied continuously in such a way that $\varphi(0, s)$ becomes negative, then $\varphi(s, 1)$ becomes positive.

Thus it follows that there exists a common point P_s of l and D, such that $\varphi(0, s)$ is negative. The greatest value of s with this property we call a, then $\varphi(0, a) = -2n\pi < 0$.

c. The least parameter value $s > a$ for which P_s is on the segment P_aP_1 of m will be denoted by b and the arc P_aP_b of D will be denoted by D'. The segment $k = P_aP_b$ of m, together with D', forms a simple closed curve C. Since $\varphi(0, a) < 0$ and $\varphi(0, b) = 0$, the index of P_0 with respect to C is not equal to zero, and hence is equal to 1, so that $\varphi(0, a) = -2\pi$. Thus the positive orientation on C is determined on D' by the order P_aP_b, hence on k by the order P_bP_a. It follows that a point Q in the half plane α near interior points of k has index 0 with respect to C. Since $\varphi(0, b) = 0$, the points on D near P_b are in the half plane β, so that the point Q may be joined to P_1 by a polygon in α not meeting C. Thus P_1 is either on C or exterior to C.

d. The last statement implies, in view of the Jordan curve theorem, that there may be determined a continuous function $f(R)$ giving an argument of the segment RP_1, where R is either in the interior of C or on C (but not in P_1 if $b = 1$) and $f(P_0) = 0$. Since $\varphi(0, s) = 0$ for any common point $P_s \neq P_a$ of D' and the segment P_0P_b, while always $\varphi(0, s) \leqq 0$, it is clear that $f(R)$ is defined, hence equal to 0, for any R on the segment P_0P_b.

It is clear that $f(P_t)$ is the value of $\varphi(t, 1)$ if no point P_s, $0 \leqq s \leqq t$, is exterior to C. Hence $\varphi(c, 1) = 0$, if P_c is the common point of D and k which has the lowest parameter value. Unless $c = a$, it follows that, for slightly lower values of s, we have $\varphi(s, 1) > 0$. But, if $c = a$, then $f(P_t) = \varphi(t, 1)$ for $0 \leqq s < b$, while obviously $f(P_t) > 0$ for some points D_t of D'. So in both cases we find values of s for which $\varphi(s, 1) > 0$. This contradiction proves (2).

IV. The Rolle-Ostrowski theorem can be derived from (2) by a very simple argument. In our terminology it says:

(3) *For any $\varepsilon > 0$ the range of $\varphi(s, t)$ over $s < t < s + \varepsilon$ is equal to the total range of $\varphi(s, t)$.*

If this is not true, then two numbers a and b, $0 < a < b \leqq 1$, can be determined, such that $\varphi(a, b)$ is either greater than or less than any value of $\varphi(s, t)$, for which $s < t < s + b - a$. This immediately contradicts (2).

V. Let D be a simple arc determined by the function P_s, $0 \leqq s \leqq 1$, let $\varphi(s, t)$, $0 \leqq s < t \leqq 1$, be its argument function, and let m be the line $P_0 P_1$. The second theorem in the second note of Ostrowski (l.c. p. 178) is to the effect that D consists of two spirals, one around each endpoint, if the following two conditions are satisfied: (I) At any intersection of D and m the curve D crosses the line m (except of course in P_0 and in P_1); (II) If P_u and P_v are two such intersections which are *successive on the arc* D, then either $\varphi(u, 1) \neq \varphi(v, 1)$ or $\varphi(0, u) \neq \varphi(0, v)$. It is easily seen that the statement remains true if (II) is modified so as to concern intersections *successive on the line* m. In the new form this theorem may be used to give a second proof of the theorem in III. In fact, in proving (2) for an arbitrary arc D we may replace the part of D for which $a \leqq s \leqq b$ by the line segment $P_a P_b$ whenever the new curve is again a simple arc and the change in value of $\varphi(0, s)$ and $\varphi(s, 1)$ is the same along the line segment as along the arc. (If P_a or P_b is an endpoint, we may drop the condition corresponding to that endpoint.) For by such a change the range of $\varphi(0, s)$ and of $\varphi(s, 1)$ is never increased. By a finite number of these changes we can reduce (2) to the case of a polygon Π_1 which satisfies condition (I), except if the original arc is a line segment. By a further finite number of changes we can replace Π_1 by a polygon Π_2 that satisfies the modified condition (II), so that Π_2 has the double spiral form described by Ostrowski. Finally, by means of two more changes of the type described, Π_2 may be replaced by a polygon Π_3 which has exactly two sides. For Π_3 the statement of (2) is trivial, so that (2) follows for arbitrary arcs.

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