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COMPATIBILITY OF THE THETA CORRESPONDENCE WITH THE WHITTAKER FUNCTORS

by Vincent Lafforgue & Sergey Lysenko

Abstract. — We prove that the global geometric theta-lifting functor for the dual pair (H, G) is compatible with the Whittaker functors, where (H, G) is one of the pairs $(\mathrm{S}\mathbb{O}_{2n}, \mathbb{S}\mathrm{p}_{2n})$, $(\mathbb{S}\mathrm{p}_{2n}, \mathrm{S}\mathbb{O}_{2n+2})$ or $(\mathbb{G}\mathrm{L}_n, \mathbb{G}\mathrm{L}_{n+1})$. That is, the composition of the theta-lifting functor from H to G with the Whittaker functor for G is isomorphic to the Whittaker functor for H.

Résumé (*Compatibilité de la thêta-correspondence avec les foncteurs de Whittaker*)
Nous démontrons que le foncteur géométrique de théta-lifting pour la paire duale (H, G) est compatible avec la normalisation de Whittaker, où (H, G) est l'une des paires $(\mathrm{S}\mathbb{O}_{2n}, \mathbb{S}\mathrm{p}_{2n})$, $(\mathbb{S}\mathrm{p}_{2n}, \mathrm{S}\mathbb{O}_{2n+2})$ ou $(\mathbb{G}\mathrm{L}_n, \mathbb{G}\mathrm{L}_{n+1})$. Plus précisément, le composé du foncteur de théta-lifting de H vers G et du foncteur de Whittaker pour G est isomorphe au foncteur de Whittaker pour H.

We prove in this note that the global geometric theta lifting for the pair (H, G) is compatible with the Whittaker normalization, where $(H, G) = (\mathrm{S}\mathbb{O}_{2n}, \mathbb{S}\mathrm{p}_{2n})$, $(\mathbb{S}\mathrm{p}_{2n}, \mathrm{S}\mathbb{O}_{2n+2})$, or $(\mathbb{G}\mathrm{L}_n, \mathbb{G}\mathrm{L}_{n+1})$. More precisely, let k be an algebraically closed field of characteristic $p > 2$. Let X be a smooth projective connected curve over k. For a stack S write $\mathrm{D}(S)$ for the derived category of étale constructible $\bar{\mathbb{Q}}_\ell$-sheaves on S. For a reductive group G over k write

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Bun_G for the stack of G-torsors on X. The usual Whittaker distribution admits a natural geometrization $\mathrm{Whit}_G : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Spec}\, k)$.

We construct an isomorphism of functors between $\mathrm{Whit}_G \circ F$ and Whit_H, where $F : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ is the theta lifting functor (cf. Theorems 1, 2 and 3).

This result at the level of functions (on $\mathrm{Bun}_H(k)$ and $\mathrm{Bun}_G(k)$ when k is a finite field) is well known since a long time and the geometrization of the argument is straightforward. We wrote this note for the following reason.

Our proof holds also for $k = \mathbb{C}$ in the setting of D-modules. In this case for a reductive group G, Beilinson and Drinfeld proposed a conjecture, which (in a form that should be made more precise) says that there exists an equivalence α_G between the derived category of D-modules on Bun_G and the derived category of $\mathscr{O}$-modules on $\mathrm{Loc}_{\check{G}}$. Here $\mathrm{Loc}_{\check{G}}$ is the stack of $\check{G}$-local systems on X, and $\check{G}$ is the Langlands dual group to G. Moreover, Whit_G should be the composition $\mathrm{D}(D\text{-mod}(\mathrm{Bun}_G)) \xrightarrow{\alpha_G} \mathrm{D}(\mathrm{Loc}_{\check{G}}, \mathscr{O}) \xrightarrow{\mathrm{R}\Gamma} \mathrm{D}(\mathrm{Spec}\, \mathbb{C})$.

A morphism $\gamma : \check{H} \to \check{G}$ gives rise to the extension of scalars morphism $\bar{\gamma} : \mathrm{Loc}_{\check{H}} \to \mathrm{Loc}_{\check{G}}$. The functor $\bar{\gamma}_* : \mathrm{D}(\mathrm{Loc}_{\check{H}}, \mathscr{O}) \to \mathrm{D}(\mathrm{Loc}_{\check{G}}, \mathscr{O})$ should give rise to the Langlands functoriality functor

$$\gamma_L = \alpha_G^{-1} \circ \bar{\gamma}_* \circ \alpha_H : \mathrm{D}(D\text{-mod}(\mathrm{Bun}_H)) \to \mathrm{D}(D\text{-mod}(\mathrm{Bun}_G))$$

compatible with the action of Hecke functors.

In the cases $(H, G) = (\mathbb{SO}_{2n}, \mathbb{S}\mathrm{p}_{2n})$, $(\mathbb{S}\mathrm{p}_{2n}, \mathbb{SO}_{2n+2})$ or $(\mathbb{GL}_n, \mathbb{GL}_{n+1})$ the compatibility of the theta lifting functor $F : \mathrm{D}(D\text{-mod}(\mathrm{Bun}_H)) \to D(D\text{-mod}(\mathrm{Bun}_G))$ with the Hecke functors ([6]) and the compatibility of F with the Whittaker functors (proved in this paper) indicate that F should be the Langlands functoriality functor.

Notation. From now on k denotes an algebraically closed field of characteristic $p > 2$, all the stacks we consider are defined over k. Let X be a smooth projective curve of genus g. Fix a prime $\ell \neq p$ and a non-trivial character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$, and denote by $\mathscr{L}_\psi$ the corresponding Artin-Schreier sheaf on $\mathbb{A}^1$. Since k is algebraically closed, we systematically ignore the Tate twists.

For a k-stack locally of finite type S write simply $\mathrm{D}(S)$ for the category introduced in ([3], Remark 3.21) and denoted $\mathrm{D}_c(S, \bar{\mathbb{Q}}_\ell)$ in *loc.cit.* It should be thought of as the unbounded derived category of constructible $\bar{\mathbb{Q}}_\ell$-sheaves on S. For $* = +, -, b$ we have the full triangulated subcategory $\mathrm{D}^*(S) \subset \mathrm{D}(S)$ denoted $\mathrm{D}_c^*(S, \bar{\mathbb{Q}}_\ell)$ in *loc.cit.* Write $\mathrm{D}^*(S)_! \subset \mathrm{D}^*(S)$ for the full subcategory of objects which are extensions by zero from some open substack of finite type. Write $\mathrm{D}^\prec(S) \subset \mathrm{D}(S)$ for the full subcategory of complexes $K \in D(S)$ such that for any open substack $U \subset S$ of finite type we have $K \mid_U \in D^-(U)$.

For any vector space (or bundle) E, we define $\mathrm{Sym}^2(E)$ and $\Lambda^2(E)$ as quotients of $E\otimes E$ (and denote by $x.y$ and $x\wedge y$ the images of $x\otimes y$) and we will use in this article the embeddings

$$\begin{array}{llll} \mathrm{Sym}^2(E) \to & E\otimes E & \text{and} \quad \Lambda^2(E)\to & E\otimes E \\ x.y \quad \mapsto & \frac{x\otimes y+y\otimes x}{2} & \qquad x\wedge y \mapsto & \frac{x\otimes y-y\otimes x}{2} \end{array} \tag{1}$$

1. Whittaker functors

Let G be a reductive group over k. We pick a maximal torus and a Borel subgroup $T\subset B\subset G$ and we denote by Δ_G the set of simple roots of G. The Whittaker functor

$$\mathrm{Whit}_G : \mathrm{D}^{\prec}(\mathrm{Bun}_G)\to \mathrm{D}^-(\mathrm{Spec}\, k)$$

is defined as follows. Write Ω for the canonical line bundle on X. Pick a T-torsor $\mathcal{F}_T$ on X with a trivial conductor, that is, for each $\check{\alpha}\in\Delta_G$ it is equipped with an isomorphism $\delta_{\check{\alpha}} : \mathcal{L}^{\check{\alpha}}_{\mathcal{F}_T}\to\Omega$. Here $\mathcal{L}^{\check{\alpha}}_{\mathcal{F}_T}$ is the line bundle obtained from $\mathcal{F}_T$ via extension of scalars $T\xrightarrow{\check{\alpha}}\mathbb{G}_m$. Let $\mathrm{Bun}_N^{\mathcal{F}_T}$ be the stack classifying a B-torsor $\mathcal{F}_B$ together with an isomorphism

$$\zeta : \mathcal{F}_B\times_B T\widetilde{\to}\mathcal{F}_T$$

Let $\epsilon : \mathrm{Bun}_N^{\mathcal{F}_T}\to\mathbb{A}^1$ be the evaluation map (cf. [1], 4.3.1 where it is denoted $ev_{\check{\omega}}$). Just recall that for each $\check{\alpha}\in\Delta_G$ the class of the extension of $\mathcal{O}$ by Ω associated to $\mathcal{F}_B$, ζ and $\delta_{\check{\alpha}}$ gives $\epsilon_{\check{\alpha}} : \mathrm{Bun}_N^{\mathcal{F}_T}\to\mathbb{A}^1$ and that $\epsilon=\sum_{\check{\alpha}\in\Delta_G}\epsilon_{\check{\alpha}}$. Write $\pi : \mathrm{Bun}_N^{\mathcal{F}_T}\to\mathrm{Bun}_G$ for the extension of scalars $(\mathcal{F}_B,\zeta)\mapsto\mathcal{F}_B\times_B G$. Set $P^0_\psi=\epsilon^*\mathcal{L}_\psi[d_N]$, where $d_N=\dim\mathrm{Bun}_N^{\mathcal{F}_T}$. Let $d_G=\dim\mathrm{Bun}_G$. As in ([7], Definition 2) for $\mathcal{F}\in\mathrm{D}^{\prec}(\mathrm{Bun}_G)$ set

$$\mathrm{Whit}_G(\mathcal{F})=\mathrm{R}\Gamma_c(\mathrm{Bun}_N^{\mathcal{F}_T}, P^0_\psi\otimes\pi^*(\mathcal{F}))[-d_G] \tag{2}$$

Remark 1. — The collection $(\mathcal{F}_T,(\delta_{\check{\alpha}})_{\check{\alpha}\in\Delta_G})$ as above exists, because k is algebraically closed, and one can take $\mathcal{F}_T=(\sqrt{\Omega})^{2\rho}$ for some square root $\sqrt{\Omega}$ of Ω. One has an exact sequence of abelian group schemes $1\to Z\to T\xrightarrow{\prod\check{\alpha}}\mathbb{G}_m^{\Delta_G}\to 1$, where Z denotes the center of G. So, two choices of the collection $(\mathcal{F}_T,(\delta_{\check{\alpha}})_{\check{\alpha}\in\Delta_G})$ are related by a point of $\mathrm{Bun}_Z(k)$ and the associated Whittaker functors are isomorphic up to the automophism of Bun_G given by tensoring with the corresponding Z-torsor.

Remark 2. — When $\mathcal{F}_T$ is fixed, the functor $\mathrm{Whit}_G : \mathrm{D}^{\prec}(\mathrm{Bun}_G)\to\mathrm{D}^-(\mathrm{Spec}\, k)$ does not depend, up to isomorphism, on the choice of the isomorphisms $(\delta_{\check{\alpha}})_{\check{\alpha}\in\Delta_G}$. That is, for any $(\lambda_{\check{\alpha}})_{\check{\alpha}\in\Delta_G}\in(k^*)^{\Delta_G}$, the functors associated to $(\mathcal{F}_T,(\delta_{\check{\alpha}})_{\check{\alpha}\in\Delta_G})$ and $(\mathcal{F}_T,(\lambda_{\check{\alpha}}\delta_{\check{\alpha}})_{\check{\alpha}\in\Delta_G})$ are isomorphic. Indeed, the two diagrams $\mathrm{Bun}_G \xleftarrow{\pi} \mathrm{Bun}_N^{\mathcal{F}_T} \xrightarrow{\epsilon} \mathbb{A}^1$ associated to $(\delta_{\check{\alpha}})_{\check{\alpha}\in\Delta_G}$ and $(\lambda_{\check{\alpha}}\delta_{\check{\alpha}})_{\check{\alpha}\in\Delta_G}$ are isomorphic for the following reason. Since k is algebraically closed, $T(k) \to (k^*)^{\Delta_G}$ is surjective. We pick any preimage $\gamma \in T(k)$ of $(\lambda_{\check{\alpha}})_{\check{\alpha}\in\Delta_G}$ and get the automorphism $(\mathcal{F}_B, \zeta) \mapsto (\mathcal{F}_B, \gamma\zeta)$ of $\mathrm{Bun}_N^{\mathcal{F}_T}$, which together with the idendity of Bun_G and $\mathbb{A}^1$ intertwines the two diagrams.

1.1. Whittaker functor for $\mathbb{GL}_n$. — For $i, j \in \mathbb{Z}$ with $i \leq j$ we denote by $\mathcal{N}_{i,j}$ the stack classifying the extensions of Ω^i by Ω^{i+1} ... by Ω^j, i.e. classifying a vector bundle E_{j-i+1} on X with a complete flag of vector subbundles $0 = E_0 \subset E_1 \subset ... \subset E_{j-i+1}$ together with isomorphisms $E_{k+1}/E_k \simeq \Omega^{j-k}$ for $k = 0, ..., j-i$. Write $\epsilon_{i,j} : \mathcal{N}_{i,j} \to \mathbb{A}^1$ for the map given by the sum of the classes in $\mathrm{Ext}^1(\mathcal{O}, \Omega) \widetilde{\to} \mathbb{A}^1$ of the extensions $0 \to E_{k+1}/E_k \to E_{k+2}/E_k \to E_{k+2}/E_{k+1} \to 0$ for $k = 0, \ldots, j-i-1$.

For $G = \mathbb{GL}_n$, we consider the diagram $\mathrm{Bun}_n \xleftarrow{\pi_{0,n-1}} \mathcal{N}_{0,n-1} \xrightarrow{\epsilon_{0,n-1}} \mathbb{A}^1$, where $\pi_{0,n-1} : \mathcal{N}_{0,n-1} \to \mathrm{Bun}_n$ is $(0 = E_0 \subset \cdots \subset E_n) \mapsto E_n$. This diagram is isomorphic to the diagram $\mathrm{Bun}_G \xleftarrow{\pi} \mathrm{Bun}_N^{\mathcal{F}_T} \xrightarrow{\epsilon} \mathbb{A}^1$ associated to the choice of $\mathcal{F}_T$ whose image in Bun_n is $\Omega^{n-1} \oplus \Omega^{n-2} \oplus \cdots \oplus \mathcal{O}$.

Therefore the functor $\mathrm{Whit}_{\mathbb{GL}_n} : \mathrm{D}^{\prec}(\mathrm{Bun}_n) \to \mathrm{D}^-(\mathrm{Spec}\, k)$ associated to the above choice of $\mathcal{F}_T$ is given by

$$\mathrm{Whit}_{\mathbb{GL}_n}(\mathcal{F}) = \mathrm{R}\Gamma_c(\mathcal{N}_{0,n-1}, \epsilon_{0,n-1}^*(\mathcal{L}_\psi) \otimes \pi_{0,n-1}^*(\mathcal{F}))[\dim \mathcal{N}_{0,n-1} - \dim \mathrm{Bun}_n].$$

Remark 3. — If E is an irreducible rank n local system on X let Aut_E be the corresponding automorphic sheaf on Bun_n (cf. [2]) normalized to be perverse. Then Aut_E is equipped with a canonical isomorphism $\mathrm{Whit}_{\mathbb{GL}_n}(\mathrm{Aut}_E) \widetilde{\to} \bar{\mathbb{Q}}_\ell$. This is our motivation for the above shift normalization in (2).

1.2. Whittaker functor for $\mathbb{Sp}_{2n}$. — Write G_n for the group scheme on X of automorphisms of $\mathcal{O}^n \oplus \Omega^n$ preserving the natural symplectic form $\wedge^2(\mathcal{O}^n \oplus \Omega^n) \to \Omega$. The stack Bun_{G_n} of G_n-torsors on X can be seen as the stack classifying vector bundles M over X of rank $2n$ equipped with a non-degenerate symplectic form $\Lambda^2 M \to \Omega$.

The diagram $\mathrm{Bun}_{G_n} \xleftarrow{\pi_{G_n}} \mathcal{N}_{G_n} \xrightarrow{\epsilon_{G_n}} \mathbb{A}^1$ constructed in the next definition is isomorphic to the diagram $\mathrm{Bun}_G \xleftarrow{\pi} \mathrm{Bun}_N^{\mathcal{F}_T} \xrightarrow{\epsilon} \mathbb{A}^1$ associated, for $G = G_n$, to the choice of $\mathcal{F}_T$ whose image in Bun_{G_n} is $L \oplus L^* \otimes \Omega$ with $L = \Omega^n \oplus \Omega^{n-1} \oplus \cdots \oplus \Omega$ (with the natural symplectic structure for which L and $L^* \otimes \Omega$ are lagrangians).

Definition 1. — Let $\mathcal{N}_{G_n}$ be the stack classifying $((L_1, ..., L_n), E)$, where $(0 = L_0 \subset L_1 \subset ... \subset L_n) \in \mathcal{N}_{1,n}$, and E is an extension of $\mathcal{O}_X$-modules

$$0 \to \mathrm{Sym}^2 L_n \to E \to \Omega \to 0 \tag{3}$$

We associate to (3) an extension

$$0 \to L_n \to M \to L_n^* \otimes \Omega \to 0 \tag{4}$$

with $M \in \mathrm{Bun}_{G_n}$ and L_n lagrangian as follows. Equip $L_n \oplus L_n^* \otimes \Omega$ with the symplectic form $(l, l^*), (u, u^*) \mapsto \langle l, u^* \rangle - \langle u, l^* \rangle$ for $l, u \in L, l^*, u^* \in L^*$. Here $\langle ., . \rangle$ is the canonical paring between L_n and L_n^*. Using (1), we consider (3) as a torsor on X under the sheaf of symmetric morphisms $L_n^* \otimes \Omega \to L_n$. The latter sheaf acts naturally on $L_n \oplus L_n^* \otimes \Omega$ preserving the symplectic form. Then M is the twisting of $L_n \oplus L_n^* \otimes \Omega$ by the above torsor. This defines a morphism $\pi_{G_n} : \mathcal{N}_{G_n} \to \mathrm{Bun}_{G_n}$.

Note that the extension of Ω by $L_n \otimes L_n$ obtained from (4) is the push-forward of (3) by the embedding $\mathrm{Sym}^2 L_n \to L_n \otimes L_n$ we have fixed in (1).

Let $\epsilon_{G_n} : \mathcal{N}_{G_n} \to \mathbb{A}^1$ denote the sum of $\epsilon_{1,n}(L_1, ..., L_n)$ with the class in $\mathrm{Ext}(\mathcal{O}, \Omega) = \mathbb{A}^1$ of the push-forward of (3) by $\mathrm{Sym}^2 L_n \to \mathrm{Sym}^2(L_n/L_{n-1}) = \Omega^2$.

The functor $\mathrm{Whit}_{G_n} : \mathrm{D}^{\prec}(\mathrm{Bun}_{G_n}) \to \mathrm{D}^-(\mathrm{Spec}\, k)$ associated to the above choice of $\mathcal{F}_T$ is given by

$$\mathrm{Whit}_{G_n}(\mathcal{F}) = \mathrm{R}\Gamma_c(\mathcal{N}_{G_n}, \epsilon_{G_n}^*(\mathcal{L}_\psi) \otimes \pi_{G_n}^*(\mathcal{F}))[d_{N(G_n)} - d_{G_n}]$$

with $d_{N(G_n)} = \dim \mathcal{N}_{G_n}$ and $d_{G_n} = \dim \mathrm{Bun}_{G_n}$.

1.3. Whittaker functor for $\mathbb{SO}_{2n}$ (first form). — Let $H_n = \mathbb{SO}_{2n}$.The stack Bun_{H_n} of H_n-torsors can be seen as the stack classifying vector bundles V over X equipped with a non-degenerate symmetric form $\mathrm{Sym}^2 V \to \mathcal{O}$ and a compatible trivialization $\det V \widetilde{\to} \mathcal{O}$.

The diagram $\mathrm{Bun}_{H_n} \xleftarrow{\pi_{H_n}} \mathcal{N}_{H_n} \xrightarrow{\epsilon_{H_n}} \mathbb{A}^1$ constructed in the next definition is isomorphic to the diagram $\mathrm{Bun}_G \xleftarrow{\pi} \mathrm{Bun}_N^{\mathcal{F}_T} \xrightarrow{\epsilon} \mathbb{A}^1$ associated, for $G = H_n$, to the choice of $\mathcal{F}_T$ whose image in Bun_{H_n} is $U \oplus U^*$ with $U = \Omega^{n-1} \oplus \Omega^{n-2} \oplus \cdots \oplus \mathcal{O}$ (with the natural symmetric structure for which U and U^* are isotropic).

Definition 2. — Let $\mathcal{N}_{H_n}$ be the stack classifying $((U_1, ..., U_n), E)$, where $(U_1, ..., U_n) \in \mathcal{N}_{0,n-1}$ (i.e. we have a filtration $0 = U_0 \subset U_1 \subset ... \subset U_n$ with $U_i/U_{i-1} \simeq \Omega^{n-i}$ for $i = 1, ..., n$), and E is an extension of $\mathcal{O}_X$-modules

$$0 \to \Lambda^2 U_n \to E \to \mathcal{O} \to 0 \tag{5}$$

We associate to (5) an extension

$$0 \to U_n \to V \to U_n^* \to 0 \tag{6}$$

with $V \in \mathrm{Bun}_{H_n}$ and U_n isotropic as follows. Equip $U_n \oplus U_n^*$ with the symmetric form given by $(u, u^*), (v, v^*) \mapsto \langle u, v^* \rangle + \langle v, u^* \rangle$ with $u, v \in U_n, u^*, v^* \in U_n^*$. Using (1), we consider (5) as a torsor under the sheaf of antisymmetric morphisms $U_n^* \to U_n$ of $\mathcal{O}_X$-modules. This sheaf acts naturally on $U_n \oplus U_n^*$ preserving the symmetric form and the trivialization of $\det(U_n \oplus U_n^*)$. Then (6) is the twisting of $U_n \oplus U_n^*$ by the above torsor. This defines a morphism $\pi_{H_n} : \mathcal{N}_{H_n} \to \mathrm{Bun}_{H_n}$.

Note that the extension of $\mathcal{O}_X$ by $U_n \otimes U_n$ obtained from (6) is the push-forward of (5) by the embedding $\Lambda^2 U_n \to U_n \otimes U_n$ fixed in (1).

For $\lambda \in k^*$ let $\epsilon_{H_n,\lambda} : \mathcal{N}_{H_n} \to \mathbb{A}^1$ be the sum of $\epsilon_{0,n-1}(U_1, ..., U_n)$ with λu, where $u \in \mathrm{Ext}(\mathcal{O}, \Omega) = \mathbb{A}^1$ is the class of the push-forward of (5) by $\Lambda^2 U_n \to \Lambda^2(U_n/U_{n-2}) = \Omega$. Set $\epsilon_{H_n} = \epsilon_{H_n,1}$.

The functor $\mathrm{Whit}_{H_n} : \mathrm{D}^{\prec}(\mathrm{Bun}_{H_n}) \to \mathrm{D}^{-}(\mathrm{Spec}\, k)$ associated to the above choice of $\mathcal{F}_T$ sends $\mathcal{F} \in D^{\prec}(\mathrm{Bun}_{H_n})$ to

$$\mathrm{Whit}_{H_n}(\mathcal{F}) = \mathrm{R}\Gamma_c(\mathcal{N}_{H_n}, \epsilon_{H_n}^*(\mathcal{L}_\psi) \otimes \pi_{H_n}^*(M))[d_{N(H_n)} - d_{H_n}] \tag{7}$$

with $d_{N(H_n)} = \dim \mathcal{N}_{H_n}$ and $d_{H_n} = \dim \mathrm{Bun}_{H_n}$. By Remark 2, if we replace in (7) ϵ_{H_n} by $\epsilon_{H_n,\lambda}$ then the functor Whit_{H_n} gets replaced by an isomorphic one.

1.4. Whittaker functor for $\mathbb{SO}_{2n}$ (second form)

DEFINITION 3. — Let $\widetilde{\mathcal{N}}_{H_n}$ be the stack classifying $(V_1 \subset \cdots \subset V_n \subset V)$, where $V \in \mathrm{Bun}_{H_n}$, $V_n \subset V$ is a subbundle, $(V_1, ..., V_n) \in \mathcal{N}_{0,n-1}$ (i.e. we have a filtration $0 = V_0 \subset V_1 \subset ... \subset V_n$ with $V_i/V_{i-1} \simeq \Omega^{n-i}$ for $i = 1, ..., n$), and the composition

$$\mathrm{Sym}^2 V_n \to \mathrm{Sym}^2 V \to \mathcal{O}$$

coincides with $\mathrm{Sym}^2 V_n \to \mathrm{Sym}^2(V_n/V_{n-1}) = \mathcal{O}$ (in particular V_{n-1} is isotropic).

The morphism $\widetilde{\pi}_{H_n} : \widetilde{\mathcal{N}}_{H_n} \to \mathrm{Bun}_{H_n}$ sends $((V_1, ..., V_n), V)$ to V. The morphism $\widetilde{\epsilon}_{H_n} : \widetilde{\mathcal{N}}_{H_n} \to \mathbb{A}^1$ is given by $\widetilde{\epsilon}_{H_n}((V_1, ..., V_n), V) = \epsilon_{0,n-1}(V_1, ..., V_n)$.

Define a morphism $\kappa : \mathcal{N}_{H_n} \to \widetilde{\mathcal{N}}_{H_n}$ as follows. Let $(U_1, ..., U_n), E) \in \mathcal{N}_{H_n}$ and let V be as in Definition 2. For $i = 1, ..., n-1$ define V_i as the image of U_i in V and V_{2n-i} as the orthogonal of V_i in V. Then we have a filtration

$$0 = V_0 \subset V_1 \subset ... \subset V_{n-1} \subset V_{n+1} \subset ... \subset V_{2n-1} \subset V_{2n} = V.$$

Recall that we have an identification $U_n/U_{n-1} \simeq \mathcal{O}$. The exact sequence $0 \to U_n/U_{n-1} \to V_{n+1}/V_{n-1} \to V_{n+1}/U_n \to 0$ admits a unique splitting s such that the image of $\mathcal{O} = V_{n+1}/U_n \xrightarrow{s} V_{n+1}/V_{n-1}$ is isotropic. Thus, V_{n+1}/V_{n-1} is canonically identified with $\mathcal{O} \oplus \mathcal{O}$ in such a way that the symmetric bilinear form $\mathrm{Sym}^2(\mathcal{O} \oplus \mathcal{O}) \to \mathcal{O}$ becomes

$$(1,0).(1,0) \mapsto 0, \ (1,0).(0,1) \mapsto 1, \ (0,1).(0,1) \mapsto 0$$

Under this identification $\mathcal{O} = U_n/U_{n-1} \to V_{n+1}/V_{n-1} = \mathcal{O} \oplus \mathcal{O}$ sends 1 to $(1,0)$.

Define V_n, equipped with $\mathcal{O} \simeq V_n/V_{n-1}$ by the property that $\mathcal{O} \simeq V_n/V_{n-1} \hookrightarrow V_{n+1}/V_{n-1}$ sends 1 to $(1, \frac{1}{2}) \in \mathcal{O} \oplus \mathcal{O}$. The following is easy to check.

LEMMA 1. — *The map* $\kappa : \mathcal{N}_{H_n} \to \widetilde{\mathcal{N}}_{H_n}$ *is an isomorphism. There exists* $\lambda \in k^*$ *such that* $\tilde{\epsilon}_{H_n} \circ \kappa = \epsilon_{H_n,\lambda}$ *and* $\tilde{\pi}_{H_n} \circ \kappa = \pi_{H_n}$.

By Remark 2, if we replace in (7) $\epsilon_{H_n}, \pi_{H_n}$ by $\tilde{\epsilon}_{H_n}, \tilde{\pi}_{H_n}$ then the functor Whit_{H_n} gets replaced by an isomorphic one.

2. Main statements

Write Bun_n for the stack of rank n vector bundles on X. Let Bun_{P_n} be the stack classifying $L \in \mathrm{Bun}_n$ and an exact sequence $0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0$. Remind the complex $S_{P,\psi}$ on Bun_{P_n} introduced in ([4], 5.2). Let $\mathcal{V}$ be the stack over Bun_n whose fibre over L is $\mathrm{Hom}(L, \Omega)$. For $\mathcal{X}_n = \mathcal{V} \times_{\mathrm{Bun}_n} \mathrm{Bun}_{P_n}$ let $p : \mathcal{X}_n \to \mathrm{Bun}_{P_n}$ be the projection. Write $q : \mathcal{X}_n \to \mathbb{A}^1$ for the map sending $s \in \mathrm{Hom}(L, \Omega)$ to the pairing of $s \otimes s \in \mathrm{Hom}(\mathrm{Sym}^2 L, \Omega^2)$ with the exact sequence $0 \to \mathrm{Sym}^2 L \to ? \to \Omega \to 0$. Let $d_{\mathcal{X}_n}$ be the "corrected" dimension of $\mathcal{X}_n$, i.e. the locally constant function $\dim \mathrm{Bun}_{P_n} - \chi(L)$. Set

$$S_{P,\psi} = p_! q^* \mathcal{L}_\psi [d_{\mathcal{X}_n}].$$

Let $\mathcal{A}$ be the line bundle on Bun_{G_n} whose fibre at M is $\det \mathrm{R}\Gamma(X, M)$. Write $\widetilde{\mathrm{Bun}}_{G_n}$ for the gerb of square roots of $\mathcal{A}$ and Aut for the theta-sheaf on $\widetilde{\mathrm{Bun}}_{G_n}$ ([4], Definition 1). The projection $\nu_n : \mathrm{Bun}_{P_n} \to \mathrm{Bun}_{G_n}$ lifts naturally to a map $\tilde{\nu}_n : \mathrm{Bun}_{P_n} \to \widetilde{\mathrm{Bun}}_{G_n}$. In what follows, we pick an isomorphism[(1)]

$$S_{P,\psi} \widetilde{\to} \tilde{\nu}_n^* \mathrm{Aut}[\dim . \mathrm{rel}(\tilde{\nu}_n))] \tag{8}$$

provided by ([5], Proposition 1). Here $\dim . \mathrm{rel}(\tilde{\nu}_n)$ is the relative dimension of $\tilde{\nu}_n$. The isomorphisms we construct below may depend on this choice.

2.1. From $\mathbb{S}\mathrm{p}_{2n}$ to $\mathbb{S}\mathbb{O}_{2n+2}$. — Let $F : \mathrm{D}^-(\mathrm{Bun}_{G_n})_! \to \mathrm{D}^{\prec}(\mathrm{Bun}_{H_{n+1}})$ be the theta lifting functor introduced in ([6], Definition 2).

THEOREM 1. — *The functors* $\mathrm{Whit}_{H_{n+1}} \circ F$ *and* Whit_{G_n} *from* $\mathrm{D}^-(\mathrm{Bun}_{G_n})_!$ *to* $\mathrm{D}^-(\mathrm{Spec}\, k)$ *are isomorphic.*

Let $\mathcal{X}$ be the stack classifying $(M, (U_1, ..., U_{n+1}), E, s)$ with $M \in \mathrm{Bun}_{G_n}$, $(U_1, ..., U_{n+1}) \in \mathcal{N}_{0,n}$ (i.e. $U_{k+1}/U_k = \Omega^{n-k}$ for $k = 0, ..., n$), E an extension $0 \to \Lambda^2 U_{n+1} \to E \to \mathcal{O} \to 0$, and $s : U_{n+1} \to M$ a morphism of $\mathcal{O}_X$-modules.

Let $\alpha_{\mathcal{X}} : \mathcal{X} \to \mathrm{Bun}_{G_n}$ be the morphism $(M, (U_1, ..., U_{n+1}), E, s) \mapsto M$. Let $\beta_{\mathcal{X}} : \mathcal{X} \to \mathbb{A}^1$ be defined as follows. For $(M, (U_1, ..., U_{n+1}), E, s) \in \mathcal{X}$,

$$\beta_{\mathcal{X}}(M, (U_1, ..., U_{n+1}), E, s) = \epsilon_{0,n}(U_1, ..., U_{n+1}) + \gamma(E) - \langle E, \Lambda^2 s \rangle$$

(1) Once $\sqrt{-1} \in k$ is chosen, this isomorphism is well defined up to a sign.

where $\gamma(E)$ is the pairing between the class of E in $\mathrm{Ext}(\mathcal{O}, \Lambda^2 U_{n+1})$ and the morphism $\Lambda^2 U_{n+1} \to \Lambda^2(U_{n+1}/U_{n-1}) = \Omega$ and $\langle E, \Lambda^2 s\rangle$ is the pairing between the class of E in $\mathrm{Ext}(\mathcal{O}, \Lambda^2 U_{n+1})$ and $\Lambda^2 s : \Lambda^2 U_{n+1} \to \Lambda^2 M$ followed by $\Lambda^2 M \to \Omega$.

Let $a_n = n(n+1)(1-g)(n-\frac{1}{2})$, this is the dimension of the stack classifying extension $0 \to \wedge^2 U_{n+1} \to ? \to \mathcal{O} \to 0$ of $\mathcal{O}_X$-modules for any fixed $(U_1, \dots, U_{n+1}) \in \mathcal{N}_{0,n}$.

Let $d_{\alpha_{\mathcal{X}}}$ denote the "corrected" relative dimension of $\alpha_{\mathcal{X}}$, that is, $d_{\alpha_{\mathcal{X}}} = a_n + \dim \mathcal{N}_{0,n} + \chi(U^*_{n+1} \otimes M)$ for any k-points $M \in \mathrm{Bun}_{G_n}$ and $(U_1, \dots, U_{n+1}) \in \mathcal{N}_{0,n}$. One checks that (8) yields for $\mathcal{F} \in \mathrm{D}^-(\mathrm{Bun}_{G_n})_!$ an isomorphism in $\mathrm{D}^-(\mathrm{Spec}\, k)$

$$\mathrm{Whit}_{H_{n+1}} \circ F(\mathcal{F}) \xrightarrow{\sim} \mathrm{R}\Gamma_c(\mathcal{X}, \alpha^*_{\mathcal{X}}(\mathcal{F}) \otimes \beta^*_{\mathcal{X}}(\mathcal{L}_\psi)[d_{\alpha_{\mathcal{X}}}])$$

We will show later that Theorem 1 is reduced to the following proposition.

Proposition 1. — *There is an isomorphism* $\alpha_{\mathcal{X}!}(\beta^*_{\mathcal{X}}(\mathcal{L}_\psi)[2a_n]) \xrightarrow{\sim} \pi_{G_n!}\epsilon^*_{G_n}(\mathcal{L}_\psi)$ *in* $\mathrm{D}^-(\mathrm{Bun}_{G_n})_!$.

The proposition is a consequence of the following lemmas. Let $\mathcal{Y}$ be the stack classifying $(M, (U_1, ..., U_{n+1}), s)$ with $M \in \mathrm{Bun}_{G_n}$, $(U_1, ..., U_{n+1}) \in \mathcal{N}_{0,n}$ (i.e. $U_{k+1}/U_k = \Omega^{n-k}$ for $k = 0, ..., n$), and $s : U_{n+1} \to M$ a morphism such that the composition $\Lambda^2 U_{n+1} \xrightarrow{\Lambda^2 s} \Lambda^2 M \to \Omega$ coincides with $\Lambda^2 U_{n+1} \to \Lambda^2(U_{n+1}/U_{n-1}) = \Omega$.

Let $\alpha_{\mathcal{Y}} : \mathcal{Y} \to \mathrm{Bun}_{G_n}$ be the morphism $(M, (U_1, ..., U_{n+1}), s) \mapsto M$. Let $\beta_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{A}^1$ be the map sending $(M, (U_1, ..., U_{n+1}), s) \in \mathcal{Y}$ to $\epsilon_{0,n}(U_1, ..., U_{n+1})$.

Lemma 2. — *There is an isomorphism* $\alpha_{\mathcal{X},!}\beta^*_{\mathcal{X}}(\mathcal{L}_\psi) = \alpha_{\mathcal{Y},!}\beta^*_{\mathcal{Y}}(\mathcal{L}_\psi)[-2a_n]$ *in* $\mathrm{D}^-(\mathrm{Bun}_{G_n})_!$.

For $i \in \{1, ..., n+1\}$ let $\mathcal{Y}_i$ denote the open subset of $\mathcal{Y}$ given by the condition that the image of U_i by s is a subbundle of M. One has open immersions $\mathcal{Y}_{n+1} \subset \mathcal{Y}_n \subset ... \subset \mathcal{Y}_1 \subset \mathcal{Y}$. Denote by $\alpha_{\mathcal{Y}_i} : \mathcal{Y}_i \to \mathrm{Bun}_{G_n}$ and $\beta_{\mathcal{Y}_i} : \mathcal{Y}_i \to \mathbb{A}^1$ the restrictions of $\alpha_{\mathcal{Y}}$ and $\beta_{\mathcal{Y}}$ to $\mathcal{Y}_i$.

Lemma 3. — *The natural maps* $\alpha_{\mathcal{Y}_{n+1},!}\beta^*_{\mathcal{Y}_{n+1}}(\mathcal{L}_\psi) \to \alpha_{\mathcal{Y}_n,!}\beta^*_{\mathcal{Y}_n}(\mathcal{L}_\psi) \to ... \to \alpha_{\mathcal{Y}_1,!}\beta^*_{\mathcal{Y}_1}(\mathcal{L}_\psi) \to \alpha_{\mathcal{Y},!}\beta^*_{\mathcal{Y}}(\mathcal{L}_\psi)$ *are isomorphisms in* $\mathrm{D}^-(\mathrm{Bun}_{G_n})_!$.

Proof. — First, one has $\mathcal{Y}_{n+1} = \mathcal{Y}_{n-1}$ thanks to the condition that the composition $\Lambda^2 U_{n+1} \xrightarrow{\Lambda^2 s} \Lambda^2 M \to \Omega$ coincides with $\Lambda^2 U_{n+1} \to \Lambda^2(U_{n+1}/U_{n-1}) = \Omega$.

Write $\mathcal{Y}_0 = \mathcal{Y}$. Let $i \in \{1, ..., n-1\}$. We are going to prove that the natural map

$$\alpha_{\mathcal{Y}_i,!}\beta^*_{\mathcal{Y}_i}(\mathcal{L}_\psi) \to \alpha_{\mathcal{Y}_{i-1},!}\beta^*_{\mathcal{Y}_{i-1}}(\mathcal{L}_\psi)$$

is an isomorphism. Set $\mathcal{Z}_i = \mathcal{Y}_{i-1} \backslash \mathcal{Y}_i$, let $\alpha_{\mathcal{Z}_i}$ and $\beta_{\mathcal{Z}_i}$ be the restrictions of $\alpha_{\mathcal{Y}_{i-1}}$ and $\beta_{\mathcal{Y}_{i-1}}$ to $\mathcal{Z}_i$. We must prove that $\alpha_{\mathcal{Z}_i,!}\beta^*_{\mathcal{Z}_i}(\mathcal{L}_\psi) = 0$.

Let $\mathcal{T}_i$ be stack classifying $(M, (U_1, U_2, ..., U_i), s_i)$ with $M \in \mathrm{Bun}_{G_n}$, $(U_1, U_2, ..., U_i) \in \mathcal{N}_{n-i+1,n}$, $s_i : U_i \to M$ such that the restriction of s_i to U_{i-1} is injective and its image is a subbundle of M, but the image of s_i is not a subbundle of M of the same rank as U_i. The map $\alpha_{\mathcal{Z}_i}$ decomposes naturally as $\mathcal{Z}_i \stackrel{\gamma_{\mathcal{Z}_i}}{\to} \mathcal{T}_i \stackrel{\alpha_{\mathcal{T}_i}}{\to} \mathrm{Bun}_{G_n}$. It suffices to show that the $*$-fibre of $\gamma_{\mathcal{Z}_i,!}\beta^*_{\mathcal{Z}_i}(\mathcal{L}_\psi)$ at any closed point $(M, (U_1, U_2, ..., U_i), s_i) \in \mathcal{T}_i$ vanishes.

The fiber $\mathcal{Q}$ of $\gamma_{\mathcal{Z}_i}$ over this point is the stack classifying $((U_1, ..., U_{n+1}), s)$, where $(U_1, ..., U_{n+1}) \in \mathcal{N}_{0,n}$ extends $(U_1, U_2, ..., U_i)$, $s : U_{n+1} \to M$ extends s_i, and the composition $\Lambda^2 U_{n+1} \stackrel{\Lambda^2 s}{\to} \Lambda^2 M \to \Omega$ coincides with $\Lambda^2 U_{n+1} \to \Lambda^2(U_{n+1}/U_{n-1}) = \Omega$.

Let F denote the smallest subbundle of M containing $s(U_i)$, its rank is i or $i-1$. Let $\mathcal{R}$ be stack classifying $((W_1, ..., W_{n+1-i}), t)$ with $(W_1, ..., W_{n+1-i}) \in \mathcal{N}_{0,n-i}$ and $t \in \mathrm{Hom}(W_{n+1-i}, M/F)$. There is a morphism $\rho : \mathcal{Q} \to \mathcal{R}$ which sends $((U_1, ..., U_{n+1}), s)$ to $((U_{i+1}/U_i, ..., U_{n+1}/U_i), \bar{s})$ where $\bar{s} : U_{n+1}/U_i \to M/F$ is the reduction of s. Let $\beta_{\mathcal{Q}} : \mathcal{Q} \to \mathbb{A}^1$ be the restriction of $\beta_{\mathcal{Z}_i}$ to $\mathcal{Q}$. It suffices to show that $\rho_!\beta^*_{\mathcal{Q}}(\mathcal{L}_\psi) = 0$.

Pick $((W_1, ..., W_{n+1-i}), t) \in \mathcal{R}$, let $\mathcal{S}$ be the fiber of ρ over

$$((W_1, ..., W_{n+1-i}), t).$$

Write $\beta_{\mathcal{S}}$ for the restriction of $\beta_{\mathcal{Q}}$ to $\mathcal{S}$. We will show that $R\Gamma_c(\mathcal{S}, \beta^*_{\mathcal{S}}(\mathcal{L}_\psi)) = 0$.

If F is of rank $i-1$ then $\mathcal{S}$ identifies with the stack classifying extensions $0 \to U_i/U_{i-1} \to ? \to U_{n+1}/U_i \to 0$ of $\mathcal{O}_X$-modules. Since $\beta_{\mathcal{S}}$ is a nontrivial character, we are done in this case.

If F is of rank i then $\mathcal{S}$ is a scheme with a free transitive action of $\mathrm{Hom}(U_{n+1}/U_i, F/s(U_i))$. Under the action of $\mathrm{Hom}(U_{n+1}/U_i, F/s(U_i))$, $\beta_{\mathcal{S}}$ changes by some character

$$\mathrm{Hom}(U_{n+1}/U_i, F/s(U_i)) \to \mathrm{Hom}(U_{i+1}/U_i, F/s(U_i)) \stackrel{\delta}{\to} \mathbb{A}^1.$$

If $D = \mathrm{div}(F/s(U_i))$ then $F/s(U_i) \widetilde{\to} \Omega^{n-i+1}(D)/\Omega^{n-i+1}$ naturally, and $\delta : \mathrm{H}^0(X, \Omega(D)/\Omega) \to \mathrm{H}^1(X, \Omega)$ is the map induced by the short exact sequence $0 \to \Omega \to \Omega(D) \to \Omega(D)/\Omega \to 0$, i.e. it is the sum of the residues. Since $D > 0$, δ is nontrivial, and we are done. □

Lemma 4. — *There is an isomorphism $\mu : \mathcal{Y}_{n+1} \to \mathcal{N}_{G_n}$ such that $\pi_{G_n} \circ \mu = \alpha_{\mathcal{Y}_{n+1}}$ and $\epsilon_{G_n} \circ \mu = \beta_{\mathcal{Y}_{n+1}}$.*

It remains to show that Proposition 1 implies Theorem 1 . By the base change theorem we have

$$\mathrm{Whit}_{G_n}(\mathcal{F}) \widetilde{\to} R\Gamma_c(\mathcal{N}_{G_n}, \epsilon^*_{G_n}(\mathcal{L}_\psi) \otimes \pi^*_{G_n}(\mathcal{F}))[d_{N(G_n)} - d_{G_n}]$$

$$\widetilde{\to} R\Gamma_c(\mathrm{Bun}_{G_n}, \pi_{G_n,!}\epsilon^*_{G_n}(\mathcal{L}_\psi)\otimes\mathcal{F})[d_{N(G_n)} - d_{G_n}]$$

and

$$R\Gamma_c(\mathcal{X}, \alpha^*_{\mathcal{X}}(\mathcal{F})\otimes\beta_{\mathcal{X}}(\mathcal{L}_\psi)[d_{\alpha_{\mathcal{X}}}])\widetilde{\to} R\Gamma_c(\mathrm{Bun}_{G_n}, \alpha_{\mathcal{X},!}(\beta^*_{\mathcal{X}}(\mathcal{L}_\psi)\otimes\mathcal{F}[d_{\alpha_{\mathcal{X}}}])).$$

It remains to prove $d_{\alpha_{\mathcal{X}}} - 2a_n = d_{N(G_n)} - d_{G_n}$. This follows from $d_{G_n} = -(1-g)n(2n+1)$, $d_{N(G_n)} - \dim\mathcal{N}_{0,n} = (1-g)(-n^2+n(n+1)(n-\frac{1}{2}))$, and $\chi(U^*_{n+1}\otimes M) = (1-g)2n^2(n+1)$ where $(U_1,\dots,U_{n+1})$ and M are closed points in $\mathcal{N}_{0,n}$ and Bun_{G_n}.

2.2. From $\mathbb{SO}_{2n}$ to $\mathbb{Sp}_{2n}$. — Let $F : \mathrm{D}^-(\mathrm{Bun}_{H_n})_! \to \mathrm{D}^{\prec}(\mathrm{Bun}_{G_n})$ be the Theta functor introduced in ([6], Definition 2).

THEOREM 2. — *The functors* $\mathrm{Whit}_{G_n}\circ F$ *and* Whit_{H_n} *from* $\mathrm{D}^-(\mathrm{Bun}_{H_n})_!$ *to* $\mathrm{D}^-(\mathrm{Spec}\, k)$ *are isomorphic.*

We use the same letters as in the last paragraph (with a different meaning), as the proof is very similar.

Let $\mathcal{X}$ be the stack classifying $(V,(L_1,...,L_n),E,s)$ with $V\in\mathrm{Bun}_{H_n}$, $(L_1,...,L_n)\in\mathcal{N}_{1,n}$ (i.e. $L_{k+1}/L_k = \Omega^{n-k}$ for $k=0,...,n-1$), an extension $0\to\mathrm{Sym}^2 L_n\to E\to\Omega\to 0$ of $\mathcal{O}_X$-modules, and a section $s : L_n\to V\otimes\Omega$.

Let $\alpha_{\mathcal{X}} : \mathcal{X}\to\mathrm{Bun}_{H_n}$ be the morphism $(V,(L_1,...,L_n),E,s)\mapsto V$. Let $\beta_{\mathcal{X}} : \mathcal{X}\to\mathbb{A}^1$ be the map sending $(V,(L_1,...,L_n),E,s)\in\mathcal{X}$ to

$$\epsilon_{1,n}(L_1,...,L_n) + \gamma(E) - \langle E, \mathrm{Sym}^2 s\rangle,$$

where $\gamma(E)$ is the pairing between the class of E in $\mathrm{Ext}^1(\Omega,\mathrm{Sym}^2 L_n)$ and the map $\mathrm{Sym}^2 L_n\to\mathrm{Sym}^2(L_n/L_{n-1}) = \Omega^2$; $\langle E,\mathrm{Sym}^2 s\rangle$ is the pairing between the class of E in $\mathrm{Ext}^1(\Omega,\mathrm{Sym}^2 L_n)$ and $\mathrm{Sym}^2 s : \mathrm{Sym}^2 L_n\to\mathrm{Sym}^2 V\otimes\Omega^2$ followed by $\mathrm{Sym}^2 V\to\mathcal{O}$.

Let $b_n = -\chi(\Omega^{-1}\otimes\mathrm{Sym}^2 L_n)$ for any k-point $(L_1,\dots,L_n)\in\mathcal{N}_{1,n}$. Write $d_{\alpha_{\mathcal{X}}}$ for the "corrected" relative dimension of $\alpha_{\mathcal{X}}$, that is,

$$d_{\alpha_{\mathcal{X}}} = \dim\mathcal{N}_{1,n} + b_n + \chi(L_n^*\otimes V\otimes\Omega)$$

for any k-points $(L_1,\dots,L_n)\in\mathcal{N}_{1,n}$ and $V\in\mathrm{Bun}_{H_n}$. One checks that (8) yields for $\mathcal{F}\in\mathrm{D}^-(\mathrm{Bun}_{H_n})_!$ an isomorphism in $\mathrm{D}^-(\mathrm{Spec}\, k)$

$$\mathrm{Whit}_{G_n}\circ F(\mathcal{F}) = \mathrm{R}\Gamma_c(\mathcal{X}, \alpha^*_{\mathcal{X}}(\mathcal{F})\otimes\beta^*_{\mathcal{X}}(\mathcal{L}_\psi))[d_{\alpha_{\mathcal{X}}}]$$

We will derive Theorem 2 from the following proposition.

PROPOSITION 2. — *There is an isomorphism* $\alpha_{\mathcal{X},!}\beta^*_{\mathcal{X}}(\mathcal{L}_\psi)[2b_n]\simeq\widetilde{\pi}_{H_n,!}\widetilde{\epsilon}^*_{H_n}(\mathcal{L}_\psi)$ *in* $\mathrm{D}^-(\mathrm{Bun}_{H_n})_!$.

Proposition 2 is reduced to the following lemmas. Let $\mathcal{Y}$ be the stack classifying $(V,(L_1,\dots,L_n),s)$ with $V\in\mathrm{Bun}_{\mathbb{SO}_{2n}}$, $(L_1,\dots,L_n)\in\mathcal{N}_{1,n}$ (i.e., $L_{k+1}/L_k=\Omega^{n-k}$ for $k=0,\dots,n-1$) and $s: L_n\to V\otimes\Omega$ a morphism such that the composition $\mathrm{Sym}^2 L_n \overset{\mathrm{Sym}^2 s}{\to} (\mathrm{Sym}^2 V)\otimes\Omega^2\to\Omega^2$ coincides with

$$\mathrm{Sym}^2 L_n\to\mathrm{Sym}^2(L_n/L_{n-1})=\Omega^2$$

Let $\alpha_{\mathcal{Y}}:\mathcal{Y}\to\mathrm{Bun}_{H_n}$ be the map $(V,(L_1,\dots,L_n),s)\mapsto V$. Let $\beta_{\mathcal{Y}}:\mathcal{Y}\to\mathbb{A}^1$ be the map sending $(V,(L_1,\dots,L_n),s)\in\mathcal{Y}$ to $\epsilon_{1,n}(L_1,\dots,L_n)$.

Lemma 5. — *There is an isomorphism* $\alpha_{\mathcal{X},!}\beta^*_{\mathcal{X}}(\mathcal{L}_\psi)\widetilde{\to}\alpha_{\mathcal{Y},!}\beta^*_{\mathcal{Y}}(\mathcal{L}_\psi)[-2b_n]$ *in* $\mathrm{D}^-(\mathrm{Bun}_{H_n})_!$.

For $i\in\{1,\dots,n\}$ let $\mathcal{Y}_i\subset\mathcal{Y}$ be the open substack given by the condition that $s(L_i)\subset V\otimes\Omega$ is a subbundle of rank i. We have inclusions $\mathcal{Y}_n\subset\mathcal{Y}_{n-1}\subset\dots\subset\mathcal{Y}_1\subset\mathcal{Y}$. Denote by $\alpha_{\mathcal{Y}_i}:\mathcal{Y}_i\to\mathrm{Bun}_{H_n}$ and $\beta_{\mathcal{Y}_i}:\mathcal{Y}_i\to\mathbb{A}^1$ the restrictions of $\alpha_{\mathcal{Y}}$ and $\beta_{\mathcal{Y}}$ to $\mathcal{Y}_i$.

As in Lemma 3, one proves

Lemma 6. — *The natural maps* $\alpha_{\mathcal{Y}_n,!}\beta^*_{\mathcal{Y}_n}(\mathcal{L}_\psi)\to\alpha_{\mathcal{Y}_{n-1},!}\beta^*_{\mathcal{Y}_{n-1}}(\mathcal{L}_\psi)\to\dots\to\alpha_{\mathcal{Y}_1,!}\beta^*_{\mathcal{Y}_1}(\mathcal{L}_\psi)\to\alpha_{\mathcal{Y},!}\beta^*_{\mathcal{Y}}(\mathcal{L}_\psi)$ *are isomorphisms in* $\mathrm{D}^-(\mathrm{Bun}_{H_n})_!$.

Lemma 7. — *There is an isomorphism* $\mu:\mathcal{Y}_n\to\widetilde{\mathcal{N}}_{\mathbb{SO}_{2n}}$ *such that* $\widetilde{\pi}_{\mathbb{SO}_{2n}}\circ\mu=\alpha_{\mathcal{Y}_n}$ *and* $\widetilde{\epsilon}_{\mathbb{SO}_{2n}}\circ\mu=\beta_{\mathcal{Y}_n}$.

Theorem 2 follows from Proposition 2 because $d_{\alpha_{\mathcal{X}}}-2b_n=d_{N(H_n)}-d_{H_n}$. Let us just indicate that $d_{N(H_n)}-\dim\mathcal{N}_{1,n}=(1-g)n(n-1)(n-\frac{3}{2})$, $\chi(L_n^*\otimes V\otimes\Omega)=(1-g)2n^3$, $b_n=(1-g)n(n+1)(n-\frac{1}{2})$ and $d_{H_n}=-(1-g)n(2n-1)$, where $(L_1,\dots,L_n)$ and V are closed points in $\mathcal{N}_{1,n}$ and Bun_{H_n}.

2.3. From $\mathbb{GL}_n$ to $\mathbb{GL}_{n+1}$. — Let $F:\mathrm{D}^-(\mathrm{Bun}_n)_!\to\mathrm{D}^{\prec}(\mathrm{Bun}_{n+1})$ be the composition of the direct image by $\mathrm{Bun}_n\to\mathrm{Bun}_n$, $L\mapsto L^*$ and the theta functor $F_{n,n+1}:\mathrm{D}^-(\mathrm{Bun}_n)_!\to\mathrm{D}^{\prec}(\mathrm{Bun}_{n+1})$ introduced in ([6], Definition 3). It is a consequence of Theorem 5 in [6] that F is compatible with Hecke functors according to the morphism of dual groups $\mathbb{GL}_n\to\mathbb{GL}_{n+1}$, $A\mapsto\begin{pmatrix}A&0\\0&1\end{pmatrix}$.

Let us recall the definition of F. Denote $\mathcal{W}$ be the classifying stack of (L,U,s) with $L\in\mathrm{Bun}_n$, $U\in\mathrm{Bun}_{n+1}$ and $s:L\to U$ a morphism. We have $(h_n,h_{n+1}):\mathcal{W}\to\mathrm{Bun}_n\times\mathrm{Bun}_{n+1},(L,U,s)\mapsto(L,U)$. Then for $\mathcal{F}\in\mathrm{D}^-(\mathrm{Bun}_n)_!$,

$$F(\mathcal{F})=h_{n+1,!}((h_n^*\mathcal{F})[\dim\mathrm{Bun}_{n+1}+\chi(L^*\otimes U)]),$$

where $\chi(L^*\otimes U)$ is considered as a locally constant function on $\mathrm{Bun}_n\times\mathrm{Bun}_{n+1}$.

THEOREM 3. — *The functors* $\mathrm{Whit}_{\mathbb{GL}_{n+1}} \circ F$ *and* $\mathrm{Whit}_{\mathbb{GL}_n}$ *from* $\mathrm{D}^-(\mathrm{Bun}_n)_!$ *to* $\mathrm{D}^-(\mathrm{Spec}\, k)$ *are isomorphic.*

Let $\mathcal{X}$ be the stack classifying $L \in \mathrm{Bun}_n$, $(U_1, \dots, U_{n+1}) \in \mathcal{N}_{0,n}$, and $s : L \to U_{n+1}$ a morphism. We have $\alpha_{\mathcal{X}} : \mathcal{X} \to \mathrm{Bun}_n$ and $\beta_{\mathcal{X}} : \mathcal{X} \to \mathbb{A}^1$ which send $(L, (U_1, \dots U_{n+1}), s)$ to L and $\epsilon_{0,n}(U_1, \dots U_{n+1})$.

We have

$$\mathrm{Whit}_{\mathbb{GL}_{n+1}} \circ F(\mathcal{F}) = \mathrm{R}\Gamma_c(\mathrm{Bun}_n, \mathcal{F} \otimes \alpha_{\mathcal{X},!}\beta_{\mathcal{X}}^*(\mathcal{L}_\psi)[\dim \mathcal{N}_{0,n} + \chi(L^* \otimes U_{n+1})]) \text{ and}$$

$$\mathrm{Whit}_{\mathbb{GL}_n}(\mathcal{F}) = \mathrm{R}\Gamma_c(\mathrm{Bun}_n, \mathcal{F} \otimes (\pi_{0,n-1})_!\epsilon_{0,n-1}^*(\mathcal{L}_\psi)[\dim \mathcal{N}_{0,n-1} - \dim \mathrm{Bun}_n]).$$

For $i \in \{0, \dots, n\}$ denote by $\mathcal{X}_i$ the open substack of $\mathcal{X}$ classifying $(L, (U_1, \dots U_{n+1}), s)$ such that the composition $L \xrightarrow{s} U_{n+1} \to U_{n+1}/U_{n+1-i}$ is surjective. We have $\mathcal{X} = \mathcal{X}_0 \supset \mathcal{X}_1 \supset \cdots \supset \mathcal{X}_n$ and we have an isomorphism $\mathcal{N}_{0,n-1} \to \mathcal{X}_n$ which sends $(E_1, \dots, E_n)$ to $(E_n, (\Omega^n, \Omega^n \oplus E_1, \dots, \Omega^n \oplus E_n), (0, \mathrm{Id}))$ with $(0, \mathrm{Id}) : E_n \to \Omega^n \oplus E_n$ the obvious inclusion.

It is easy to compute that for $L = E_n$ with $(E_1, \dots, E_n) \in \mathcal{N}_{0,n-1}$ and $(U_1, \dots, U_{n+1}) \in \mathcal{N}_{0,n}$ we have $\dim \mathcal{N}_{0,n} + \chi(L^* \otimes U_{n+1}) = \dim \mathcal{N}_{0,n-1} - \dim \mathrm{Bun}_n$.

Therefore we are reduced to the following lemma. We denote by $\alpha_{\mathcal{X}_i} : \mathcal{X}_i \to \mathrm{Bun}_n$ and $\beta_{\mathcal{X}_i} : \mathcal{X}_i \to \mathbb{A}^1$ the restrictions of $\alpha_{\mathcal{X}}$ and $\beta_{\mathcal{X}}$ to $\mathcal{X}_i$.

LEMMA 8. — *The natural maps* $\alpha_{\mathcal{X}_n,!}\beta_{\mathcal{X}_n}^*(\mathcal{L}_\psi) \to \alpha_{\mathcal{X}_{n-1},!}\beta_{\mathcal{X}_{n-1}}^*(\mathcal{L}_\psi) \to \dots \to \alpha_{\mathcal{X}_1,!}\beta_{\mathcal{X}_1}^*(\mathcal{L}_\psi) \to \alpha_{\mathcal{X},!}\beta_{\mathcal{X}}^*(\mathcal{L}_\psi)$ *are isomorphisms in* $\mathrm{D}^-(\mathrm{Bun}_n)_!$.

Proof. — We recall that $\mathcal{X} = \mathcal{X}_0$. Let $i \in \{1, \dots, n\}$. We are going to prove that the natural map

$$\alpha_{\mathcal{X}_i,!}\beta_{\mathcal{X}_i}^*(\mathcal{L}_\psi) \to \alpha_{\mathcal{X}_{i-1},!}\beta_{\mathcal{X}_{i-1}}^*(\mathcal{L}_\psi)$$

is an isomorphism. Set $\mathcal{Z}_i = \mathcal{X}_{i-1} \backslash \mathcal{X}_i$, let $\alpha_{\mathcal{Z}_i}$ and $\beta_{\mathcal{Z}_i}$ be the restrictions of $\alpha_{\mathcal{X}_{i-1}}$ and $\beta_{\mathcal{X}_{i-1}}$ to $\mathcal{Z}_i$. We must prove that $\alpha_{\mathcal{Z}_i,!}\beta_{\mathcal{Z}_i}^*(\mathcal{L}_\psi) = 0$.

Let $\mathcal{T}_i$ be stack classifying $(L, (V_1, V_2, \dots, V_i), t)$ with $L \in \mathrm{Bun}_n$, $(V_1, V_2, \dots, V_i) \in \mathcal{N}_{0,i-1}$, $t : L \to V_i$ such that the composition $L \xrightarrow{t} V_i \to V_i/V_1$ is surjective but t is not surjective. The map $\alpha_{\mathcal{Z}_i}$ decomposes naturally as $\mathcal{Z}_i \xrightarrow{\gamma_{\mathcal{Z}_i}} \mathcal{T}_i \xrightarrow{\alpha_{\mathcal{T}_i}} \mathrm{Bun}_n$ where $\gamma_{\mathcal{Z}_i}(L, (U_1, \dots U_{n+1}), s) = (L, (U_{n+2-i}/U_{n+1-i}, \dots, U_{n+1}/U_{n+1-i}), \bar{s})$ and $\alpha_{\mathcal{T}_i}(L, (U_1, \dots U_{n+1}), s) = L$. It suffices to show that the $*$-fibre of $\gamma_{\mathcal{Z}_i,!}\beta_{\mathcal{Z}_i}^*(\mathcal{L}_\psi)$ at any closed point $(L, (V_1, V_2, \dots, V_i), t) \in \mathcal{T}_i$ vanishes.

Let us choose a closed point $(L, (V_1, V_2, \dots, V_i), t) \in \mathcal{T}_i$ and define $L'' = \mathrm{Ker}\, t$ and L' the kernel of the composition $L \xrightarrow{t} V_i \to V_i/V_1$. Then L' is a subbundle of L of rank $n+1-i$ and L'' is a subbundle of L of rank $n+1-i$ or $n-i$.

The fiber $\mathcal{Q}$ of γ_{Z_i} over this closed point is the stack classifying $((U_1,...,U_{n+1}),s)$, with an isomorphism between U_{n+1}/U_{n+1-i} and V_i sending $U_{j+n+1-i}/U_{n+1-i}$ to V_j for any $j \in \{0,\dots,i\}$ and $s : L \to U_{n+1}$ such that the composition $L \xrightarrow{s} U_{n+1} \to U_{n+1}/U_{n+1-i} \simeq V_i$ is t. Let $\mathcal{R}$ be stack classifying $((U_1,\dots,U_{n+1-i}),s_i)$ with $(U_1,\dots,U_{n+1-i}) \in \mathcal{N}_{i,n}$ and $s_i \in \mathrm{Hom}(L'',U_{n+1-i})$. There is a morphism $\rho : \mathcal{Q} \to \mathcal{R}$ which sends $((U_1,...,U_{n+1}),s)$ to $((U_1,\dots,U_{n+1-i}),s_i)$ where s_i is the restriction of s to L''. Let $\beta_{\mathcal{Q}} : \mathcal{Q} \to \mathbb{A}^1$ be the restriction of β_{Z_i} to $\mathcal{Q}$. It suffices to show that $\rho_!\beta_{\mathcal{Q}}^*(\mathcal{L}_\psi) = 0$.

Pick $((U_1,\dots,U_{n+1-i}),s_i) \in \mathcal{R}$, let $\mathcal{S}$ be the fiber of ρ over $((U_1,\dots,U_{n+1-i}),s_i)$. Write $\beta_{\mathcal{S}}$ for the restriction of $\beta_{\mathcal{Q}}$ to $\mathcal{S}$. We will show that $R\Gamma_c(\mathcal{S},\beta_{\mathcal{S}}^*(\mathcal{L}_\psi)) = 0$.

If $L' = L''$ we have an exact sequence $0 \to L/L'' \to U_{n+1}/U_{n+1-i} \to U_{n+2-i}/U_{n+1-i} \to 0$, and $\mathcal{S}$ identifies with the stack classifying extensions $0 \to U_{n+1-i} \to ? \to U_{n+2-i}/U_{n+1-i} \to 0$ of $\mathcal{O}_X$-modules. Since $\beta_{\mathcal{S}}$ is a nontrivial character, we are done in this case.

If L'/L'' is a line bundle then $\mathcal{S}$ is a scheme with a free transitive action of the H^0 of the cone of the morphism of complexes of k-vector spaces

$$\mathrm{RHom}(U_{n+1}/U_{n+1-i},U_{n+1-i}) \to \mathrm{RHom}(L/L'',U_{n+1-i})$$

which is also the cone of the morphism of complexes

$$\mathrm{RHom}(U_{n+2-i}/U_{n+1-i},U_{n+1-i}) \to \mathrm{RHom}(L'/L'',U_{n+1-i})$$

and whose cohomology is concentrated in degree 0. The last morphism of complexes comes from the non zero morphism $L'/L'' \to U_{n+2-i}/U_{n+1-i} = \Omega^{i-1}$ which identifies L'/L'' to $\Omega^{i-1}(-D)$ for some effective non zero divisor D. Therefore the H^0 of this cone is equal to

$$H^0(X, U_{n+1-i}\otimes\Omega^{1-i}(D)/U_{n+1-i}\otimes\Omega^{1-i})$$

and $\beta_{\mathcal{S}}^*(\mathcal{L}_\psi)$ transforms under this action through the character

$$H^0(X, U_{n+1-i}\otimes\Omega^{1-i}(D)/U_{n+1-i}\otimes\Omega^{1-i}) \to$$
$$H^0(X,(U_{n+1-i}/U_{n-i})\otimes\Omega^{1-i}(D)/(U_{n+1-i}/U_{n-i})\otimes\Omega^{1-i}) = H^0(X,\Omega(D)/\Omega) \xrightarrow{\sigma} \mathbb{A}^1$$

where σ is the sum of the residues. Since D is non zero, σ is a non zero character and we are done. □

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