BULLETIN DE LA S. M. F.

P. ERDÖS
M. JOÓ
I. JOÓ

On a problem of Tamas Varga

Bulletin de la S. M. F., tome 120, nº 4 (1992), p. 507-521
<http://www.numdam.org/item?id=BSMF_1992__120_4_507_0>

Bull. Soc. math. France,
120, 1992, p. 507–521.

ON A PROBLEM OF TAMAS VARGA

BY

P. ERDÖS, M. JOÓ AND I. JOÓ (*)

RÉSUMÉ. — Dans la première partie, on considère des propriétés quantitatives des nombres q où un développement en base q de 1 possède un nombre non borné de chiffres 0 consécutifs. Dans la seconde partie, on étudie la distribution des sommes finies $\sum \varepsilon_i q^i$, où $\varepsilon_i = 0$ ou 1 pour des valeurs spéciales de q. La troisième partie est consacrée à l'étude de la distribution des chiffres dans les développements gloutons des nombres x presque partout dans $[0,1]$. Finalement, on pose des problèmes ouverts.

ABSTRACT. — In the first part we investigate the quantitative properties of the numbers q for which there exists an expansion of 1 in base q where the length of consecutive 0-digits is not bounded. In the second part we study the distribution of the finite sums $\sum \varepsilon_i q^i$, $\varepsilon_i = 0$ or 1 for special values q. The third part is devoted to the study of the digit distribution of the greedy expansion of a.e. $x \in [0,1]$. Finally we give some open problems.

Dedicated to academician Vera T. SÓS
on the occasion of her birthday

During his marvellous mathematical teaching activity Tamás VARGA found a lot of deep new problems. We mention the following one : in a heads or tails game repeated n times how long sequences of consecutive heads can be found ? In other words, if we consider the dyadic expansion

$$x = \sum_{1}^{\infty} \frac{\varepsilon_k(x)}{2^k}$$

of a randomly chosen number $0 \le x \le 1$, what can be asserted about the

(*) Texte reçu le 3 juillet 1991, révisé le 30 septembre 1991.
P. ERDÖS and I. JOÓ, Mathematical Institute of the Hungarian Academy of Sciences, Reáltanoda u. 13–15, 1053 Budapest, Hungary. Email : 42436joo@ella.uucp.hu
M. JOÓ, 309 Payne Str Auburn, Alabama 36830, U.S.A.
Classification AMS : primary 10 A 30 ; secondary 10 K 10.

Keywords : expansions, distribution of digits.

longest sequence of consecutive 0-digits (resp. 1-digits) between the first n digits? This problem has thoroughly been investigated by many authors, see e.g. [1] and [2].

1.—In the paper [4], one of the authors modified the problem as follows. Let $1 < q < 2$ be an arbitrary number and consider the expansions of the number 1 of the following type :

$$1 = \sum_{i=1}^{\infty} \frac{1}{q^{n_i}}, \qquad n_i \in \mathbb{N} \text{ are different (natural numbers).} \tag{1}$$

For some values q this expansion is not unique so the uniqueness problem can be investigated as well (see [5], [6]). Recently the second and third author proved in [7] that in the case $q = \sqrt{2}$ there exists an expansion 1 with the property :

$$\sup_i (n_{i+1} - n_i) = \infty. \tag{2}$$

The authors of the present paper, V. KOMORNIK and M. HORVÁTH solved the uniqueness problem of the expansion (1) in [5] and [6], further P. ERDŐS and I. JOÓ studied in [8] the properties of the sequence $(n_{i+1} - n_i)$.

In this paper consider the following properties of the expansion (1)

$$\sup_i \frac{n_i}{i} = \infty, \tag{2'}$$

$$\lim_i (n_{i+1} - n_i) = \infty, \tag{3}$$

$$\lim_i \frac{n_i}{i} = \infty. \tag{3'}$$

Obviously (2′) ⇒ (2) and (3) ⇒ (3′), further the reverse statements do not hold in general.

THEOREM 1 (cf. [8]). — *The set*

$$A := \Big\{ q \in \,]1,2[: \text{ there exists an expansion } (1) \text{ satisfying } (2) \Big\}$$

is residual and of full measure in $]1,2[$.

Problem 1. — Does the statement of the THEOREM 1 remain true after substituting (2′) in place of (2) in the definition of A?

THEOREM 2. — *The set*

$$B := \Big\{ q \in \,]1,2[\colon \text{ there exists an expansion (1) satisfying } (3') \Big\}$$

is of first category and of measure zero.

Proof. — It is enough to make the proof for the sets

$$B \cap \,]1+\delta, 2[, \qquad \delta > 0.$$

Consider arbitrary numbers $1+\delta < q_1 < q_2 < 2$, and sufficiently large $t \in \mathbb{N}$:

$$1 = \sum_{i=1}^{n} \frac{\varepsilon_i}{q_1^i} = \sum_{i=1}^{n} \frac{\varepsilon_i}{q_2^i} + \frac{1}{q_2^{n+t}} \cdot \tag{4}$$

It follows from $1+\delta < q_1$ that there exists $k \le c(\delta)$ with $\varepsilon_k = 1$. Consequently

$$\begin{aligned} \frac{1}{q_2^{n+t}} &= \sum_{i=1}^{n} \varepsilon_i \Big(\frac{1}{q_1^i} - \frac{1}{q_2^i} \Big) \ge \frac{1}{q_1^k} - \frac{1}{q_2^k} \\ &= \frac{\big[q_1 + (q_2 - q_1)\big]^k - q_1^k}{q_1^k q_2^k} \ge \frac{k(q_2 - q_1)}{q_1 q_2^k} \ge \frac{q_2 - q_1}{q_2^{k+1}} \end{aligned}$$

i.e.

$$q_2 - q_1 \le \frac{1}{q_2^{n+t-k-1}} \le \frac{1}{q_2^{n+t-c(\delta)}} \le \frac{1}{(1+\delta)^{n+t-c(\delta)}} \cdot \tag{5}$$

Denote B_n the set of those $1+\delta < q < 2$ for which there exists an expansion of 1 satisfying $n_i/i \ge t$ for $n_i > n$. Take a number $N > 2n$. We see that between $\frac{1}{2}N$ and N there exist $\ge \frac{1}{3}t$ consecutive zeros for any $q \in B_n$. Indeed, assume the contrary. Then between $\frac{1}{2}N$ and N there exist

$$\ge \frac{\frac{1}{2}N}{\frac{1}{3}t} = \frac{3}{2}\,\frac{N}{t} \text{ 1-digits}$$

and then $i \ge \frac{3}{2}N/t$ and $n_i \le N$ would imply that $n_i/i \le \frac{2}{3}t$. The contradiction shows that there exists $\ge \frac{1}{3}t$ consecutive zeros between $\frac{1}{2}N$ and N ; hence q can be covered by an interval of length

$$\le (1+\delta)^{-N/2-t/3+c(\delta)},$$

see (5) above. For any $q \in B_n$ and N we get an interval; the number of such intervals is not greater than N times the number of the sequences $\varepsilon_1, \dots, \varepsilon_N$ with $n_i \geq it$ and $n_i > n$. In particular $\varepsilon_1 + \ldots + \varepsilon_N \leq N/t$ (if $N > nt$). The number of choices of N/t digits from $\varepsilon_1, \dots, \varepsilon_N$ is

$$\begin{aligned}
\binom{N}{[N/t]} &= \frac{N(N-1)\cdots(n-[N/t]+1)}{[N/t]\,!} \\
&\leq c\,\frac{N^{[N/t]}}{\big([N/t]e^{-1}\big)^{[N/t]}\sqrt{[N/t]}} \\
&\leq c\,\frac{N^{[N/t]}}{\sqrt{[N/t]}\,\big(N/(2te)\big)^{[N/t]}} \\
&\leq c\,\sqrt{t/N}\,(2te)^{N/t} \\
&\leq c\sqrt{t}\,2^{c(1+\ln t)N/t} \\
&\leq c\,\sqrt{t}\,2^{N\varepsilon(t)} \qquad \big(\varepsilon(t)\to 0 \text{ when } t\to\infty\big).
\end{aligned}$$

Hence the sum of the length of the above intervals covering B_n is not greater than $c\sqrt{t}\,(1+\delta)^{c(\delta)-t/3}\cdot N\big(2^{\varepsilon(t)}/\sqrt{1+\delta}\big)^N$. Given $\delta > 0$ we can choose $t \geq t(\delta)$ satisfying $2^{\varepsilon(t)}/\sqrt{1+\delta} < 1$. If we fix t and let N tend to infinity, we see that the set B_n can be covered by finite systems of intervals of arbitrary small length sum. Consequently B_n is nowhere dense and of measure zero.

Since $B \subset \bigcup_1^\infty B_n$ the proof of THEOREM 2 is complete. □

Remark. — By (3) ⇒ (3′) the same statement holds with (3) instead of (3′).

THEOREM 3. — *Define the set*

$$C := \Big\{q \in\,]1,2[\ :\ \textit{there exists an expansion } (3) \textit{ satisfying } (1)\Big\},$$

For any interval $I \subset\,]1,2[$ the intersection $I \cap C$ has $2^{\aleph_0}$ many points.

Proof. — Take any value $1 < q_0 < 2$ and any expansion $1 = \sum \varepsilon_i/q_0^i$. The set of q for which there exists an expansion of 1 starting with $\varepsilon_1, \dots, \varepsilon_n$, forms an interval whose length tends to zero when $n \to \infty$. This can be verified just as in (5). Consequently it is enough to prove that given q_0 and n arbitrary we have $2^{\aleph_0}$ many $q \in C$ whose "good" expansion starts with $\varepsilon_1, \ldots, \varepsilon_n$. Fix a sequence $n < n_1 < n_2 \ldots$ satisfying (3) and construct a set $\mathcal{P}$ of infinite subsets of the set $\{n_1, n_2, \ldots\}$ such that $P_1, P_2 \in \mathcal{P}$ implies $P_1 \subset P_2$ or $P_2 \subset P_1$ and there are $2^{\aleph_0}$ elements of $\mathcal{P}$. This can be done in the usual way mapping the set $\{n_k\}$ onto the set of rational numbers $\mathbb{Q}$ in a one-to-one way and then consider the sets $\mathbb{Q} \cap]-\infty, x[$, $x \in \mathbb{R}$. Now for every $P \in \mathcal{P}$ it corresponds to a $q = q_P$ by the rule

$$1 = \sum_{i=1}^{n} \frac{\varepsilon_i}{q^i} + \sum_{n_i \in P} \frac{1}{q^{n_i}} \cdot$$

Then $q_P \in C$ and for different P the value q_P is also different (in case $P_1 \subset P_2$ we have $q_{P_1} < q_{P_2}$).

THEOREM 3 is proved. ∎

2. — Now consider the following problem. For given $1 < q < 2$ define the sets

$$A_n := A_n(q) := \Big\{ \sum_{i=0}^{n-1} \varepsilon_i q^i : \varepsilon_i = 0 \text{ or } 1 \Big\}, \quad n = 1, 2, \ldots$$

If we arrange the sums A_n in a sequence $y_1^{(n)} \leq \ldots \leq y_{2^n}^{(n)}$, we can write

$$A_n = \Big\{ y_k^{(n)} : 1 \leq k \leq 2^n \Big\}.$$

LEMMA 1. — *We have* $y_{k+1}^{(n)} - y_k^{(n)} \leq 1$ *for all* k *and* n.

Proof. — Almost the same is proved in [6]. It runs as follows. We apply induction on n. If $n = 1$ then $A_n = \{0, 1\}$ so the statement is true. Suppose it for A_n and prove for A_{n+1}. Obviously we have

$$A_{n+1} = A_n \cup (q^n + A_n). \tag{6}$$

Now if in A_n there is an element larger than q^n, the smallest element of $q^n + A_n$, then the inductional hypothesis gives the statement by (6). If not, we have to check that the distance between the largest element of A_n and q^n is not larger than 1, i.e.

$$(1 + q + q^2 + \ldots + q^{n-1}) + 1 \geq q^n \quad \text{i.e.} \quad \frac{q^n - 1}{q - 1} \geq q^n - 1.$$

But this is true since $1 < q < 2$.

LEMMA 2. — *The polynomial*

$$P_r(x) := x^{r+1} - \sum_{k=0}^{r} x^k, \quad r \geq 1$$

has exactly one zero ξ_r *in* $]1,2[$ *and* $\xi_r \to 2$ *monotone increasingly as* $r \to \infty$.

Proof. — Define the polynomial Q_r by

$$P_r(x) = x^{r+1} - \frac{x^{r+1}-1}{x-1} = \frac{x^{r+2}-2x^{r+1}+1}{x-1} := \frac{Q_r(x)}{x-1}\cdot$$

We see that the polynomial Q_r decreases for $1 \leq x \leq x_0$, increases for $x_0 \leq x \leq 2$, where $x_0 = 2(r+1)/(r+2)$, further $Q_r(1) = 0, Q_r(2) = 1$. It shows that $Q_r(x)$ has exactly one zero ξ_r in $]1,2[$ and $x_0 < \xi_r < 2$. It implies at once that $\xi_r \to 2$ as $r \to \infty$. On the other hand

$$Q_r(\xi_{r-1}) = \xi_{r-1}^{r+2} - 2\xi_{r-1}^{r+1} + 1 = 1 - \xi_{r-1} < 0$$

which shows that $\xi_{r-1} < \xi_r$.

LEMMA 3. — *Let* $n \geq 1$, $q = \xi_r$ *for some* $r \geq 1$ *and* $A_n = A_n(q)$. *Then we have* $A_n \cap]q^n, 1+q^n[= \emptyset$.

Proof. — By $q = \xi_r$ we have $P_r(q) = 0$, i.e.

$$1 = \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{r+1}}\cdot \tag{7}$$

Iterating this we get the other representation

$$1 = \sum_{\substack{k=1 \\ k \nmid r+1}}^{\infty} \frac{1}{q^k}\cdot \tag{8}$$

Next we show that

$$1 \geq \sum_{\substack{k=1 \\ k \neq r}}^{\infty} \frac{1}{q^k}, \tag{9}$$

and equality holds for $r = 1$. Indeed, we can transform the numbers $q^{-r}, q^{-2r-1}, q^{-3r-2}$, etc. of (8) by the aid of (7) to obtain

$$
\begin{aligned}
1 = \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{r-1}} + \frac{1}{q^{r+1}} + \Big(\frac{2}{q^{r+2}} + \ldots + \frac{2}{q^{2r}}\Big) \\
+ \frac{1}{q^{2r+1}} + \frac{1}{q^{2r+2}} + \Big(\frac{2}{q^{2r+3}} + \cdots + \frac{2}{q^{3r+1}}\Big) \\
+ \frac{1}{q^{3r+2}} + \frac{1}{q^{3r+3}} + \Big(\frac{2}{q^{3r+4}} + \cdots + \frac{2}{q^{4r+2}}\Big) + \cdots
\end{aligned}
$$

which implies (9). This shows that if $y \in A_n$ and $y > q^n$ then the first r digits $\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_{n-r}$ of

$$y = \sum_{i=1}^{n-1} \varepsilon_i q^i$$

must be 1 (otherwise $y < q^n$). If $\varepsilon_{n-r-1} = 1$ then $q^n = \sum_{n-r-1}^{n-1} \varepsilon_i q^i$ and hence $y > q^n$ clearly implies $y \geq q^n + 1$. If $\varepsilon_{n-r-1} = 0$ then define

$$y_1 := y - \sum_{i=n-r}^{n-1} q^i \in A_{n-r-1}.$$

Since $q^{n-1} + \ldots + q^{n-r} = q^n - q^{n-r-1}$, this implies

$$q^{n-r-1} < y_1 < q^{n-r-1} + 1.$$

Iterating this process we finally find a value $1 \leq n \leq r+1$ and $y \in A_n$, $q^n < y < 1 + q^n$. But this is impossible since $n \leq r+1$ implies that $q^n \geq 1 + q + \ldots + q^{n-1}$. The contradiction proves the LEMMA 3. $\square$

Now introduce the Fibonacci-type sequence $F_n^{(k)}$ by the recursion :

$$
F_n^{(k)} = \begin{cases} 0 & \text{for } n < 0, \\ \displaystyle\sum_{i=1}^{k} F_{n-i}^{(k)} + 1 & \text{for } n \geq 0. \end{cases} \tag{10}
$$

We see that

$$
\begin{aligned}
&F_n^{(k)} = 2^n \quad \text{for} \quad 0 \leq n \leq k, && F_{k+1}^{(k)} = 2^{k+1} - 1, \\
&F_{k+2}^{(k)} = 2^{k+2} - 3, && F_{k+3}^{(k)} = 2^{k+3} - 11.
\end{aligned}
$$

THEOREM 4. — *Let $n, r = 1, 2 \dots$ and $q = \xi_r$. Then*

$$\text{(a)} \qquad |A_n(q)| = F_n^{(r+1)},$$

$$\text{(b)} \qquad \min_{\substack{1 \le k \le 2^n \\ y_{k+1}^{(n)} \ne y_k^{(n)}}} \left(y_{k+1}^{(n)} - y_k^{(n)}\right) \ge \frac{1}{q},$$

and equality holds for $n \ge r + 1$.

Proof. — Consider first the case $n \le r$. Then

$$q^n - \left(1 + q + \dots + q^{n-1}\right) \ge \frac{1}{q} > 0$$

hence

$$A_n \cap (q^n + A_n) = \emptyset$$

and then from the obvious relation

$$A_{n+1} = A_n \cup (q^n + A_n)$$

we see at once that

$$|A_{n+1}| = 2\,|A_n| = \dots = 2^n\,|A_1| = 2^{n+1} = F_{n+1}^{r+1}$$

further (b) is also true and equality holds only for $n = r + 1$.

Now let $n \ge r + 1$. Then A_n and $q^n + A_n$ has nonempty intersection since $q^n = q^{n-1} + \dots + q^{n-r-1}$. We show that in this case the sets A_n and $q^n + A_n$ has overlapping maximal possible. Namely every $y \in A_n$, $y \ge q^n$ belongs to $q^n + A_n$. More precisely :

$$(*) \qquad \begin{cases} \text{Every } y \in A_n, y \ge q^n \text{ has an expansion} \\ y = q^{n-1} + \dots + q^{n-r-1} + \displaystyle\sum_{k=0}^{n-r-2} \varepsilon_k q^k. \end{cases}$$

To prove (*) we apply induction on n. For $n = r + 1$ the only element $y \in A_n$ with $y \ge q^n$ is $q^n = q^{n-1} + \dots + q + 1$. Suppose (*) for n and prove it for $n + 1$. Let $y \in A_{n+1}$ and $y \ge q^{n+1}$. If in the expansion $y = \sum_{k=0}^{n} \varepsilon_k q^k$ we have $\varepsilon_0 = 0$, then applying the induction hypothesis to $y/q \in A_n$, $y/q \ge q^n$ we are ready. If $y = q^{n+1}$ then $y = q^n + q^{n-1} + \dots + q^{n-r}$ which is also a good representation. Finally if $y > q^{n+1}$ and $\varepsilon_0 = 1$ then by Lemma 3, $y - 1 \ge q^{n+1}$, hence we can apply the induction hypothesis for $y-1/q \in A_n, y-1/q \geq q^n$ so (*) holds indeed. Consequently $A_{n+1} = 2A_n - A_{n-r-1}$, if $n \geq r+1$. So we can prove (a) by induction as follows :

$$\begin{aligned}|A_{n+1}| &= 2F_n^{(r+1)} - F_{n-r-1}^{(r+1)} \\ &= F_n^{(r+1)} + \Big(1 + \sum_{i=1}^{r+1} F_{n-i}^{(r+1)}\Big) - F_{n-r-1}^{(r+1)} \\ &= 1 + \sum_{i=1}^{r+1} F_{n+1-i}^{(r+1)} = F_{n+1}^{(r+1)}.\end{aligned}$$

The proof of (b) for $n \geq r+1$ is obvious : in A_n and in $A_n + q^n$ the minimal distance is $1/q$ and they overlap maximally hence in A_{n+1} the minimal distance is also $1/q$. THEOREM 4 is proved. □

3. — In the following part of this paper we consider two other problems related to the papers of ERDŐS, RÉNYI [3] and ERDŐS, RÉVÉSZ [1]. To formulate the first one, fix a number $1 < q < 2$ and expand any number $0 \leq x \leq 1$ by the so-called *greedy expansion*

$$x = \sum_1^\infty \frac{\varepsilon_n(x)}{q^n}, \qquad \varepsilon_n(x) = \begin{cases} 0 \\ 1. \end{cases}$$

We assert that

THEOREM 5. — *There exists a constant $c > 0$ with the following properties. Consider the set of those $x \in [0,1]$ for which the greedy expansion of x contains a sequence of $\geq c \log n$ consecutive 0-digits between the first n digits $\varepsilon_1(x), \ldots, \varepsilon_n(x)$ for all indices $n \geq n_0(x)$. This set has full measure in $[0,1]$.*

The second problem arises in a heads or tails game with an asymmetric piece of money. We represent it as a random variable whose value is zero with probability $p, 0 < p < 1$ and 1 with probability $q = 1-p$. Consider a sequence $x_1, x_2, \ldots$ of independent random variables with such distributions. Introduce the quantities

$$\alpha_n := \log n - \log\log\log n + K$$

with some constant $K < 0$ to be specified later. We prove the

THEOREM 6. — *The following event has probability 1 : between the first n digits $x_1, \ldots, x_n$ there exist α_n consecutive 0 digits for sufficiently large $n > n_0$; here $n_0 = n_0(\omega)$ may depend on the concrete value of the sequence $\big(x_n(\omega)\big)_1^\infty$.*

We mention the following open

Problem 2. — THEOREM 5 does not remain true for large $c > 0$ (this is the case if $q = 2$, see ERDŐS, RÉNYI [3]).

For the proof of THEOREMS 5 and 6 we need some lemmas. Denote

$$P(\varepsilon_1, \ldots \varepsilon_n) = \left|\left\{x \in [0,1] : \varepsilon_1 = \varepsilon_1(x), \ldots, \varepsilon_n = \varepsilon_n(x)\right\}\right|$$

the probability of the event that the greedy expansion of x begins with the digits $\varepsilon_1, \ldots, \varepsilon_n$.

LEMMA 4.

(a) $$P(\varepsilon_1, \ldots, \varepsilon_n, 1) \le \frac{1}{q-1} P(\varepsilon_1, \ldots, \varepsilon_n, 0),$$

(b) $$P(\varepsilon_1, \ldots, \varepsilon_n) \le \frac{q}{q-1} P(\varepsilon_1, \ldots, \varepsilon_n, 0).$$

Proof. — (b) follows form (a) since

$$\begin{aligned} P(\varepsilon_1, \ldots, \varepsilon_n) &= P(\varepsilon_1, \ldots, \varepsilon_n, 0) + P(\varepsilon_1, \ldots, \varepsilon_n, 1) \\ &\le \frac{q}{q-1} P(\varepsilon_1, \ldots, \varepsilon_n, 0). \end{aligned}$$

To see (a) denote I_n the length of the segment

$$\left\{x \in [0,1] : \varepsilon_1 = \varepsilon_1(x), \ldots, \ \varepsilon_n = \varepsilon_n(x)\right\}.$$

The left endpoint is $x = \sum_{k=1}^{n} \frac{\varepsilon_k}{q^k}$. Hence :

i) If $I_n < \frac{1}{q^{n+1}}$, then

$$P(\varepsilon_1, \ldots, \varepsilon_n, 0) = P(\varepsilon_1, \ldots, \varepsilon_n), \quad P(\varepsilon_1, \ldots, \varepsilon_n, 1) = 0.$$

ii) If $I_n \in \left[\frac{1}{q^{n+1}}, \frac{1}{q^{n+1}} \frac{q}{q-1}\right[$, then

$$P(\varepsilon_1, \ldots, \varepsilon_n, 0) = \frac{1}{q^{n+1}}, \quad P(\varepsilon_1, \ldots, \varepsilon_n, 1) = P(\varepsilon_1, \ldots, \varepsilon_n) - \frac{1}{q^{n+1}}$$

and hence (a) follows. □

From LEMMA 4 we obtain immediately the

LEMMA 5.

$$P\big(\varepsilon_1,\ldots,\varepsilon_n,\overset{1}{\overline{0}},\ldots,\overset{\alpha_n}{\overline{0}}\big) \geq \Big(\frac{q-1}{q}\Big)^{\alpha_n} P(\varepsilon_1,\ldots,\varepsilon_n).$$

Now denote $S_k(x) := \sum_{j=1}^{k} \frac{\varepsilon_j(x)}{q^j}$. We obtain from LEMMA 5 by induction (in $[n/\alpha_n]$) the

LEMMA 6.

$$\Big|\Big\{x : S_{\ell\alpha_n}(x) \neq S_{(\ell+1)\alpha_n}(x),\ \ell = 0,1,\ldots,\Big[\frac{n}{\alpha_n}\Big]\Big\}\Big| \leq \Big(1-\Big(\frac{q-1}{q}\Big)^{\alpha_n}\Big)^{[n/\alpha_n]+1}$$

Proof of the Theorem 5. — Let

$$\alpha_n := \log n - \log\log n - \log\log\log n - K$$

where log denotes the logarithm of base $q/(q-1)$ and $K = K(q) > 0$ is a constant "large enough". Then

$$\Big(1-\Big(\frac{q-1}{q}\Big)^{\alpha_n}\Big)^{(q/(q-1))^{\alpha_n}} \to \frac{1}{e} \qquad (\text{as } n \to \infty),$$

hence

$$\Big(1-\Big(\frac{q-1}{q}\Big)^{\alpha_n}\Big)^{(q/(q-1))^{\alpha_n}} \leq \frac{1}{e}.$$

(We know that $\sqrt[k+1]{1-(1-k^{-1})^k} \leq \frac{k}{k+1}$ i.e.

$$(1-k^{-1})^k \leq \Big(1-\frac{1}{k+1}\Big)^{k+1}.)$$

We have

$$\Big(1-\Big(\frac{q-1}{q}\Big)^{\alpha_n}\Big)^{[n/\alpha_n]+1} \leq e^{-((q-1)/q)^{\alpha_n}([n/\alpha_n]+1)}.$$

The exponent can be estimated as follows if $n > n_0$:

$$\begin{aligned}
-\Big(\frac{q-1}{q}\Big)^{\alpha_n}\Big(\Big[\frac{n}{\alpha_n}\Big]+1\Big) &\leq -\frac{1}{2}\Big(\frac{q-1}{q}\Big)^{\alpha_n}\frac{n}{\alpha_n}\\
&= -\frac{1}{2}\Big(\frac{q}{q-1}\Big)^{K}\frac{\log n\log\log n}{n}\,\frac{n}{\alpha_n}\\
&\leq -\frac{1}{3}\Big(\frac{q}{q-1}\Big)^{K}\log\log n\\
&\leq -R\log\log n\,;
\end{aligned}$$

but if $K = K(q)$ is large enough, then the condition $n > n_0$ can be omitted. We obtained that

$$|P_n| = \left|\left\{x : S_{\ell\alpha_n}(x) \neq S_{(\ell+1)\alpha_n}(x),\ \ell = 0, 1, \dots, [n/\alpha_n]\right\}\right| \le \frac{1}{(\log n)^2}$$

is R is large enough, i.e. K is large enough. This means that

$$\sum |P_{[(\frac{q}{q-1})^m]}| < \infty$$

and according to the Borel-Cantelli lemma almost every $x \in [0,1]$ belongs to finitely many set $P_{[(q/(q-1))^m]}$, i.e. for a.e. $x \in [0,1]$ the first $[(q/(q-1))^m]$ digit contains 0-sequence of length

$$(*) \qquad cm \quad \text{if} \quad m > m_0(x).$$

Now if

$$\left[\left(\frac{q}{q-1}\right)^m\right] \le n < \left[\left(\frac{q}{q-1}\right)^{m+1}\right],$$

then $m \asymp \log n$, i.e. it follows from (*) that for a.e. $x \in [0,1]$ among the first n digits there exists 0-sequence of length $\ge c \log n$. Theorem 5 is proved. □

Proof the Theorem 6. — We need some lemmas.

LEMMA 7. — *The probability of the event that a sequence of length $2n$ contains a sequence of zeros of length n is $p^n(1 + nq)$.*

Proof. — Consider the sequence of n consecutive zeros with minimal first index. The probability of the event that this minimal index is the first is p^n, the probability of the event that it is k $(2 \le k \le n+1)$ is qp^n because the $(k-1)$th digit must be equal to 1. Hence LEMMA 7 follows. □

LEMMA 8. — *Let $0 < \alpha_n < n$ be arbitrary and consider the sets*

$$B_k := (S_{k+\alpha_n} = S_k), \qquad (k = 0, 1, \dots, n - \alpha_n),$$

$$C_\ell := \bigcup_{k=\ell\alpha_n}^{(\ell+1)\alpha_n} B_k, \qquad \bigl(\ell = 0, 1, \dots, 2([n/\alpha_n] - 1)\bigr),$$

$$D_n := D := \bigcup_{\ell=0}^{[n/\alpha_n]-1} C_{2\ell}.$$

Then the probability of $\overline{D}$ is

$$\big[1 - p^{\alpha_n}(1 + \alpha_n q)\big]^{[n/(2\alpha_n)]}.$$

Proof. — The events $C_{2\ell}$ are independent and one of them has probability $p^{\alpha_n}(1 + \alpha_n q)$ by LEMMA 7. □

Now we give upper estimate for the probability of $\overline{D}$. Because

$$p^{\alpha_n}(1 + \alpha_n q) \to 0 \quad \text{as} \quad \alpha_n \to \infty,$$

hence

$$\Big[\big(1 - p^{\alpha_n}(1 + \alpha_n q)\big)^{(p^{\alpha_n}(1+\alpha_n q))^{-1}}\Big]^{p^{\alpha_n}(1+\alpha_n q)[n/\alpha_n]} \le \Big(\frac{1}{e}\Big)^{p^{\alpha_n}(1+\alpha_n q)[n/\alpha_n]} =: W.$$

Let $\alpha_n := \log n - \log\log\log n + K$ with some constant K, where log is of base $1/p$. Then we have

$$p^{\alpha_n} = p^K \, \frac{\log\log n}{n}$$

hence the exponent of $1/e$ in W is

$$\ge p^K \, \frac{\log\log n}{n} \, q \log n \, \frac{n}{4\log n} = \tfrac{1}{4} \log\log n \, p^K q,$$

consequently

$$W \le \Big[\Big(\frac{1}{e}\Big)^{\log\log n}\Big]^{p^K q/4}.$$

Choose $-K$ to be large enough, then the probability of $\overline{D}_{1/p^n}$ can be estimated as follows :

$$\big|\overline{D}_{1/p^n}\big| \le \Big[\Big(\frac{1}{e}\Big)^{\log\log(1/p^n)}\Big]^{p^K q/4} \le \frac{c_1}{n^{c_2 p^K}},$$

hence

$$\sum \big|\overline{D}_{1/p^n}\big| \le c_1 \sum \frac{1}{n^{c_2 p^K}} < \infty$$

so according to Borel-Cantelli lemma, almost every x belongs to finitely many $\overline{D}_{1/p^n}$ only i.e. for every $n > n_0(x)$ the first n digits contain consecutive 0-s of length α_n. THEOREM 6 is proved. □

At last we state the following questions.

Problem 3. — Investigate the behaviour of n_{i+1}/n_i for the greedy, lazy or arbitrary expansions (see [8]).

Problem 4. — Is the set investigated in THEOREM 5 residual in $[0,1]$?

Problem 5. — If q is not the root of the equations

$$1 = \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{r+1}}, \qquad r = 1, 2, \ldots$$

then

$$\inf(y_{n+1} - y_n) = 0,$$

where y_n is the strictly increasing list of the values

$$\sum_{i=1}^{k} \varepsilon_i q^i \quad k = 1, 2, \ldots; \qquad \varepsilon_i = \begin{cases} 0 \\ 1. \end{cases}$$

Problem 6. — The statement analogous to THEOREM 5 with lazy expansions and consecutive 1 digits.

Problem 7. — THEOREM 3 for greedy expansion.

Problem 8. — By [6], THEOREM 2 the set of q for which the greedy expansion of 1 contains consecutive 0-sequences of length $\geq \log_2 m$ between the first m digits for infinitely many m, is residual and of full measure in $]1, 2[$. Does it remain true if we require $\geq c \log m$ consecutive 0 between the first m digits for every $m \geq m(q)$ (the constant $c > 0$ can be chosen appropriately small)?

Problem 9. — In [14] we showed, among others, that the value q defined by

$$1 = \sum_{i=1}^{9} \frac{1}{q^i} + \sum_{j=1}^{n} \frac{1}{q^{9+10j}} + \sum_{k=1}^{\infty} \frac{1}{q^{9+10n+5k}} \qquad (n \geq 1)$$

has the property that 1 has exactly $n+1$ expansions. Describe the set of all q's having this property.

The third author is indebted to Professor Christiane FROUGNY (Paris) for her valuable discussion (in Budapest, 1991, May) on the topic of this paper and the connections of these problems with those in the theory of finite automata [9, 10, 11].

BIBLIOGRAPHIE

[1] ERDÖS (P.), RÉVÉSZ (P.). — On a problem of Tamás Varga (in Hungarian, *Mat. Lapok*, t. **24**, 1973, p. 273–282.

[2] ERDÖS (P.), RÉVÉSZ (P.). — On the length of the longest head-run, Topics in information theory, Keszthely (Hungary), *Colloq. Math. Soc. János Bolyai*, t. **16**, 1975, p. 219–228.

[3] ERDÖS (P.), RÉNYI (A.). — On a new law of large numbers, *J. Analyse Math.*, t. **22**, 1970, p. 103–111.

[4] JOÓ (I). — On Riesz bases, *Ann. Univ. Sci. Budapest Sect. Math.*, t. **31**, 1988, p. 141–153.

[5] ERDÖS (P.), HORVÁTH (M.) and JOÓ (I.). — On the uniqueness of the expansion $1 = \sum q^{-n_i}$, *Acta Math. Hungar.*, t. **58**, **3–4**, 1991.

[6] ERDÖS (P.), JOÓ (I.) and KOMORNIK (V.). — Characterization of the unique expansion $1 = \sum q^{-n_i}$ and related problems, *Bull. Soc. Math. France*, t. **118**, 1990, p. 377–390.

[7] JOÓ (I.) and JOÓ (M.). — On an arithmetical property of $\sqrt{2}$, *Publ. Math.*, t. **39/1-2**, 1991, p. 87-89.

[8] ERDÖS (P.) and JOÓ (I.). — On the expansion $1 = \Sigma q^{n_i}$, *Period. Math. Hungar.*, t. **23**, **1**, 1991, p. 27-30.

[9] FROUGNY (Ch.). — *Representation of numbers and finite automata.* — Math. Systems Theory, *to appear.*

[10] FROUGNY (Ch.). — *On the expansion of integers in non-integer basis.* — Manuscript, 1991.

[11] RAUZY (G.). — *Sur le développement en base non entière.* — Colloque de Théorie des Nombres, Alger, 1989.

[12] BOGMÉR (A.), HORVÁTH (M.) and SÖVEGJÁRTÓ (A.). — On some problems of I. Joó, *Acta Math. Hung.*, t. **58**, **1–2**, 1991.

[13] RÉNYI (A.). — Representation for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hungar.*, t. **8**, 1957, p. 173–179.

[14] ERDÖS (P.) and JOÓ (I.). — *On the number of expansions* $1 = \sum q^{n_i}$, Preprint of the Math. Inst. of the Hung. Acad. of Sci., n° 30, 1991.