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HENRY HELSON

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SECTIONS IN FUNCTION SPACES

BY

HENRY HELSON (*)

[University of California, Berkeley]

RÉSUMÉ. — La non-existence de moyennes d'un certain type dans certains espaces de suites ou de fonctions réelles est démontrée par la construction de cocycles additifs sur un système dynamique qui satisfont des conditions de croissance convenables.

SUMMARY. — The fact that certain spaces of real-valued sequences or functions do not possess a certain kind of mean is proved by constructing non-trivial additive cocycles satisfying appropriate growth conditions on a dynamical system.

1. In this paper we prove that function spaces do not have means of a certain kind by constructing non-trivial additive cocycles of slow growth on dynamical systems. We give a new criterion to show these cocycles are not coboundaries. A recent result of HAMACHI, OKA and OSIKAWA ([1], [2]) is used to sharpen our result, and a new proof is given of their Theorem. Finally, we try to formulate our method in a very general setting, where the problem of replacing a cocycle by a cohomologous one of simpler form leads to questions with a metamathematical flavor.

2. Let L be a topological vector space of real-valued sequences that contains constant sequences and is invariant under translation. A *section* of L will mean a Borel subset L_0 that is invariant under translation, and such that each f in L has a unique representation $f_0 + a$, with f_0 in L_0 and a real. We ask whether particular sequence spaces have sections.

If L is l^∞, then the set of all sequences f such that $\limsup_{n \to \infty} f(n) = 0$ constitutes a section. (Note that a section is not required to be a subspace.) Similarly, L has a section if it consists of sequences having some generalized limit at infinity, providing the limiting process is *effective* so that L_0 is a Borel set. If L contains sequences tending to infinity no ordinary limiting process can be applicable; our result will be that no section exists in some such spaces.

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Henry HELSON, Department of mathematics, University of California, Berkeley, Ca. 94720, États-Unis.

Let L_0 be a section of L. Define a real function Φ on L by setting $\Phi(f_0+a) = a$ for f_0 in L_0 and a real. Then Φ is invariant under translation, and $\Phi(f+a) = \Phi(f)+a$ for all f in L and all a. If L is a Polish space, then Φ is a Borel function. For the mapping from L onto $L_0 \times R$ that carries f_0+a to (f_0, a) is a Borel isomorphism by Souslin's theorem, and the mapping from (f_0, a) to a is continuous.

Conversely, if Φ is a Borel function on L with the properties mentioned, then the set of all f such that $\Phi(f) = 0$ is a section.

A *lattice* in L is a Borel subset L_1 invariant under translation and such that each element of L has a unique representation f_1+a with f_1 in L_1 and $0 \leqslant a < 1$. If L_0 is a section, then the union of the sets L_0+n $(n = 0, \pm 1, \ldots)$ is a lattice. The spaces we study do not even contain lattices. This is a little unexpected on account of the analogy with an irrational flow on a torus. This flow has no cross-section but does have the analogue of a lattice: a Borel set intersecting each orbit in an arithmetic progression. (The annihilator of any non-zero character has this property.)

Our main result about sections and lattices is the following theorem.

THEOREM 1. — *Let ρ be a positive even sequence tending to infinity such that $\rho(n)/\rho(n+1)$ is bounded from* 0 *and from* ∞. *Let L be the Banach space of sequences f for which the norm*

$$\|f\| = \sup_n |f(n)\rho(n)^{-1}| \tag{1}$$

is finite. Then L has no lattice, and a fortiori *no section.*

This theorem will imply the same result for other spaces of sequences and functions. The proof depends on this criterion for the non-triviality of additive cocycles on a dynamical system.

THEOREM 2. — *(X, B, τ, μ) is an ergodic dynamical system. G is a locally compact abelian group and v is a measurable function from X to G taking finitely many values $a_1, \ldots, a_n$. Let G_0 be the subgroup of G generated by the a_j. Denote by r the smallest positive integer of the form Σk_j where the k_j are integers and $\Sigma k_j a_j = 0$, or $r = 0$ if no such sum is positive. Assume G_0 is closed in G and $r > 1$, and further that τ^r is ergodic. Then v is not a coboundary; that is, v does not have the form $w(\tau x)-w(x)$ with w measurable from X to G.*

In the next section we show the connection between Theorem 1 and non-triviality of cocycles. In paragraph 4, we use Theorem 1 to prove the non-existence of lattices in various spaces. Theorem 2 is proved in paragraph 5. The cocycles needed to prove Theorem 1 are constructed in paragraph 6. A version of the Theorem of HAMACHI, OKA and OSIKAWA is stated and proved in paragraph 7. In the last section Theorem 1 is put into a more general perspective.

I am grateful to the referee for a simplification in the proof of Theorem 2, and for the proof, under mild assumptions on G, that the case $r = 0$ cannot occur.

3. (X, B, τ, μ) is a *dynamical system* if X is a set, B a σ-field of subsets of X, τ a bimeasurable automorphism of X, and μ a probability measure on B invariant under τ. For v a real measurable function on X define

$$\begin{aligned} v(n, x) &= \textstyle\sum_0^{n-1} v(\tau^k x) \qquad (n = 1, 2, \ldots), \\ v(m+n, x) &= v(m, x) + v(n, \tau^m x) \qquad \text{(all integers } m, n). \end{aligned} \tag{2}$$

This is an *additive cocycle* on X.

Let v be a cocycle on X that, as a function of n, belongs to a space L for almost every x. Suppose L has a section L_0 with functional Φ. If we define $w(x) = \Phi \circ v(., x)$ and apply Φ to both sides of the equation

$$v(n+1, x) = v(n, \tau x) + v(x), \tag{3}$$

we get

$$w(x) = w(\tau x) + v(x), \tag{4}$$

because Φ is invariant under translation and commutes with addition of the constant $v(x)$. If w is measurable, this shows v is a coboundary.

The space L of Theorem 1 is a Banach space, so Φ is a Borel function. Thus w will be measurable if we show that the mapping from x to $v(., x)$ is a measurable function from X to L. For any sequence f in L and positive number k the set of x such that

$$\sup_n |v(n, x) - f(n)| \rho(n)^{-1} \leqslant k \tag{5}$$

is measurable. That is, the inverse image of each ball in L is measurable, so the mapping is measurable.

Thus if L has a section, then each cocycle with values in L is a coboundary. To prove that L has no section it will suffice to find a cocycle on any dynamical system with values in L that is not a coboundary.

Suppose L has a lattice L_1. Define a functional Ψ on L by setting

$$\Psi(f_1+a) = \exp 2\pi i a \quad \text{for } f_1 \text{ in } L_1 \text{ and } 0 \leqslant a < 1.$$

Then

$$\Psi(f+a) = \Psi(f) \exp 2\pi i a \quad \text{for all } f \text{ in } L \text{ and real } a.$$

Define $q(x) = \Psi \circ v(., x)$, a function of modulus one on X. Applying Ψ to (3) we find

$$q(x) = q(\tau x) \exp 2\pi i v(x), \tag{6}$$

the multiplicative analogue of (4). Thus $\exp 2\pi i v$ is a coboundary in a multiplicative sense. By the same argument, $\exp i\lambda v$ is a multiplicative coboundary for every real λ.

The Theorem of HAMACHI, OKA, OSIKAWA ([1], [2]) referred to above asserts that v must then be an additive coboundary, provided the dynamical system satisfies certain mild hypotheses. Thus L has no lattice either, if there is a non-trivial additive cocycle on a suitable dynamical system having values in L.

4. Theorem 1, once proved, will imply the non-existence of lattices in other spaces. If L is embedded as a Borel subset in L', and if L has no lattice, then L' cannot have one either. For the intersection of a lattice in L' with L forms a lattice in L. Here is an application of this remark.

THEOREM 3. — *Let γ be a positive sequence satisfying*

$$\sum \gamma(n) < \infty, \qquad \gamma(n)/\gamma(n+1) \textit{ bounded from } 0 \textit{ and } \infty. \tag{7}$$

Then the weight space l^p_γ has no lattice for $1 \leqslant p < \infty$.

The hypothesis implies that l^p_γ contains constant sequences and admits translation. By elementary means, we can find a sequence ρ satisfying the hypotheses of Theorem 1 and such that

$$\sum \rho(n)^p \gamma(n) < \infty. \tag{8}$$

Thus the space L of Theorem 1 based on this function ρ is contained in l^p_γ, and the result follows from Theorem 1.

Sequence spaces are naturally embedded in spaces of functions defined on the line, by identifying a sequence with a function constant on intervals between successive integers. Thus Theorems 1 and 3 have analogues for function spaces on the line.

Here is one consequence of the continuous version of Theorem 3. For f a real function in L^p on the circle, let F be the harmonic extension of f to the disc. We ask for a functional Φ defined on L^p with some of the properties of $\lim_{z\to 1} F(z)$. Ordinarily a generalized limit is required to be linear, and to extend the ordinary limit. We require instead that Φ should be a Borel functional invariant under conformal maps of the disc that leave $z = 1$ fixed, and satisfying

$$\Phi(f+a) = \Phi(f)+a \quad \text{for all } f \text{ and } a \text{ real.}$$

Our result is that for p finite no such functional exists.

5. We prove Theorem 2.

Since G_0 is countable and closed, its topology as a subset of G is discrete. Choose α to satisfy $\alpha^r = 1$, $\alpha \neq 1$. Then a character is defined on G_0 by setting

$$\chi(\textstyle\sum_1^n k_j a_j) = \alpha^{k_j}. \tag{9}$$

We extend χ to be a continuous character on G. If v is a coboundary, say $v(x) = w(\tau x)-w(x)$, we have

$$\begin{cases} \chi(w(\tau\, x)-w(x)) = \alpha \neq 1, \\ \chi(w(\tau^r x)-w(x)) = \alpha^r = 1. \end{cases} \tag{10}$$

Since τ^r is ergodic the second equation implies that $\chi \circ w$ is almost everywhere a constant; but the first equation shows that this is not the case. Hence v is not a coboundary.

In our application v will be a real function with values 1, -1, which gives $r = 2$. For this case the proof above can be given even more simply.

6. THEOREM 4. — *Let (X, B, τ, μ) be an aperiodic dynamical system, and let ρ be any positive sequence tending to infinity. There is a measurable function v on X taking the values* 1, -1, *such that $v(n, x) = 0(\rho(n))$ for almost every x.*

First we use this result to complete the proof of Theorem 1. Choose a dynamical system such that τ^2 is ergodic and $L^2(X, \mu)$ is separable. Theorem 2 shows that the cocycle of Theorem 4 is not an additive coboundary. But the values of the cocycle are in L for almost every x, so L has no section. The theorem of the next part gives the stronger statement that $\exp i\lambda v$ is non-trivial in the multiplicative sense for some real λ, so that L does not even have a lattice.

We prove Theorem 4. Construct a Rohlin tower for the dynamical system with two floors, whose union we call T_1, and a residual set C_1 of measure at most ε_1. X is the disjoint union of T_1 and C_1.

A transformation τ_1 is induced on C_1 by defining $\tau_1 x$ to be the first $\tau^k x$ $(k > 0)$ belonging to C_1. The restriction μ_1 of μ to C_1 is invariant under τ_1, and the new dynamical system is aperiodic. Now represent C_1 as a tower with two floors (whose union is T_2) and a residual set C_2 of measure at most ε_2. X is the disjoint union of T_1, T_2 and C_2.

We continue inductively. Choose the numbers ε_k tending to zero, so that X is the disjoint union of T_1, T_2, ... Define $v(x)$ to be 1 on the lower floor of each T_k, and -1 on each upper floor. We shall prove that v has the property of the theorem if ε_k tends to zero rapidly enough.

Without loss of generality assume $\rho(n)$ is non-decreasing and takes integer values. Let N_k be the smallest N such that $\rho(N) \geqslant k+1$.

LEMMA. — *$|v(n, x)| \leqslant k < \rho(N_k)$ for $1 \leqslant n \leqslant N_k$ except for x belonging to a set E_k of measure at most $\varepsilon_k N_k$.*

Let x be a point such that $\tau^j x$ belongs to $T_1 \cup \ldots \cup T_k$ for $0 \leqslant j < N_k$. In the sum $v(n, x) = \Sigma_0^{n-1} v(\tau^j x)$, we group together the terms such that $\tau^j x$ is in a particular T_i. Adjacent terms in the group have opposite sign by the definition of v and the properties of a Rohlin tower. Thus each group contributes 1, 0 or -1 to the sum, whose modulus can therefore not exceed k. The exceptional set E_k consists of points x such that $\tau^j x$ is in C_k for some $j < N_k$, of measure at most $\varepsilon_k N_k$. This proves the lemma.

For $N_k < n < N_{k+1}$ we have except in E_{k+1},

$$|v(n, x)| \leqslant k+1 = \rho(N_k) = \rho(n). \tag{11}$$

Thus $|v(n, x)| = O(\rho(n))$ for each x not in infinitely many E_k. Let the ε_k satisfy

$$\textstyle\sum_1^\infty \varepsilon_k N_k < \infty; \tag{12}$$

the Borel-Cantelli lemma asserts that the points in infinitely many E_k form a null set. This completes the proof of the theorem.

Certain of the non-trivial cocycles constructed in [3] have the property that $|v(n, x)| = O(n^\varepsilon)$ almost everywhere, for any positive fixed ε. G. RAUZY has investigated the growth of $v(n, x)$ when X is $[0, 1)$, $\tau x = x+\alpha \pmod 1$ with α irrational, and $v(x) = 1$ on $[0, 1/2)$, -1 on $[1/2, 1)$. For suitable α, the order can be as small as $\log n$ but, by known results in Diophantine approximation, never of smaller order.

7. THEOREM 5. — *Let* (X, B, τ, μ) *be an ergodic dynamical system such that* $L^2(X, \mu)$ *is separable. Suppose* v *is a real measurable function on* X *with the property that* $\exp i\lambda v$ *is a multiplicative coboundary for each real* λ. *Then* v *is an additive coboundary.*

This statement was communicated by William PARRY, who observed that it is a version of the result presented as Theorem 3 in [1] and Proposition 1 in [2].

For each real λ there is a unitary function q_γ on X such that

$$\exp i\lambda v(n, x) = q_\lambda(\tau^n x)\bar{q}_\lambda(x) \quad \text{a.e. } (n = 1, 2, \ldots). \tag{13}$$

Since τ is ergodic each q_λ is determined up to a multiplicative constant, and $q_{\lambda+\zeta}$ is a constant multiple of $q_\lambda q_\zeta$. Let $M(\lambda)$ be the mean value of q_λ. By the ergodic theorem

$$p_\lambda(x) = \lim_{N\to\infty} N^{-1}\textstyle\sum_1^N \exp i\lambda v(n, x) \tag{14}$$

exists a. e., and equals $M(\lambda)\bar{q}_\lambda(x)$ a. e. for each λ. The function $p_\lambda(x)$ is measurable on $R\times X$ relative to the product of the Borel field on R with B. Hence the set E of λ where $M(\lambda) \neq 0$ is a Borel subset of R.

It is not difficult to prove that E has positive measure. The proof can be avoided if we multiply (13) by $h(\tau^n x)$, so that $M(\lambda)$ is the mean value of hq_λ. It is immediate that $M(\lambda) \neq 0$ for λ in a set of positive measure, for some h in $L^2(X, \mu)$. The separability of $L^2(X, \mu)$ is necessary at this point.

Now define $u_\lambda(x, y) = q_\lambda(x)\bar{q}_\lambda(y)$. We have $u_{\lambda+\zeta} = u_\lambda u_\zeta$ almost everywhere on $X\times X$ for each real λ and ζ. Also $u_\lambda(x, y) = p_\lambda(x)^{-1}p_\lambda(y)$ if λ is in E. Thus u_λ is a measurable mapping of E to $L^2(X\times X)$. By the functional equation u_λ is measurable as well on any translate of E. The family of measurable subsets of R on which u_λ is measurable forms a σ-algebra, and it is not hard to see that it contains all Borel subsets of R. By changing $u_\lambda(x, y)$ on null sets of (x, y) for each λ, we can assume the function is measurable as a numerical function on $R\times X\times X$.

By the Fubini theorem, there is a y_0 so that for almost every x

$$u_{\lambda+\zeta}(x, y_0) = u_\lambda(x, y_0)u_\zeta(x, y_0) \quad \text{for almost every } (\lambda, \zeta). \tag{15}$$

For such x there is a real number $w(x)$ such that $u_\lambda(x, y_0) = \exp i\lambda w(x)$ (almost all λ), and the function w so defined is measurable. Thus we have

$$\exp i\lambda v(x) = u_\lambda(\tau x, y_0)\bar{u}_\lambda(x, y_0) = \exp i\lambda[w(\tau x) - w(x)] \tag{16}$$

for almost all (λ, x). By continuity, the relation holds identically in λ for almost every x. Thus we have shown that v is a coboundary.

The theorem has an interesting reformulation. Let B be the Bohr group, dual to the discrete real line R_d. B_0 is the distinguished one-parameter subgroup of B whose elements e_t (t real) are defined by

$$e_t(\lambda) = e^{it\lambda} \qquad (\lambda \text{ in } R_d). \tag{17}$$

A measurable function h from X to B is called *coherent* if $h(\tau x) - h(x)$ is in B_0 for almost every x. For example, h is coherent if its values lie in a single coset of B_0. Our result essentially says that every coherent function has this form.

A difficulty arises from the fact that B has no countable neighborhood base at 0. Say that a mapping k from X to B is a *null function* if $\lambda \circ k(x) = 1$ a. e. on X for each λ in R_d. (Unfortunately we cannot say that $k(x) = 0$ a. e.) We can prove the following theorem.

THEOREM 6. — *Under the hypotheses of Theorem* 5, *every coherent mapping h from X to B differs by a coherent null function from a function taking its values almost everywhere in a single coset of B_0.*

Define a *coordinate function* on B_0 by setting $c(e_t) = t$. Given a coherent mapping h, set $v(x) = c[h(\tau x) - h(x)]$. Then

$$\exp i\lambda v(x) = \lambda \circ h(\tau x)/\lambda \circ h(x),$$

a multiplicative coboundary for each λ. By Theorem 5 there is a measurable real function w such that $v(x) = w(\tau x) - w(x)$ a. e. Set $\tilde{w}(x) = e_{w(x)}$. Then the last equation means $\tilde{w}(\tau x) - \tilde{w}(x) = h(\tau x) - h(x)$ a. e. That is, the function $k(x) = \tilde{w}(x) - h(x)$ is invariant under τ. Since the dynamical system is ergodic, $\lambda \circ k(x)$ is constant a. e. for each λ. This constant is a multiplicative function of λ, and so defines an element y of B:

$$\lambda \circ k(x) = y(\lambda) \quad \text{a.e. for each } \lambda.$$

Thus $k(x) - y$ is an invariant null function, and the theorem is proved.

The hypothesis of separability in Theorems 5 and 6 cannot be omitted. Take for X the group dual to the discrete circle and for τ an ergodic translation. Then $\exp i\lambda$ (that is, $v(x)$ identically 1) is a multiplicative coboundary for each λ, but v is not an additive coboundary. In the other language, there is a coherent mapping h from X to B not taking values in a single coset of B_0. For each character x of the discrete circle, let $h(x)$ be the character of R_d defined by $h(x)(\lambda) = x(\exp i\lambda)$.

The idea involed in Theorem 6 can be carried further v. Let Γ be a *countable* dense subgroup of the line. Give Γ the discrete topology, and call its dual K. Define the distinguished subgroup $K_0 = (e_t)$ as above. To the coherent mapping h from X into K associate the real function $v(x) = c[h(\tau x) - h(x)]$. For each λ in Γ, $\exp i\lambda v = \lambda \circ h(\tau x)/\lambda \circ h(x)$, a multiplicative coboundary. It is not difficult to show, using the fact that Γ is countable, that every function v such that $\exp i\lambda v$ is a multiplicative coboundary for all λ in Γ is thus obtained from a coherent mapping h of X to K. Two coherent mappings determine the same v if and only if they differ by a constant element of K. Finally, v is an additive coboundary if and only if h takes its values in a single coset of K_0. The fundamental theorem of H. DYE on weak equivalence of dynamical systems shows that in ordinary cases there are coherent mappings whose values do *not* lie in a coset of K_0. Thus Theorem 5 can be viewed as the statement in a dramatic way that Dye's theorem cannot be extended to a particular non-separable context.

Our remarks lead to this result: *if* $\exp i\lambda v$ *is a multiplicative coboundary for each* λ *in* Γ, *a dense subgroups of* R, *and if* ζ *is any non-zero element of* Γ, *then* v *is additively cohomologous to a function* v' *whose values are multiples of* $2\pi/\zeta$. For each element of K has a unique representation in the form $y + e_t$ with $y(\zeta) = 1$ and $0 \leqslant t < 2\pi/\zeta$. Let h be the coherent mapping of X to K associated with v; define $r(y+e_t) = y$ and $h' = r \circ h$. Then h' is coherent and the corresponding function v' is cohomologous to v (because h'-h has values in K_0). But $h'(\tau x) - h'(x) = e_u$ where u is a multiple of $2\pi/\zeta$, which proves the assertion.

In the same way, any effective procedure for changing the value of $h(x)$ within the same coset of K_0 leads to a function cohomologous to v. We think of $h(\tau^n x)$ as a function of n with values in a line that has a scale of distance but no origin. An "effective procedure" is a Borel operation that transforms sequences in some space to sequences in another, preferably smaller space. The operation should commute with translation in n, and with translation in the range space. The functionals Φ of paragraph 2 transformed sequences to constants, which we identify with constant sequences, and were required to have exactly these properties. Now we want to generalize Theorem 1 to cases where the operation is allowed to be of more general type.

8. A *cone* will be a family of real sequences containing all constant sequences, invariant under translation, closed under addition and multiplication by positive scalars, and endowed with a Borel structure consistent with these operations. A *homomorphism* from a cone L to a cone M will be a Borel mapping from L to M that commutes with translation and with the addition of constants.

A cone is *measurable* if the coordinate functions are Borel functions on the cone; and if in addition each additive cocycle $v(n, x)$ belonging to the cone as a function of n for each x defines a mesurable mapping from X to the cone.

THEOREM 7. — *Suppose a cocycle $v(n, x)$ takes values in a measurable cone L, and suppose there is a homomorphism of L into a measurable cone M. Then v is cohomologous to a cocycle with values in M.*

Let $w(., x)$ be the element of M obtained by applying the homomorphism to $v(., x)$. The properties of a homomorphism imply that

$$v(x) = w(k+1, x) - w(k, \tau x) \quad \text{for each } k.$$

The function $w(x) = w(0, x)$ is measurable on X, and by induction we find

$$w(n, x) = v(n, x) + w(\tau^n x).$$

This sequence is in M for each x, so

$$w(n, x) - w(x) = v(n, x) + [w(\tau^n x) - w(x)]$$

is also in M, and this is a cocycle cohomologous to v as required.

This theorem enables us to prove that homomorphisms between certain cones do not exist, by constructing appropriate cocycles.

THEOREM 8. — *Let L be the space of Theorem* 1, *M_0 the cone of constant sequences, and M_1 the cone of non-decreasing sequences in the topology of uniform convergence, or the topology of pointwise convergence. There is no homomorphism of L into M_1, nor of M_1 into M_0.*

All the cones are measurable. Take $v = 1$ on any dynamical system. Then $v(n, x)$ is in M_1 for each x. If there were a homomorphism of M_1 to M_0 v would be a coboundary, which is not the case.

Let v be the function of Theorem 4 on a dynamical system such that v is not a coboundary. If there is a homomorphism from L to M_1, then for some measurable function w we have $v(x) + w(\tau x) - w(x) \geqslant 0$ a. e. By the ergodic theorem,

$$\lim_{n \to \infty} n^{-1} [v(n, x) + w(\tau^n x) - w(x)] \tag{18}$$

exists a. e., and is evidently non-negative. Since v has mean value 0 by construction, $\lim n^{-1} v(n, x) = 0$ a. e. Also $\lim n^{-1} w(x) = 0$. Hence $\lim n^{-1} w(\tau^n x)$ exists a. e. and equals the limit in (18).

Now $\liminf | w(\tau^n x)$ is finite a. e. Thus the limit in (18) is 0 a. e. It follows that the non-negative function $v(x)+w(\tau x)-w(x)$ has integral 0 and therefore vanishes a. e. That is, v is a coboundary. This contradiction shows there is no homomorphism from L to M_1.

Let P be the space of all real polynomials, restricted to the integers, with the topology of pointwise convergence. Let P_k be the subspace of polynomials of degree at most k (so that P_0 is the same as M_0 above).

THEOREM 9. — *There is no homomorphism of L, the space of Theorem* 1, *into P, and so* a fortiori *there is none into P_k for any* $k = 0, 1, 2, \ldots$

The case $k = 0$ was Theorem 1. Suppose there is a homomorphism of L into P_k for some positive k. Let v be a non-trivial cocycle with values in L, say the one constructed in Theorem 4. As in the proof of Theorem 7

$$v(n, x)+w(\tau^n x) = w(n, x) = \textstyle\sum_0^k a_j(x)\, n^j, \tag{19}$$

where $w(., x)$ is the image of $v(., x)$ under the homomorphism. Let a_p be the last coefficient that is not zero a. e. Then $p > 0$ because v is not a coboundary. Divide (19) by n^p, and let n tend to ∞. Since v has mean value 0, the first term on the left tends to 0. For the second term, $\limsup \geqslant 0$ and $\liminf \leqslant 0$. Hence $a_p = 0$, a contradiction. This proves there is no homomorphism of L to P_k.

Finally suppose there is a homomorphism of L to P. We have (19) again, but the degree $k(x)$ depends on x and can be arbitrarily large. However replacing n by $n+1$ in (19) shows that the degree is invariant under τ. Thus $k(x)$ is an invariant function, and we verify that it is measurable. If the dynamical system is ergodic, k is constant; that is, the homomorphism carries $v(., x)$ to a single P_k for almost every x. This was shown to be impossible in the first part of the proof.

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