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MINIMAL PURE SUBGROUPS IN PRIMARY GROUPS ;

BY

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Throughout all groups are assumed to be primary abelian groups, and all topological references are to the p-adic topology. By a subsocle of a group we mean a subgroup of the socle. Thus S is a subsocle of G if S is a subgroup of G and if $px = 0$ for all x in S. Let H be a subgroup of G. If among the pure subgroups of G which contain H there exists a minimal one, we say that H is contained in, or is imbedded in, a minimal pure subgroup in G. B. CHARLES studied minimal pure subgroups in [1]; he asserted that each of the conditions

(1) H is a subsocle of G

and

(2) There is a pure subgroup of G contained in H which is dense in H

is sufficient for the existence of a minimal pure subgroup for H in G provided G is without elements of infinite height. Head showed in [4] that condition (2) is not sufficient, and one of the authors showed in [6] that neither is condition (1).

In this paper we characterize the groups G in which each subgroup is imbedded in a minimal pure subgroup. The characterization is : G is the sum of a divisible and a bounded group. We give a short proof of a theorem of IRWIN and WALKER [5] and give a solution to a new generalization of Fuchs' Problem 4. Some results are also given concerning minimal pure subgroups for subsocles.

It was shown in [3] that most groups have neat dense subgroups which do not contain basic subgroups. The following theorem shows, however, that if a neat subgroup has a dense subsocle, then it must contain a basic subgroup.

THEOREM 1. — *Let S be a dense subsocle of G, $\overline{S} = G[p]$. If H is maximal in G with respect to $H[p] = S$, then H is pure and dense in G.*

PROOF. — Let H be maximal in G with respect to $H[p] = S$. Then H is neat in G, that is, $H \cap pG = pH$. We need to show that $H \cap p^n G = p^n H$ for all natural numbers n; our proof is by induction. Assume that $H \cap p^n G = p^n H$ and suppose that $p^{n+1} x \in H$. Since H is neat, there is an $h_0 \in H$ such that $p^{n+1} x = p h_0$. The element $p^n x - h_0$ is in $G[p]$. Since S is dense in $G[p]$, there is an $s \in S$ such that $p^n x - h_0 - s$ is in $p^n G$. By the induction hypothesis, there is an $h_1 \in H$ such that $p^n h_1 = h_0 + s$. Thus $p^{n+1} h_1 = p h_0 = p^{n+1} x$ and H is pure.

Since H is pure, any element of order p in G/H can be represented by an element of order p in G. Therefore, the density of $H[p] = S$ in $G[p]$ implies that each element of order p in G/H has infinite height. Hence G/H is divisible, that is, H is dense in G.

COROLLARY 1 (IRWIN and WALKER [5]). — *Let N be a subgroup of G^1, the elements of infinite height in G. If H is maximal in G with respect to $H \cap N = 0$, then H is pure in G.*

PROOF. — The maximality of H implies that H is neat. Thus H cannot be enlarged without enlarging its socle. Since $G[p] = H[p] + N[p]$, $H[p]$ is dense in $G[p]$.

One may generalize problem 4 in [2] by replacing the subgroup G^1 by an arbitrary fully invariant subgroup. The solution to the generalized problem is contained in the following corollary and a well known result of Szele.

COROLLARY 2. — *Let F be a fully invariant subgroup of G and let A be a subgroup of G such that $A \cap F = 0$. Then A is contained in a pure subgroup H of G such that $H \cap F = 0$.*

PROOF. — If $F \subseteq G^1$, the conclusion follows from the preceding corollary. Assume that F is not contained in G^1. Let $\sum B_n$ be the standard decomposition of a basic subgroup B of G into homogeneous groups B_n. Define $A_1 = G$ and $A_{n+1} = \{B_{n+1}, B_{n+2}, \ldots, p^n G\}$ for $n \geq 1$. Then $G = B_1 + B_2 + \ldots + B_n + A_{n+1}$. Since F is fully invariant with elements of finite height in G, $F[p] = A_m[p]$ where m is the smallest positive integer such that $F \cap B_m \neq 0$.

It follows from [2] (theorem 22.2) that A_m is an absolute direct summand of G. Hence if H is maximal with respect to $H \cap F = 0$, then H is maximal with respect to $H \cap A_m = 0$ and is a direct summand of G; in particular, H is pure in G.

The following theorem, which is of independent interest, (eventually) implies that most groups have subgroups which are not imbedded in minimal pure subgroups.

THEOREM 2. — *Let L be a subgroup of G. If H is a minimal pure subgroup of G containing L, then $H = A + K$ where A is bounded and $K[p] = L[p]$.*

PROOF. — There is no pure subgroup of H properly between L and H. It follows from theorem 1 that every subsocle of H which contains $L[p]$ is closed in $H[p]$.

Define $S_n = L[p] \cap p^n H$ and let $S_n = Q_n + S_{n+1}$ for $n = 0, 1, 2, \ldots$. The height in H of each nonzero element of Q_n is exactly n. Moreover, if C_n is (zero or) a direct sum of cyclic groups or order p^n such that $C_n[p] = Q_{n-1}$, then $C = \sum C_n$ is pure in H. Extend C to a basic subgroup $B = A + C$ of H.

Suppose that there is an element x of order p in $A \cap L$. Since x is in A, it has finite height t in H. The closure (in H) of $C[p]$ contains $L[p]$. Thus $x = p^{t+1}h + c$ where $c \in C[p]$ and $h \in H$. This implies that $x - c$ has height greater than t in H and, consequently, in B since B is pure in H. This is impossible since $B = A + C$, so $A \cap L = 0$.

Assume that A is unbounded. Then is has a proper basic subgroup A_1. Since $B = A + C$ is basic in H, $B_1 = A_1 + C$ is basic in H. Thus

$$\overline{A_1[p] + L[p]} \supseteq \overline{B_1[p]} = H[p].$$

Since this contradicts the fact that $A_1[p] + L[p]$ is a proper closed subsocle of H, we conclude that A is bounded.

Let $p^m A = 0$. An argument similar to the one given above for the proof that $A \cap L = 0$ shows that $A \cap \{C, p^m H\} = 0$. Now we have that

$$H = \{B, p^m H\} = \{A + C, p^m H\} = A + \{C, p^m H\}.$$

Define $K = \{C, p^m H\}$. The purity of C implies that

$$K[p] = \{C[p], p^m H[p]\}.$$

Since $L[p]$ is closed in the socle of H, $C[p]$ does not have limit points in the socle of H outside of $L[p]$. But $p^m C[p]$ is dense in $p^m H[p]$ since $p^m C$ is basic in $p^m H$. Thus $p^m H[p] \subseteq L[p]$ and therefore $K[p] \subseteq L[p]$. Since $H[p] = A[p] + K[p]$ and since $A \cap L = 0$, it follows that $K[p] = L[p]$.

PROPOSITION 1. — *If each subgroup of G is contained in a minimal pure subgroup of G, then G has a bounded basic subgroup.*

PROOF. — Suppose that $B = \sum B_n$ is a basic subgroup of G where $B_n \neq 0$ for infinitely many n and is a homogeneous group of degree n. Choose a sequence $n(i)$ of positive integers such that $n(i+1) - n(i) \geqq 2$ and such that $B_{n(i)} \neq 0$. Define

$$t(i) = n(2i+1) - n(2i) - 1$$

and let

$$L = \sum_{i=1}^{\infty} \{ b_{n(2i)} + p^{t(i)} b_{n(2i+1)} \}$$

where $\{ b_{n(i)} \}$ is a nonzero direct summand of $B_{n(i)}$. Suppose that H is a minimal pure subgroup of G containing L. By theorem 2, $H = A + K$ where A is bounded and $K[p] = L[p]$.

Let $p^m A = 0$. Then $p^m H[p] \subseteq L[p]$. Let $G = \{ b_{n(2i)} \} + \{ b_{n(2i+1)} \} + G_0$. Since $p^{n(2i+1)-1} b_{n(2i+1)}$ is in L, there is an element $h_0 = j b_{n(2i)} + b_{n(2i+1)} + g_0$ in H where j is an integer, $g_0 \in G_0$, and

$$p^{n(2i+1)-1} h_0 = p^{n(2i+1)-1} b_{n(2i+1)}.$$

Now the element

$$h_1 = (b_{n(2i)} + p^{t(i)} b_{n(2i+1)}) - p^{t(i)} h_0 = b_{n(2i)} - p^{t(i)} (j b_{n(2i)} + g_0)$$

is in H. Since $p^{n(2i)-1} h_1 = p^{n(2i)-1} b_{n(2i)}$ and since $p^m H[p] \subseteq L[p]$, we conclude that L contains $p^{n(2i)-1} b_{n(2i)}$ if $i \geqq m$. However, it is immediate from the definition of L that this is impossible, so L is not contained in a minimal pure subgroup of G.

PROPOSITION 2. — *If G is a bounded group, each subgroup of G is contained in a minimal pure subgroup of G.*

PROOF. — Our proof is by induction on n where $p^n G = 0$. If $p G = 0$, every subgroup is pure. Suppose that L is a subgroup of G and that $p^{n+1} G = 0$. Since a homogeneous subgroup of G of degree $n+1$ is an absolute direct summand, we may assume that $p^n G \subseteq L$.

Let

$$L = L_{n+1} + C_n,$$
$$p G \cap C_n = L_n + C_{n-1},$$
$$\ldots\ldots\ldots\ldots,$$
$$p^n G \cap C_1 = L_1,$$

where L_i is a homogeneous group of degree i with L_{n+1} chosen maximal in L and L_i chosen maximal in $p^{n+1-i} G \cap C_i$ for $i = n, n-1, \ldots, 1$. Observe that there are homogeneous subgroups B_i of G of degree $n+1$

such that $B_i \supseteq L_i$ and $B_i[p] = L_i[p]$. Define $B = \sum B_i$. Then $B[p] = p^n G \cap L = p^n G$.

Since B is an absolute direct summand of G, there are decompositions

$$\{L, B\} = K + B$$

and

$$G = H + B$$

such that $H \supseteq K$. Since $p^n G = B[p]$, $p^n H = 0$. By the induction hypothesis, K is contained in a minimal pure subgroup A of H. We prove that $A + B$ is a minimal pure subgroup of G containing L.

Suppose that S is a pure subgroup of $A + B$ containing L. We wish to show that $S = A + B$. Proceeding by induction, assume that $p^i A \subseteq S$ and that $p^{i+1} B \subseteq S$. From these two conditions it follows that $p^i B \subseteq S$, and it remains to show that $p^{i-1} A \subseteq S$. Routine considerations show that it suffices to prove that $p^{i-1} A[p] \subseteq S$.

Let $T = S \cap p^{i-1} A[p]$ and let $p^{i-1} A[p] = T + R$. Assume that $R \neq 0$. Choose a pure subgroup $R^\star$ of A such that $R^\star[p] = R$. Observe that $R^\star$ is homogeneous of degree i. From the construction of B, it can be shown that $p^{i-1} A \cap K \subseteq \{L, p^i B\}$. From this fact it follows that $R^\star \cap \{p^i A, K\} = 0$. Choose a subgroup $F \supseteq \{p^i A, K\}$ and maximal in A with respect to $F \cap R^\star = 0$. Since A is minimal pure for K in H, F cannot be pure in A. Hence $R^\star + F$ is a proper subgroup of A. Choose an element $a \in A$ such that $a \notin R^\star + F$ and such that $pa \in R^\star + F$. Letting $pa = r^\star + f$ where $r^\star \in R^\star$ and $f \in F$, we obtain contradictory statements : $r^\star$ has height zero in $R^\star$; and $p^{i-1} r^\star = 0$. We conclude that $R = 0$, that is, $p^{i-1} A[p] \subseteq S$.

COROLLARY 3. — *Let L be a subgroup of G. If the heights (computed in G) of the elements of L are bounded, then L is contained in a minimal pure subgroup (direct summand) of G.*

PROOF. — There is a positive integer n such that $L \cap p^n G = 0$. The group $p^n G$ is a fully invariant subgroup of G. Apply corollary 2 and proposition 2.

Now consider the case where G is the sum of a divisible group D and a bounded group B, $G = D + B$. Let L be a subgroup of G. In order to show that L is contained in a minimal pure subgroup of G, we may assume that $D[p] \subseteq L$ since a divisible subgroup is an absolute direct summand. In this case, H is minimal pure for L if H/D is minimal pure for $\{L, D\}/D$ in G/D, a bounded group. This completes the proof of

THEOREM 3. — *Each subgroup of G is contained in a minimal pure subgroup of G if and only if G is the sum of a divisible group and a bounded group.*

We now turn our attention to the question of the existence of minimal pure subgroups for subsocles. Theorem 2 shows that if a subsocle S is imbedded in a minimal pure subgroup in G, then S supports a pure subgroup, that is, there is a pure subgroup H of G such that $H[p] = S$. Thus the question of whether or not a subsocle is imbedded in a minimal pure subgroup is just the question of whether or not that subsocle supports a pure subgroup. It is well known that every subsocle of a bounded group supports a pure subgroup.

PROPOSITION 3. — *Let* $S = \bigcup S_i$ *be the union of an ascending sequence of subsocles* S_i *of* G. *If* $S_i \cap p^i G = 0$ *for* $i = 1, 2, \ldots$, *then* S *supports a pure subgroup. Indeed,* S *supports a direct summand of a basic subgroup.*

PROOF. — Since S_i is contained in a bounded direct summand *of* G, it supports a pure subgroup H_i of G. But $\{H_i, S_{i+1}\} \cap p^{i+1} G = 0$; hence $\{H_i, S_{i+1}\}$ is contained in a bounded direct summand B_{i+1} of G. Since H_i is bounded and pure in B_{i+1}, it is a direct summand of B_{i+1}; let $B_{i+1} = H_i + A_{i+1}$. Then

$$S_{i+1} = H_i[p] + (A_{i+1} \cap S_{i+1}).$$

But $A_{i+1} \cap S_{i+1}$ supports a pure subgroup C_{i+1} in A_{i+1} since A_{i+1} is bounded. Let $H_{i+1} = H_i + C_{i+1}$. The union H of the ascending sequence of pure subgroups H_i of G is a pure subgroup of G with $H[p] = S$. Kulikov's criteria shows that H is a direct sum of cyclic groups (and therefore a direct summand of a basic subgroup of G).

COROLLARY 4. — *If* G *is a direct sum of cyclic groups, then each subsocle* S *supports a pure subgroup.*

PROOF. — Let $G = \sum B_i$ where B_i is (zero or) a homogeneous group of degree i and let $S_i = (B_1 + B_2 + \ldots + B_i) \cap S$. The conditions of proposition 3 are satisfied.

Following established terminology, we say that G is a closed group if it is the primary part of a complete direct sum of cyclic groups [2].

PROPOSITION 4. — *Each subsocle of a closed group supports a pure subgroup.*

PROOF. — Let S be a subsocle of a closed group G. Choose S_i such that $S \cap p^i G = S_i + (p^{i+1} G \cap S)$ for $i = 0, 1, \ldots$. Let $T_0 = 0$, $T_i = S_0 + S_1 + \ldots + S_{i-1}$ if $i \geqq 1$, and let $T = \bigcup T_i$. By proposition 3, T supports a direct summand B_1 of a basic subgroup B of G, $B = B_1 + B_2$. Since G is a closed group, $G = \overline{B}_1 + \overline{B}_2$. Since T is dense in S, $S \subseteq \overline{T}$. But $\overline{T} = \overline{\overline{B}_1[p]} = \overline{B}_1[p]$. Thus S is a dense subsocle of $\overline{B}_1$, a direct summand of G. The proof is completed by theorem 1.

THEOREM 4. — *If $G = A + B$ where A is a direct sum of cyclic groups and B is a closed group, then each subsocle of G supports a pure subgroup.*

PROOF. — By theorem 1, it suffices to prove that each closed subsocle of G supports a pure subgroup. Let S be a closed subsocle of G and let $S' = S \cap B$. Then S is a closed subsocle of B. By proposition 4, S' supports a pure subgroup C of B. Since S is closed, C is closed in B (and therefore is a closed group). Hence C is a direct summand of B; let $B = C + K$. Then $S = S \cap (A + K) + S'$. Notice that $S \cap K = 0$. Define $S_i = (A_1 + A_2 + \ldots + A_i + K) \cap S$ where $A = \sum A_i$ is the standard decomposition of A. Then $S \cap (A + K) = \bigcup S_i$ and $S_i \cap p^i(A + K) = 0$. Thus by proposition 3, there is a pure subgroup of $A + K$ with $S \cap (A + K)$ as its socle, and the theorem is proved.

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