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MOVING FRAMES

BY

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Introduction

The method of moving frames has a long history. It is at the heart of kinematics and its conscious application to differential geometry could be traced back at least to SERRET and FRENET[2]. In his famous "Leçons sur la théorie des surfaces," based on lectures at the Sorbonne 1882–1885, DARBOUX had the revolutionary idea using frames depending on two parameters and integrability conditions turn up. DARBOUX's method was generalized to arbitrary Lie groups by Émile COTTON [14] for the search of differential invariants, whose work should be considered a forerunner of the general method of moving frames.

CARTAN's first paper on moving frames was published in 1910 [7]. He immediately observed that DARBOUX's partial derivatives should be combined into the "Maurer-Cartan forms" and DARBOUX's integrability conditions are essentially the Maurer-Cartan equations. This extends the method to the case where the ambient space is acted on by any Lie group and, more generally, by an infinite pseudogroup in CARTAN's sense, such as the pseudogroup of all complex analytic automorphisms in n variables. The emphasis

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[2] The so-called Serret-Frenet formulas were published by J.A. SERRET in MONGE's "Application de l'Analyse à la Géométrie", 5th edition (1850), p. 566, and by F. FRENET in 1847

on the Maurer-Cartan forms and their counterparts in infinite pseudo-groups cannot be over-estimated. For in this way we dispense with the local coordinates and these forms even in the case of the infinite pseudo-groups, are defined up to a transformation of a finite-dimensional Lie group, the gauge group, as it is called nowadays.

In almost all his works on differential geometry CARTAN used moving frames as the local tool. His general theory was developed in [8] and [9], the latter being based on notes edited by J. LERAY of a course given at the Sorbonne in 1931–1932. While we gather here today to honor Élie CARTAN, it is my feeling that his method of moving frames has been bypassed. As a principal reason I will quote Hermann WEYL [24] in his review of [9] :

"All of the author's books, the present one not excepted, are highly stimulating, full of original viewpoints, and profuse in interesting geometric details. CARTAN is undoubtedly the greatest living master of differential geometry... Nevertheless, I must admit that I found the book, like most of CARTAN's papers, hard reading..."

In this paper I wish to give a review of this method. I believe it is best appreciated through examples, and I will begin with some applications, old and new.

1. Riemannian geometry of two dimensions

Let M be a two-dimensional oriented Riemannian manifold. Let SM be its unit tangent bundle. A unit tangent vector can be denoted by xe_1, x being the origin. It determines uniquely the orthonormal frame xe_1e_2 consistent with the orientation of M. Thus SM can be identified with the (orthonormal) frame bundle of M, and we have the projection

$$\pi : SM \longrightarrow M, \tag{1}$$

sending xe_1e_2 to x. If $x\omega_1\omega_2$ is the dual coframe of xe_1e_2, the metric on M can be written

$$ds^2 = \omega_1^2 + \omega_2^2. \tag{2}$$

The space SM can also be interpreted as the space of all decompositions (2) of ds^2 as a sum of squares, with $\omega_1 \wedge \omega_2 > 0$.

The fundamental theorem of local Riemannian geometry says that there is a uniquely defined form ω_{12} in SM, the connection form, satisfying the equations

$$\begin{aligned} d\omega_1 &= \omega_{12} \wedge \omega_2, \\ d\omega_2 &= \omega_1 \wedge \omega_{12}. \end{aligned} \tag{3}$$

Classically this is expressed as the unique determination of the Levi-Civita connection without torsion. Taking the exterior derivatives of (3), one derives

$$d\omega_{12} = -K\omega_1 \wedge \omega_2, \tag{4}$$

where K is the Gaussian curvature. Formulas (3) and (4) contain all the information on local Riemannian geometry in two dimensions.

They give global consequences as well. A little meditation convinces one that (4) must be the formal basis of the Gauss-Bonnet formula, and this is indeed the case. It turns out that the proof of the n-dimensional Gauss-Bonnet formula can be based on this idea and can be reduced to a problem of the exterior differential calculus [11]. Interpreted in another way, (4) says that the form ω_{12} in SM has its exterior derivative in the base manifold M. This notion is called a transgression. It seems to me that this concept was not known to GAUSS.

The formalism also adapts well with variational problems. In CARTAN's original introduction of the exterior differential calculus, as a generalization of earlier works of DARBOUX and FROBENIUS, he introduced two differentials d and δ, which are commutative :

$$d\delta = \delta d. \tag{5}$$

Then the exterior product of the one-forms α and β is defined by

$$(\alpha \wedge \beta)(d, \delta) = \alpha(d)\beta(\delta) - \beta(d)\alpha(\delta) \tag{6}$$

and the exterior derivative of α was called the "bilinear covariant" and was defined by

$$d\alpha(d, \delta) = d\alpha(\delta) - \delta\alpha(d). \tag{7}$$

If we have a family of curves on M, we can let d be the variation along the curves and choose the frames so that $\omega_2(d) = 0$, (which means that e_1 are the unit tangent vectors and $\omega_1(d)$ is the element of arc), and we let δ be the variation among the curves.Then the first equation of (3) gives

$$d\omega_1(\delta) - \delta\omega_1(d) = \omega_{12}(d)\omega_2(\delta). \tag{8}$$

The second term in (8) is the first variation of the arc length, from which it is easily derived that the equations of the geodesics are

$$\omega_2 = \omega_{12} = 0. \tag{9}$$

Pursuing this method further, we can find the formula for the second variation of arc length and JACOBI's equation, etc.

2. Isometric deformation of surfaces

The isometry or isometric deformation of surfaces is a fundamental problem in classical surface theory. In 1867 BONNET studied these problems under the following additional conditions [4].

A) preservation of the lines of curvature;

B) preservation of the principal curvatures,

assuming in each case that the surfaces are free from umbilics. Since the Gaussian curvature is invariant under an isometry, condition B) is equivalent to :

B') preservation of the mean curvature.

BONNET solved Problem A), but not B). On B) he gave, however, an interesting theorem (see below).

A more interesting problem is a family of isometric surfaces, such that the isometries satisfy the condition A) or B). Such a family is called non-trivial, if it is not generated by one of its surfaces through rigid motions.

When these problems are formulated analytically, they give over-determined systems of partial differential equations, where the number of equations exceeds the number of unknown functions. Significant conclusions can be drawn only through repeated prolongations. In both problems fifth-order jets have to be used and the calculations are long. We will state the theorems, and refer the proofs to [5, pp. 269-284] and [12] respectively.

THEOREM 2.1 (Bonnet). — *A non-trivial family of isometric surfaces of non-zero Gaussian curvature, preserving the lines of curvature, is a family of cylindrical molding surfaces.*

The cylindrical molding surfaces can be kinematically described as follows : Take a cylinder Z and a curve C on one of its tangent planes. A molding surface is the locus described by C as a tangent plane rolls about Z.

THEOREM 2.2. — *A non-trivial family of isometric surfaces preserving the principal curvatures is one of the following :*

α) *(the general case) a family of surfaces of constant mean curvature;*

β) *(the exceptional case) a family of surfaces of non-constant mean curvature.*

They depend on six arbitrary constants and have the properties :

$\beta 1$) *they are W-surfaces;*

$\beta 2$) *the metric*

$$d\hat{s}^2 = (\operatorname{grad} H)^2 ds^2/(H^2 - K),$$

where ds^2 is the metric of the surface and H and K are its mean curvature and Gaussian curvature respectively, has Gaussian curvature equal to -1.

The case α) was proved by BONNET, using a Lie transformation. In [12] we gave a complete solution of Problem B, an outstanding unsolved problem in classical surface theory, to which CARTAN made important contributions. We made use of the method of moving frames, with the connection form ω_{12} playing a decisive role.

3. Minimal two-spheres in a complex Grassmann manifold

There has been a lot of recent works on harmonic mappings of surfaces or minimal surfaces in the complex projective space $P_n(\mathbf{C})$ or the Grassmann manifold $G_{2,n+1}$, which we interpret as the space of all lines in $P_n(\mathbf{C})$. I shall try to imagine how CARTAN would approach the problem, using projective geometry [10]. The points of $P_n(\mathbf{C})$ will be identified with their homogeneous coordinate vectors $Z \in C_{n+1} - \{0\}$, and the conditions we shall deal with are to be invariant under the change $Z \to \lambda Z$. The space $P_n(\mathbf{C})$ is provided with the Study-Fubini metric, and the metric notions are valid.

Given a point $Z \in P_n(\mathbf{C})$, all the points of $P_n(\mathbf{C})$ orthogonal to Z form a hyperplane $Z^{\perp} \in P_n^*(\mathbf{C})$, which is the dual space of all hyperplanes of $P_n(\mathbf{C})$. A surface $Z : M \to P_n(\mathbf{C})$ gives rise to a dual surface $Z^{\perp} : M \to P_n^*(\mathbf{C})$, and vice versa. It is easily seen that Z is minimal if and only if $Z^{\perp}$ is minimal. We make use of the complex structure on M induced by its metric. The orthogonal projection of ∂Z (resp $\overline{\partial} Z$) into $Z^{\perp}$ is a form of type (1,0) (resp. (0,1)) with value in $Z^{\perp}$ and defines a point A (resp. B) $\in Z^{\perp} \subset P_n(\mathbf{C})$. It is easily verified that A and B are orthogonal points. The fundamental theorem on minimal two-spheres in $P_n(\mathbf{C})$ says that *if $M = S^2$ is not a holomorphic or an anti-holomorphic curve, then both A and B describe minimal two-spheres.*

Applying this theorem, one gets a complete description of the minimal S^2 in $P_n(\mathbf{C})$ as follows : *Let C be a rational curve in $P_n(\mathbf{C})$, which does not belong to a lower-dimensional space, and let $Z_0 Z_1 \dots Z_n$, $Z_0 \in C$, be its Frenet frame. Then every Z_i, $0 \le i \le n$, describes a minimal two-sphere. Conversely every minimal two-sphere in $P_n(\mathbf{C})$ which does not belong to $P_m(\mathbf{C})$, $m < n$, is obtained in this way.*

Using the method of moving frames, we immediately stumble on invariant complex-valued forms of type $(k, 0)$, $k \ge 3$. The structure equations imply that they are holomorphic. On a two-sphere they must be zero. This is the key step of the proof. For details *cf.* [25]. The theorem was first stated by A.M. DIN and W.J. ZAKRZEWSKI [15], while the first mathemacian's proof was given by J. EELLS and J. WOOD [16].

The corresponding problem of minimal S^2 in $G_{2,4}$ was solved by J. RAMANATHAN [23]. J. WOLFSON and I have extended the above ideas to the more general case of $G_{2,n+1}$, and I wish to give a brief description of our results [13].

Consider a minimal surface

$$L : M \longrightarrow G_{2,n+1}, \tag{10}$$

which we regard as a manifold of lines in $P_n(\mathbf{C})$ of two real dimensions. The space $L(x)^{\perp}$ orthogonal to $L(x)$, $x \in M$, is of complex dimension $n-2$. It is an element of $G_{n-1,n+1}$, the Grassmann manifold of all projective spaces of dimension $n-2$ in $P_n(\mathbf{C})$. $L(x)^{\perp}$, $x \in M$, defines a surface in $G_{n-1,n+1}$, and L is minimal if and only if $L^{\perp}$ is minimal; they are called dual minimal surfaces.

As above, if $Z \in L$, ∂Z (resp. $\overline{\partial} Z$) defines a point A (resp.B) $\in L^{\perp}$. This defines mappings

$$A, B : L \longrightarrow L^{\perp}, \tag{11}$$

which are projective collineations. Reversing the role of L and $L^{\perp}$ (and of ∂ and $\overline{\partial}$), we have also the collineations

$$A^{\perp}, B^{\perp} : L^{\perp} \longrightarrow L. \tag{11a}$$

We shall call A, B, $A^{\perp}$, $B^{\perp}$ the *fundamental collineations.* In particular, we have

$$B^{\perp} \circ A : L \longrightarrow L. \tag{11b}$$

The determinant $\det(B^{\perp} \circ A)$ gives a form of type (4,0) on M, which is holomorphic by the structure equations. Thus, if $M = S^2$, we have

$$\det(B^{\perp} \circ A) = 0. \tag{12}$$

This is the first global result, using the fact that the surface is S^2 and has no non-zero holomorphic differential forms of degree > 0.

This result is sufficient to give a solution of our problem for $n = 3$. For in this case $\dim L^{\perp} = 1$ and condition (12) becomes

$$(\det B)(\det A) = 0. \tag{13}$$

Hence one of the determinants is zero. Suppose $\det A = 0$, $A \neq 0$. Then the collineations A and $A^{\perp}$ are degenerate, and the images $A(L)$ and $A^{\perp}(L^{\perp})$ are points.

From the fact that L in (10) is a minimal surface it follows by standard calculations that $A(L)$ and $A^{\perp}(L^{\perp})$ describe minimal surfaces in $P_3(\mathbf{C})$. The line joining $A(L)$ and $A^{\perp}(L^{\perp})$ has an orthogonal line λ. There are points Z_0, $Z_3 \in \lambda$ such that Z_0 describes a holomorphic curve and Z_3 an antiholomorphic curve and Z_0, $A(L)$, $A^{\perp}(L^{\perp})$, Z_3 are the vertices of the Frenet frame of Z_0.

For $n = 4$ we have $\dim L^{\perp} = 2$. In the generic case the lines $A(L)$ and $B(L)$ meet in a point P; P describes a minimal surface in $P_4(\mathbf{C})$.

Our main idea is to make use of constructions of projective geometry to derive from the minimal surfaces (10) in $G_{2,n+1}$, minimal surfaces of $P_n(\mathbf{C})$, thus reducing to a problem already solved. This requires several geometric constructions whose details we refer to [13]. It suffices to remark that we need to consider the osculating spaces of higher order to exhaust the ambient space, which involve the jets of higher order of the mapping.

4. General theory*

The general problem to be treated is the following : Let G be a Lie group and $H \subset G$ a closed subgroup. Let $N = G/H$ be the homogeneous space of left cosets, acted on by G by left multiplication. The fundamental problem is to find all the local invariants of a smooth submanifold

$$f : M \longrightarrow N, \tag{14}$$

i.e., if

$$f^* : M^* \longrightarrow N \tag{15}$$

is another submanifold, to find the necessary and sufficient conditions that there exist a local diffeomorphism

$$T : M \longrightarrow M^* \tag{16}$$

and an element $g \in G$ such that

$$f^* \circ T = g \circ f. \tag{17}$$

If these conditions are satisfied, M and M^* are said to be *congruent.* A related problem, the so-called fixed parametrization problem, is to take $M = M^*$ and $T =$ identity, and to ask for the existence of $g \in G$ such that (17) holds.

* I wish to thank D. BERNARD and G. JENSEN for their comments on this section.

We denote by f_k the k-th order jet defined by f. If T exists, and if for each point $x \in M$ there exists $g(x) \in G$ such that

$$(f^* \circ T)_k(x) = (g(x) \circ f)_k(x), \tag{18}$$

we say that f and f^* are G-deformable of order k.

Consider the diagram

$$\begin{array}{ccc} & & G \\ & & \big\downarrow \pi \\ M & \xrightarrow{f} & G/H, \end{array} \tag{19}$$

when π sends an element $g \in G$ to the coset gH. Let G and H be of dimensions r and s respectively, and let $\omega^1, \ldots, \omega^r$ be the left-invariant Maurer-Cartan forms in G, such that

$$\omega^{s+1} = \cdots = \omega^r = 0 \tag{20}$$

defines the foliation of G by the left cosets of H. A moving frame is a local (smooth) map

$$U \xrightarrow{F} G, \tag{21}$$

where U is a neighborhood in M, such that $f = \pi \circ F$, i.e., it assigns to each point $x \in U$ a point of the coset $f(x) \in G/H$. If $\dim M = m$ and if $\alpha^1, \ldots, \alpha^m$ are linearly independent one-forms on M, we have

$$F^*\omega^\lambda = \sum_i a_i^\lambda \alpha^i, \qquad 1 \le i \le m, s+1 \le \lambda \le r. \tag{22}$$

A new moving frame is given by $\widetilde{F} = Fh$, the latter being the right multiplication of F by $h : U \to H$. Under such a change $F^*\omega^\lambda$ undergoes a transformation of the adjoint group $\mathrm{ad}(H)$, and we have the change

$$A = (a_i^\lambda) \longrightarrow \widetilde{A} = SAT, \tag{23}$$

where S is the transformation induced on ω^λ by $\mathrm{ad}(h)$, $h \in H$, and T is any non-singular $(n \times n)$-matrix. The invariants of A under this change are the first-order invariants of the submanifold M.

The simplest case, and this is the case in most applications, is when $\mathrm{ad}(H)$ acts transitively on the Grassmann bundle of m-planes of the tangent bundle of G/H. Then the Maurer-Cartan forms of G can be so chosen that

$$F^*\omega^\lambda = 0, \qquad m+s+1 \le \lambda \le r, \tag{24}$$

and that $F^*\omega^{s+1}, \ldots, F^*\omega^{s+m}$ are linearly independent. The main step is to take the exterior derivative of (24) and make use of the Maurer-Cartan structure equations. This will lead to the second-order invariants of M.

The analysis of the general case depends on the behavior of the invariants and has generally to be divided into many cases. As CARTAN himself remarked, the method provides a *practical and rapid* mechanism leading to the invariants, which could be functions or differential forms, exterior or ordinary. It is effective in both classical problems and problems where higher-order jets are involved. A characteristic feature is the exclusive use of differential forms, so that local coordinates do not appear. This makes the method useful in both local and global problems, as illustrated in the applications in the last two sections. Personally I find the effectiveness and superiority of the method in classical problems just as noteworthy as the general theory.

CARTAN tried to reduce the congruence problem to that of contact. Let $N^{(0)} = N$, $N^{(q+1)} =$ the bundle of tangent m-planes to $N^{(q)}$, $q \geq 0$. For the submanifold (14) let $f^{(0)} = f$ and let $f^{(q+1)}$ be the prolongation of $f^{(q)}$, i.e., $f^{(q+1)} : M \to N^{(q+1)}$ is defined by

$$f^{(q+1)}(x) = f^{(q)}_*(T_x M), \quad x \in M. \tag{25}$$

Then the submanifolds (14) and (15) are said to have a k-th order contact at $x \in M$ and $x^* \in M^*$, if and only if

$$f^{(k)}(x) = f^{*(k)}(x^*). \tag{26}$$

They are said to have a G-contact of order k at x and x^*, if there exists a transformation $g \in G$ such that f and $g \circ f^*$ have k-th order contact at x and $g(x^*)$.

In [21] JENSEN proved the following theorem :

THEOREM. — *Suppose f and f^* have the same contact type. Then there exists a k with the following property : if there is a diffeomorphism* (16) *such that f and f^* have G-contact of order k at x and $T(x)$ for every $x \in M$, then f and f^* are congruent, i.e.,* (17) *is satisfied.*

Recently in [20] S. HÜCKEL proved a generalized congruence theorem, which is even valid for non-transitive actions of G and from which JENSEN's theorem may be deduced. The condition of "same contact type" is replaced by the condition that certain osculating spaces of order $k-1$ (relative to the action of G) have a constant dimension, and the a priori existence of a diffeomorphism T is replaced by the "continuity of G-contact of order k", a topological property of the space $\Gamma_k \subset M \times M^* \times G$ of triples (x, x^*, g) such that $g \in G$ realizes the G-contact of order k between M at x and M^* at x^*.

The importance of the method depends to a certain extent on the significance of structures with a higher order of smoothness. The question whether the conclusions in § 2 are true with a lower degree of smoothness is a valid one, but perhaps not very interesting. In the study of partial differential equations the step usually goes backwards, which is to consider generalized solutions with less smoothness.

There is strong reason why differential forms furnish the right tool for the study of submanifolds. This is because in the last analysis the group G plays the fundamental role; its Maurer-Cartan forms and Maurer-Cartan equations are pulled back in a natural way by the moving frame.

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