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HOLOMORPHIC SYMMETRIES

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In this paper we present a collection of results concerning holomorphic S^1-actions on complex analytic varieties and their boundaries. It is well established that complex manifolds admitting such actions are special in nature, and if the action is locally generic (topologically), then the manifold is quite special. (*See* [BB], [CL], [$CS_{1,2}$], [F], [L], and [So] for example. We shall see here that this continues to be true for singular spaces and their boundaries.

We shall show that any maximally complex submanifold of dimension $2p-1>1$ *in* $\mathbb{C}^n$ which admits an intrinsic S^1-action transversal to the CR-structure is, in fact, *algebraic* in the sense that it is embedded as a hypersurface in a complete affine algebraic variety. This variety has at most one singular point and admits an (extrinsic) linear $\mathbb{C}^*$-action which restricts to become the given action on the hypersurface. As a consequence we show that when $p=n-1$, any two such manifolds $M, M' \subset \mathbb{C}^n$ having isomorphic ring structures in their Kohn-Rossi cohomology, are diffeomorphic. In fact, there is an ambient C^∞ diffeomorphism $f:\mathbb{C}^n \to \mathbb{C}^n$ with $f(M)=M'$. For example, if $H_{KR}^{**}(M)=0$ (and M admits a transversal S'-action) then M is diffeomorphic to S^{2n-3}, and if it happens to lie in the unit sphere S^{2n-1}, it is unknotted in that sphere. In the absence of an S^1-action these assertions are utterly untrue.

One of the more basic results of the paper is a fixed-point formula for holomorphic S^1-actions on a compact complex analytic space X. In particular, it is shown that

$$\chi(X)=\chi(X^{S^1})$$

where X^{S^1} denotes the fixed point set of the action. This formula allows us to compute the Euler characteristic of pieces of given degree of the Chow variety of complex projective space. The result can be summarized as follows. Let $\mathscr{C}_{p,d,n}$ denote the set of (positive) complex analytic cycles of dimension p and degree d in $P^n(\mathbb{C})$. Consider the formal power series

$$Q_{p,n}(t)=\sum_{d=0}^{\infty}\chi(\mathscr{C}_{p,d,n})\,t^d$$

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Then the thorem is that

$$Q_{p,n}(t)=\left(\frac{1}{1-t}\right)^{\binom{n+1}{p+1}}$$

for $0 \leqq p \leqq n$ and all n. This is equivalent to the statement that $\chi(\mathscr{C}_{p,d,n})=\binom{\nu+d-1}{d}$ where $\nu=\binom{n+1}{p+1}$ for all p, d, n.

Similar calculations are carried out for products of projective spaces, and again rational functions appear.

The paper is organized as follows:

1. Some fundamental extension theorems.
2. The algebraicity of CR-manifolds with automorphisms.
3. Kohn-Rossi cohomology and smooth equivalence.
4. A fixed-point formula for analytic spaces.
5. An application to cycle spaces.

1. Some fundamental extension theorems

In this section we shall discuss some elementary facts concerning automorphism groups of complex analytic varieties and their boundaries. We shall then prove some basic results concerning the extension of automorphism groups from boundaries to the interior and from S^1 to $\mathbb{C}^\times$ or $\Delta^\times$.

Recall that a C^1 submanifold M of a complex manifold X is said to be *maximally complex*, or simply *MC*, if

$$\operatorname{codim}_{\mathbb{R}}(T_x M \cap J(T_x M))=1 \quad \text{for all } x \in M \tag{1.1}$$

where J denotes the almost complex structure of X, and the codimension refers to M. It is proved in [HL_1] that if M is compact, oriented and of dimension >1, and if X is Stein, then maximal complexity implies that M forms the boundary of a holomorphic p-chain in X.

Suppose now that M is maximally complex and set $\mathscr{H}_x=T_x M \cap J(T_x M)$ for $x \in M$. Note that each $\mathscr{H}_x$ is a complex subspace, i. e., is J_x-invariant.

DEFINITION 1.2. — A smooth S^1-action (of class C^1) on M is said to be *holomorphic* if it preserves the family of subspaces $\mathscr{H} \subset TM$ and commutes with J. It is said to be *transversal* if, in addition, the vector field V which generates the action, is transversal to $\mathscr{H}$ at all points of M.

In the C^∞ case, the condition of being holomorphic can be written in terms of the generating vector field as:

$$(1.3) \qquad \begin{cases} \mathscr{L}_V : & \Gamma(\mathscr{H}) \to \Gamma(\mathscr{H}) \\ & \mathscr{L}_V(J) \equiv 0 \end{cases}$$

where $\Gamma(\mathscr{H})$ denotes the subspace of smooth vector fields with values in $\mathscr{H}$ and where $\mathscr{L}_V$ denotes the Lie derivative.

Note that a transversal holomorphic action is locally free. Furthermore away from the singular orbits the orbit space M/S^1 is smooth and inherits naturally an almost complex structure from M. To see this, note that the projection $\pi : M \to M/S^1$ identifies $\mathscr{H}_x$ with $T_{\pi x}(M/S^1)$ and that the endomorphism $J_{\pi x}$ on $T_{\pi x}(M/S^1)$ is independent of the preimage point because $(\mathscr{H}, J)$ is S^1-invariant. It is interesting to note the following.

PROPOSITION 1.4. — *If* M *admits a transversal holomorphic action of* S^1, *the almost complex structure induced on the regular points of* M/S^1 *is integrable.*

Proof. — For simplicity we assume, that everything is C^∞. We recall that by the Newlander-Nirenberg Theorem an almost complex structure J on a smooth manifold $\mathscr{M}$ is integrable if and only if $[\Gamma^{1,0}, \Gamma^{1,0}] \subset \Gamma^{1,0}$, where $\Gamma^{1,0} \equiv \{ W \in \Gamma(T\mathscr{M} \otimes \mathbb{C}) : JW = iW \}$ is the set of (1, 0)-*vector fields* on $\mathscr{M}$. Letting $M \subset X$ be as above, we define

$$\Gamma^{1,0}(\mathscr{H}) \equiv \{ W \in \Gamma(\mathscr{H} \otimes \mathbb{C}) : JW = iW \}.$$

The integrability of the structure on X easily implies that

$$(1.5) \qquad [\Gamma^{1,0}(\mathscr{H}), \Gamma^{1,0}(\mathscr{H})] \subset \Gamma^{1,0}(\mathscr{H}).$$

Each (1, 0)-vector field W on M/S^1 lifts uniquely to an S^1-invariant element $\tilde{W} \in \Gamma 1, 0(\mathscr{H})$. Given two such fields W_1 and W_2, the $\mathscr{H} \otimes \mathbb{C}$-component of $[\tilde{W}_1, \tilde{W}_2]$ is exactly $[W_1, W_2]$. It follows from (1.5) that $[W_1, W_2]$, and therefore also $[W_1, W_2]$, are fields of type (1, 0). ■

The complex analyticity of M/S^1 extends also to the singular points. This is a consequence of the following.

PROPOSITION 1.6 (S. Webster). — *Let* M *be as above and define an almost complex structure* $\mathscr{J}$ *on* $M \times S^1$ *by the requirements:*

$$\mathscr{J}|_{\mathscr{H}} = J; \qquad \mathscr{J}(V) = \frac{\partial}{\partial\theta}; \qquad \mathscr{J}\left(\frac{\partial}{\partial\theta}\right) = -V.$$

Then $\mathscr{J}$ *is integrable on* $M \times S^1$.

Proof. — One checks straightforwardly using (1.3) and (1.5) that $[\Gamma^{1,0}, \Gamma^{1,0}] \subset \Gamma^{1,0}$. ■

The action of the complex group $S^1 \times S^1$ on $M \times S^1$ is holomorphic; hence, we have that $(M \times S^1)/(S^1 \times S^1) \cong M/S^1$ is naturally an analytic space.

We now consider S^1-actions on complex analytic spaces. We begin by recalling the usual definitions in the non-singular case. Let Y be a complex manifold which for convenience we assume to be compact, and let $S^1 \equiv \{z \in \mathbb{C} : |z| = 1\}$. A smooth action $\varphi : S^1 \times Y \to Y$ is said to be *holomorphic* if the transformation $\varphi_t \equiv \varphi(e^{it}, .) : Y \to Y$ is holomorphic for all t. This means that for all t we have

$$(1.7) \qquad (\varphi_t)_* \circ J = J \circ (\varphi_t)_*$$

where J is the almost complex structure of Y. If V is the vector field generating the action on Y, then this condition can be rewritten as

$$(1.8) \qquad \mathscr{L}_V(J) = 0.$$

From this equation we see that $[V, JV] = \mathscr{L}_V(JV) = J\,\mathscr{L}_V(V) = 0$. Therefore, the flow ψ_s generated by the vector field JV commutes with the action, i. e., $\psi_s \circ \varphi_t = \varphi_t \circ \psi_s$ for all $s, t \in \mathbb{R}$. This allows us to define an action

$$(1.9) \qquad \Phi : \quad \mathbb{C}^\times \times Y \to Y$$

of the multiplicative group $\mathbb{C}^\times \equiv \mathbb{C}\{0\}$, by setting

$$\Phi(z) = \psi_{-\log|z|} \circ \varphi_{\arg(z)}$$

It is easily seen that the map Φ is holomorphic.

Much the same is true when Y is not compact. However, the flow Φ is, in this case, only locally defined.

Suppose now that Y is a complex analytic space, and let $\mathcal{O}_Y$ denote the sheaf of germs of holomorphic functions on Y. Denote by $\mathcal{O}_Y^w$ the sheaf of germs of *weakly* holomorphic functions on Y, that is, the sheaf of germs of bounded functions which are holomorphic on the regular set of Y (*cf.* [GR]). Of course we have $\mathcal{O}_Y \subset \mathcal{O}_Y^w$, and the space Y is called *normal* if $\mathcal{O}_Y = \mathcal{O}_Y^w$.

We denote by reg(Y) the set of regular points of Y, and by sing(Y) = Y − reg(Y) the set of singular points.

Definition 1.10. — Let φ_t be a continuous action of S^1 on a complex analytic space Y, and suppose it preserves the set reg(Y). Then this action is called *holomorphic* if

$$\varphi_t^* \mathcal{O}_Y = \mathcal{O}_Y \quad \text{for all } t.$$

It is called *weakly holomorphic* if

$$\varphi_t^* \mathcal{O}_Y^w = \mathcal{O}_Y^w \quad \text{for all } t.$$

Suppose now that M is a compact oriented maximally complex submanifold of dimension $2p-1>1$ in a Stein manifold X. Then by [HL] we know that M forms the boundary of a unique holomorphic p-chain T in X. In particular, the set $Y \equiv supp\, T$ *defines a purely* p-dimensional complex subvariety in X − M, and it defines a smooth submanifold-with-boundary at almost all points of M. We shall say that M is the *boundary* of Y and write $M=\partial Y$. We shall denote by $\operatorname{reg}(\partial Y)$ the points of boundary regularity.

The following result asserts that any intrinsic holomorphic S^1-action on M extends to Y.

THEOREM 1.11. — *Let* Y *be a compact complex analytic subvariety with* C^1 *boundary* ∂Y *in a Stein manifold* X. *Assume that* Y *is of pure dimension* $p>1$ *and that* ∂Y *is connected. Then any holomorphic* S^1*-action on* ∂Y *extends to a weakly holomorphic action on* Y.

This extended action defines a continuous map $\varphi : S^1 \times Y \to Y$ *which is real analytic at all points* (t, x) *with* $x \in \operatorname{reg}(Y)-\partial Y$. *Furthermore, if both* ∂Y *and the generating vector field for the action are of class* C^k, *for some* $k \geqq 1$, *then the extended action* $\varphi : S^1 \times Y \to Y$ *is of class* C^k *in a neighborhood of* $\operatorname{reg}(\partial Y)$.

Proof. — Fix t, and consider the graph $\Gamma_t = \{(x, \varphi_t(x)) \in X \times X : x \in \partial Y\}$ of the diffeomorphism $\varphi_t : \partial Y \to \partial Y$. Since φ_t preserves the CR-structure on ∂Y (i. e., since φ_t is holomorphic), this submanifold Γ_t is maximally complex. Consequently, by the main results in [HL], there exists a unique p-dimensional subvariety $Y_t \subset X \times X$ xith $\partial Y_t = \Gamma_t$.

Consider now the projections $p_1 : X \times X \to X$ and $p_2 : X \times X \to X$ onto the first and second factors respectively. For each k, $p_k|_{\partial Y_t} : \partial Y_t \to \partial Y$ is a CR-diffeomorphism. It follows from the uniqueness results in [HL] that for each k, $p_k(Y_t)=Y$. Furthermore, the maps $p_k|_{Y_t} : Y_t \to Y$ are one-to-one outside a compact subvariety of $Y_t - \partial Y_t$, *i. e., outside a finite set of points. It follows immediately that each* p_k is globally one-to-one on Y_t. Thus the map $\tilde{\varphi}_t \equiv p_2 \circ (p_1|_{Y_t})^{-1} : Y \to Y$ is a homeomorphism which is holomorphic on the regular points of Y and smooth up to the boundary, a. e. We clearly have that $\tilde{\varphi}_t|_{\partial Y} = \varphi_t$, and by the uniqueness of the extension (corresponding to the uniqueness of its graph), we conclude that $\tilde{\varphi}_t \circ \tilde{\varphi}_s = \tilde{\varphi}_{t+s}$ for all t and s.

We now consider the regularity properties of the map $\varphi : S^1 \times Y \to Y$. To begin we choose an embedding $X \subset \mathbb{C}^N$ and for each $t \in S^1$, we consider the map $\varphi_t : Y \to Y \subset X \subset \mathbb{C}^N$ to be $\mathbb{C}^N$-valued. Each of these maps is continuous on Y and weakly holomorphic in the interior. In particular $\| \varphi_t - \varphi_s \|$ satisfies a maximum principle on Y. This fact, together with the continuity of φ on $S^1 \times \partial Y$, proves the global continuity of φ on $S^1 \times Y$. Applying the maximum principle to $(1/h)(\varphi_{t+h} - \varphi_t)$ and using the fact that φ is C^1 on ∂Y shows that we have uniform convergence to a limit

$$V = \lim_{h \to 0^+} \frac{1}{h} (\varphi_{t+h} - \varphi_t)$$

which is continuous on Y and weakly holomorphic in the interior. At points of $\operatorname{reg}(Y) \cup \partial Y$, V can be considered as a tangent vector field to Y, and as such, it is clearly the generator of the action φ. When considered as a $\mathbb{C}^N$-valued function, V is holomorphic on $\operatorname{reg}(Y) - \partial Y$, and therefore the action is real analytic at such points.

If ∂Y and $V|_{\partial Y}$ are of class C^k, then from the boundary regularity results in, say, [HL], § 5, we know that V is of class C^k up to the boundary. Therefore, at all manifold points it generates a C^k-flow. ■

THEOREM 1.12. — *Let* Y *be as in Theorem* 1.11 *and suppose there is a transversal holomorphic action of* S^1 *on* Y. *Then the extended action on* Y *has exactly one fixed-point. This point, say* p, *may be a singular point of* Y, *however,* $Y-p$ *is a smooth manifold with boundary.*

Note. — If the action is not transversal, this conclusion fails to be true even in the pseudoconvex case. For example, consider the manifolds $S^2=\{x\in\mathbb{R}^3: x|=1\}$ and $S^2_{\mathbb{C}}=\{z\in\mathbb{C}^3: z.z=1\}$, and the S^1-action on $\mathbb{C}^3$ given by $\varphi_t(z_1, z_2, z_3) =(z_1 \operatorname{cost}-z_2 \operatorname{sint}, z_1 \operatorname{sint}+z_1 \operatorname{cost}, z_3)$. This action preserves S^2 and $S^2_{\mathbb{C}}$.
Since it is unitary, it also preserves each manifold $(S^2_{\mathbb{C}})_\varepsilon \equiv \{z\in S^2_{\mathbb{C}}: \operatorname{dist}(z, S^2)\leqq\varepsilon\}$ for $\varepsilon>0$. Note that $\partial(S^2_{\mathbb{C}})_\varepsilon$ is pseudoconvex and the S^1-action is free here. However, on the interior of $S^2_{\mathbb{C}}$ the action has *two* fixed-points.

Proof. — Let V be the vector field that generates the action of S^1 on $\partial Y\cup\operatorname{reg}(Y)$. By replacing V with $-V$ if necessary we can assume that JV points interior to Y at some point of ∂Y. Since V is transversal, we conclude that JV must point strictly into the interior of Y at all points of ∂Y.

Let $F\subset Y$ denote the set of fixed-points of the S^1-action. Clearly F is a compact analytic subvariety of $Y-\partial Y$, and so F consists of a finite number of points, say, $p_1, \ldots, p_m$.

Consider now any point $y\in\operatorname{reg}(Y)\cup\operatorname{reg}(\partial Y)$, and let γ denote the orbit of y under the S^1-action. From elementary ordinary differential equations we know that there is a neighborhood $\mathscr{U}$ of γ in Y and a non-empty time interval $[0, a)$ so that the vector field JV can be integrated over this interval at all poionts of $\mathscr{U}$. When $y\notin\partial Y$, this interval can be extended to $(-a, a)$.

If φ_t denotes the S^1-action and ψ_S denotes the JV-flow as above, then the complex analytic flow

$$\Phi_z=\psi_{-\log|z|}\circ\varphi_{\arg(z)} \tag{1.13}$$

is defined at each point of $\mathscr{U}$ for all z in the set $\{z\in\mathbb{C}: r<|z|\leqq 1\}$ (or $\{z\in\mathbb{C}: r<|z|<1/r\}$ when $y\notin\partial Y$) where $r=e^{-a}$.

Suppose now that $y\in Y-\partial Y$ is a singular point. Then y must lie in the regular set of some S^1-invariant stratum Σ of the singular set of Y. The vector fields V and (therefore) JV are both tangent to Σ at regular points, so the construction above applies to give us an analytic flow (1.13) defined in some neighborhood of the orbit of y in Σ.

Let us now fix any point $y\in(Y-\partial Y)\cup\operatorname{reg}(\partial Y)$ and define r_y to be the infimum of the set of real numbers $r>0$ such that the holomorphic map

$$f_y(z)\equiv\Phi_z(y)$$

is defined for $r<|z|\leqq 1$. We claim that $r_y=0$. To see this, suppose that $r_y>0$ and note that in the region $A=\{z\in\mathbb{C}: r_y<|z|\leqq 1\}$ the function f_y is holomorphic and has the property that

$$|rf_y'(re^{i\theta})|=\left|\frac{\partial}{\partial\theta}f_y(re^{i\theta})\right|=|V|. \tag{1.14}$$

Recall from the proof of 1.11 that the extended vector field V is defined and continuous on all of the compact set Y. Hence there is a constant c so that $|V| \leqq c$ on Y. Therefore, from 1.14 we conclude that for any two points $z_1, z_2 \in A$ we have

$$|f_y(z_2) - f_y(z_1)| = \left| \int_{z_1}^{z_2} f_y'(\zeta)\, d\zeta \right| \leqq \frac{\pi c}{r_y} |z_2 - z_1|$$

and so f_y extends continuously to the *closed* annulus $\bar{A}$. Note that the image of f_y on the inner circle is just the orbit: $f_y(r_y e^{it}) = \varphi_t(f_y(r_y)) = \varphi_t(y')$, for $0 \leqq t \leqq 2\pi$, of the point $y' \equiv f_y(r_y) \in Y - \partial Y$. The corresponding holomorphic map $f_{y'}(z) = \Phi_z(y')$ is now defined in the region $\{z : 1 - \varepsilon < |z| < 1 + \varepsilon\}$ for some $\varepsilon > 0$, and clearly $f_{y'}(z) = f_y(r_y z)$ for $|z| = 1$. It follows that $f_{y'}(z) = f_y(r_y z)$ in a neighborhood of the unit circle, and so $f_{y'}$ provides an extension of f_y in violation of the minimality of r_y. We conclude that $r_y = 0$ as claimed.

We now have the holomorphic map $f_y(z)$ defined and bounded in the punctured disk:

$$\Delta^\times = \{z \in \mathbb{C} : 0 < |z| \leqq 1\}.$$

It follows from elementary theory that $-f_y$ extends holomorphically across the origin. The limiting point $f_y(0)$ is clearly a fixed-point of the S^1-action.

Let us summarize the situation at this point. Set $Y^\frown \equiv (Y - \partial Y) \cup \operatorname{reg}(\partial Y)$. Then we have defined a map

$$\Phi : \quad \Delta \times Y^\frown \to Y^\frown$$

which when restricted to $\Delta^\times \times (Y - \partial Y)$ defines a weakly holomorphic action of the analytic semi-group $\Delta^\times$ on the analytic variety $Y - \partial Y$ which extends the given S^1-action. Since for each $y \in Y^\frown$ the map $f_y : \Delta \to Y$ ($\subset \mathbb{C}^N$ for some N) given by $f_y(z) = \Phi(z, y)$ is holomorphic, we have that

$$\Phi(z, y) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\Phi(\zeta, y)}{\zeta - z}\, d\zeta \tag{1.16}$$

for all such y. Note that since $\Phi(z, y)$ is smooth for $(z, y) \in S^1 \times \partial Y$, the right hand side of (1.16) defines a map on all of $\Delta \times \partial Y$ which is smooth up to the boundary and is holomorphic in z for each fixed $y \in \partial Y$. The map $\Phi : \Delta \times \partial Y \to Y$ is evidently surjective. Furthermore, the differential of Φ at $\{1\} \times \partial Y$ is surjective since the lines $t \to \Phi(t, y)$ are the integral curves of the vector field JV and JV is everywhere transversal to ∂Y. It now follows easily that in a neighborhood of ∂Y, the set Y is a smooth manifold with boundary, i. e., $\operatorname{reg}(\partial Y) = \partial Y$ and so $Y^\frown = Y$.

The fixed-point set F of the action is exactly the set $\Phi(0 \times Y)$. Since Y is connected and F is finite, we see that F consists of exactly one point. All other points are in the image of $\Delta^\times \times \partial Y$ and are therefore regular points. ■

In the course of proving Theorem 1.12 we have also proved the following.

THEOREM 1.17. — *Let* Y *be as in Theorem* 1.11. *Then any transversal holomorphic action of* S^1 *on* ∂Y *extends to a weakly holomorphic representation of the analytic semi-group* $\Delta^\times$ *as a semi-group of analytic embeddings of* Y *into itself. This action has a single fixed-point p, and given any neighborhood* $\mathscr{U}$ *of p in* Y, *there is an* $\varepsilon > 0$ *so that* $\Phi_z(Y) \subset \mathscr{U}$ *for all z with* $|z| < \varepsilon$.

It should be noted that the assumption of transversality in Theorems 1.12 and 1.17 is necessary. Consider, for example the variety $Y = \{A \in SL_2(\mathbb{C}) : \operatorname{tr}(A\bar{A}^t) \leqq 1\}$, and consider the S^1-action on Y given by setting

$$\varphi_t\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e^{it}a & e^{it}b \\ e^{-it}c & e^{-it}d \end{pmatrix}.$$

This action is holomorphic on Y and its boundary but is *not* transversal on ∂Y. Note that the action has no fixed points in Y.

2. The algebraicity of CR-manifolds with automorphisms

Throughout the section we assume $M \subset \mathbb{C}^n$ to be a compact oriented submanifold which is oriented and of class C^1. Our main result is the following.

THEOREM 2.1. — *Let* $M \subset \mathbb{C}^n$ *be maximally complex and of dimension* $2p-1 > 1$, *and suppose* M *admits a transversal holomorphic* S^1-*action. Then there exists a holomorphic equivariant embedding* $M \subsetneq Y$ *as a hypersurface in a p-dimensional algebraic variety* $Y \subset \mathbb{C}^n$ *with a linear* $\mathbb{C}^\times$-*action.*

Proof. — Let $Y_0 \subset \mathbb{C}^n$ denote the *p*-dimensional variety with boundary $\partial Y_0 = M$. By Theorem 1.12 we know that the S^1-action extends to a weakly holomorphic action on Y_0 with a single fixed point which we may assume to be $0 \in \mathbb{C}^n$. Furthermore, the variety is smooth outside of 0.

By Theorem 1.17 the S^1-action complexifies to become a weakly holomorphic semi-group $\Delta^\times$ of contractions. This action lifts naturally to a holomorphic action on the normalization $\tilde{Y}_0$ of Y_0. We now appeal to the following known fact whose proof we include for completeness' sake.

PROPOSITION 2.2. — *Let* (W, 0) *be a germ of a normal isolated singularity on a complex analytic space. Assume* W *admits a holomorphic* S^1-*action with* 0 *as an isolated fixed-point. Then there exists a linear action on* $\mathbb{C}^N$ (*for some* N) *and an* S^1-*equivariant embedding*

$$(W, 0) \subsetneq (\mathbb{C}^N, 0).$$

Proof. — The action on (W, 0) induces a linear action on $(\mathcal{O}_w)_0$ which preserves the maximal ideal $\mathscr{M} \equiv \{f \in (\mathcal{O}_w)_0 : f(0) = 0\}$ and also the ideal $\mathscr{M}^2$. This gives a linear S^1-action on the space $\mathscr{M}/\mathscr{M}^2 \cong \mathbb{C}^N$. Any choice of functions $f_1, \ldots, f_N \in \mathscr{M}$ such that $\langle f_1 \rangle, \ldots, \langle f_N \rangle$ form a basis of $\mathscr{M}/\mathscr{M}^2$, gives us an embedding

$F=(f_1, \ldots, f_N): (W, 0) \hookrightarrow (\mathbb{C}^N, 0)$ where $(W, 0)$ now denotes some fixed representative of the germ. We want to choose an F of this type which is S^1-equivariant.

To begin we fix one such embedding F on some representation $(W, 0)$ of the isolated singularity. We then choose a smooth measure μ on $W-\{0\}$ which is S^1-invariant and "decently behaved" at 0. We do this by taking a resolution $(\hat{W}, D) \to (W, 0)$ of the singularity, and then averaging the volume form of a riemannian metric on $\hat{W}$.

Consider now the Hilbert space $L^2(W, \mu)$ and the S^1-invariant subspace $H \equiv \{\varphi \in L^2(W, \mu) : \varphi \text{ is holomorphic}\}$.

LEMMA 2.3. — *The subspace* H *is closed in* $L^2(W, \mu)$. *Furthermore,* L^2*-convergence of a sequence* $\{\varphi_n\}_{n=1}^{\infty}$ *in* H, *implies uniform convergence of* $\{\varphi_n\}_{n=1}^{\infty}$ *on compact subsets of* W.

Proof. — Our first observation is that any $\varphi \in L^2(W, \mu)$ which is holomorphic on $W-\{0\}$, extends automatically to a holomorphic function on W. To see this lift φ to the resolution $(\hat{W}, D) \to (W, 0)$ and apply the usual L^2-extension theorem to get a holomorphic extension $\hat{\varphi}$ of φ to $\hat{W}$. Since $(W, 0)$ is normal, $\hat{\varphi}$ is the pull-back of a holomorphic function φ on W.

Suppose now that $\{\varphi_n\}_{n=1}^{\infty}$ is a sequence in H which converges in $L^2(W, \mu)$ to a limit φ. It is standard that in $\operatorname{reg}(W) = W-\{0\}$, φ is holomorphic and $\varphi_n \to \varphi$ uniformly on compact subsets. Hence, by the preceding paragraph, $\varphi \in H$. Furthermore, applying the same principle on the resolution $\hat{W}$, we see that $\hat{\varphi}_n \to \hat{\varphi}$ uniformly on compact subsets of $\hat{W}$, and so $\varphi_n \to \varphi$ uniformly on compact subsets of W. ∎

We now consider the closed S^1-invariant subspace $H_0 \equiv \{\varphi \in H : \varphi(0)=0\}$ and the S^1-equivariant maps:

$$\begin{tikzcd} H_0 \arrow[r] \arrow[rr, bend right, "\psi"'] & \mathcal{M} \arrow[r] & \mathcal{M}/\mathcal{M}^2. \end{tikzcd}$$

This composition is surjective, since we may assume $f_1, \ldots, f_N \in H_0$. The subspace $\ker \psi$ is closed, S^1-invariant and of codimension N. Hence, the subspace $E \equiv (\ker \psi)^{\perp}$ is N-dimensional and S^1-invariant, and the restriction of ψ gives an S^1*-equivariant isomorphism*

$$\psi : \quad E \xrightarrow{\approx} \mathcal{M}/\mathcal{M}^2.$$

Choosing an orthonormal basis $\{\tilde{f}_1, \ldots, \tilde{f}_N\}$ of E we get a map

$$\tilde{F} = (\tilde{f}_1, \ldots, \tilde{f}_N) : W \to \mathbb{C}^N$$

such that

$$\tilde{F}(\tau z) = L(\tau) \tilde{F}(z)$$

where $L : S^1 \to U_N$ is the unitary representation of S^1 on E in this basis. Of course, for a proper choice of basis we have that

$$\tilde{F}(\tau z) = (\tau^{k_1} \tilde{f}_1(z), \ldots, \tau^{k_N} \tilde{f}_N(z)) \tag{2.4}$$

for relatively prime integers $k_1 \leqq \ldots \leqq k_N$. This proves Proposition 2.2. ■

We return now to the main argument. By 2.2 we can choose a neighborhood W of 0 in $\tilde{Y}_0$ which admits a holomorphic embedding $\tilde{F}: W \hookrightarrow \mathbb{C}^N$ which is $\Delta^\times$-equivariant with respect to a linear $\mathbb{C}^\times$-action as in (2.4). (Here we have $\Delta^\times \subset \mathbb{C}^\times$ in the obvious way.) Since $\Delta^\times$ acts by contractions on $\tilde{Y}_0$ we must have

$$(2.5) \qquad k_j \geqq 0 \quad \text{for} \quad j=1, \ldots, N.$$

By 1.15 there is a contraction $\tau_0 \in \Delta^\times$ such that

$$\varphi_{\tau_0}(\tilde{Y}_0) \subset W$$

where φ_τ denotes the action on $\tilde{Y}_0$. Composing with $\tilde{F}$ gives us an equivariant embedding of *all* of $\tilde{Y}_0$, and in particular of $\partial\tilde{Y}_0 = M$, into $\mathbb{C}^N$.

Let $Y' = \tilde{F}(\varphi_{\tau_0}(\tilde{Y}_0))$ denote the image of this embedding. Then $L(\tau)(Y') \subseteqq Y'$ for all $\tau \in \Delta^\times \subset \mathbb{C}^\times$, where $L(\tau)$ is the linear $\mathbb{C}^\times$-action above. Hence, each image $L(\tau)(Y')$ for $\tau^{-1} \in \Delta^\times$ is an analytic extension of Y', and one easily checks that the monotone union

$$Y \equiv \bigcup_{n=1}^{\infty} L(n)(Y')$$

is a proper analytic subvariety of $\mathbb{C}^N$. It follows from the explicit form of the action (2.4) and the positivity of the exponents (2.5) that the volume function $v(r) \equiv \mathrm{vol}(\{z \in Y : \|z\| \leqq r\})$ has polynomial growth. Hence, Y must be algebraic. ■

Remark 2.6. — We note that the re-embedding of M which takes place in Theorem 2.1 is, in general, necessary. If $Y \subset \mathbb{C}^n$ is a variety with an isolated singularity which is not normal, then intrinsic S^1-actions on reg(Y) do not necessarily extend to holomorphic actions on Y. A simple example is given by the curve $C = \{(x, y) \in \mathbb{C}^2 : x^5 + y^5 + x^3 y^3 = 0\}$. The piece $C' = C \cap \{(x, y) : 0 < |x|^2 + |y|^2 \leqq 1\}$ is holomorphically equivalent to $\Delta^\times$, and so it carries an intrinsic $\Delta^\times$-action. However, by the condition of Saito [S] the singularity at 0 admits no holomorphic S^1-action.

On the other hand, if the isolated singularity 0 on Y is normal, then any weakly holomorphic action on Y is actually holomorphic. The proof of Proposition 2.2 then easily adapts to show that there exists a change of ambient coordinates at 0, and a linear $\mathbb{C}^\times$-action on these coordinates which preserves the germ of Y and restricts to the given $\Delta^\times$-action. [The new coordinates will be of the form $(\mathscr{M}/\mathscr{M}^2) \oplus \mathbb{C}^k$ with Y embedded in the first factor.] Every isolated singularity on a hypersurface of dimension > 1 is normal, so we have the following corollary.

COROLLARY 2.7. — *Let* $M \subset \mathbb{C}^{n+1}$ *be a maximally complex submanifold of dimension* $2n-1 > 1$, *and suppose* M *admits a transversal holomorphic* S^1-*action. Then after a holomorphic change of coordinates in* $\mathbb{C}^{n+1}$, M *is contained in an affine algebraic hypersurface* $Y \subset \mathbb{C}^{n+1}$. *The hypersurface* Y *has at most one singular point. It also has a* $\mathbb{C}^\times$-*action and the embedding* $M \subset Y$ *is* S^1-*equivariant.*

3. Kohn-Rossi cohomology and smooth equivalence

In [KR], J. J. Kohn and H. Rossi introduced and studied certain $\bar{\partial}_b$-cohomology groups $H_{KR}^{p,q}(M)$ of the boundary M of a complex analytic variety. When M is the boundary of a hypersurface in $\mathbb{C}^{n+1}$ with isolated singularities, the second author [Y] showed that certain $H_{KR}^{p,q}(M)$ carry the structure of an algebra. Specifically, each of the groups $H_{KR}^{p,q}(M)$, for $p+q=n-1$ and $1 \leqq q \leqq n-2$, is isomorphic to the direct sum of the moduli algebras, of the singular points of the variety.

The main point of this section is to show that for boundaries with transversal automorphisms the "Kohn-Rossi algebra" determines the manifold M and its embedding in $\mathbb{C}^{n+1}$ up to diffeomorphisms.

We assume throughout the section that our manifolds are compact smooth orientable and connected, and that S^1-actions are holomorphic.

THEOREM 3.1. — *Let* $M \subset \mathbb{C}^{n+1}$ *and* $M' \subset \mathbb{C}^{n+1}$ *be pseudoconvex* MC-*manifolds with transversal* S^1-*actions, each of dimension* $2n-1>1$. *Suppose there exists an algebra isomorphism*

$$H_{KR}^{p,q}(M) \cong H_{KR}^{p,q}(M')$$

for any (p, q) *as above. Then there exists a diffeomorphism* $f: \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ *with*

$$f(M)=M'.$$

This result has the following immediate corollary which is also a direct consequence of [Y] and [MY].

COROLLARY 3.2. — *Let* $M \subset \mathbb{C}^{n+1}$ *be as above and suppose that* $H_{KR}^{**}(M)=0$. *Then* M *is diffeomorphic to the standard sphere. Furthermore, if* $M \subset S^{2n+1} \equiv \{ z \in \mathbb{C}^{n+1} : \| z \| = 1 \}$, *then this embedding is isotopic to the standard one.*

It is well known that the vanishing of singular cohomology $H^*(M)=0$ is not sufficient for the above conclusion. The Brieskorn spheres:

$$M_d \equiv \{ z \in \mathbb{C}^{2n} : \| z \| = 1 \text{ and } z_1^d + \sum_{k>1} z_k^2 = 0 \}$$

are often exotic, and even when they are not, they are knotted in S^{4n-1}.

On the other hand, in the absence of an S^1-action even the vanishing of $H_{KR}^{**}(M)$ is not sufficient for the conclusion of 3.2. In fact, let $V_0 = \{ z \in \mathbb{C}^1 : p(z) = 0 \}$ be *any* algebraic hypersurface with an isolated singularity at the origin, and choose $\varepsilon > 0$ so that $\nabla p(z) \neq 0$ for $0 < |z| \leqq \varepsilon$ and so that V is transversal to $S(\varepsilon) \equiv \{ z \in \mathbb{C}^n : |z| = \varepsilon \}$. Consider the pseudoconvex manifolds $M_t \equiv \{ z \in \mathbb{C}^n : p(z) = t \} \cap S(\varepsilon)$. These manifolds are mutually diffeomorphic for all t sufficiently small. However, the results of [Y] show that $H_{KR}^{**}(M_t) = 0$ for $t \neq 0$. Thus, the Kohn-Rossi cohomology can be perturbed away without losing any essential property except the S^1-action. This discussion applies for example to the Brieskorn spheres above.

It is possible to construct two pseudoconvex MC-manifolds M, M′ of dimension $2n-1$ in $\mathbb{C}^n$ with $H^{**}_{KR}(M) \cong H^{**}_{KR}(M') \neq 0$, and with isomorphic algebraic structures, but which are not even homotopy equivalent as manifolds. Such a pair is given as follows. Let $p(x, y) = x^2(y-1)^2 + y^2(x-1)(x-\alpha)$ where $|\alpha - 1|$ is non-zero and small, and set

$$M_\varepsilon = \{(x, y, z_1, \ldots, z_{n-2}) : p(x, y) + \sum_j z_j^2 = 0\} \cap S(\varepsilon)$$

Then $M = M_\varepsilon$ and $M' = M_{1/\varepsilon}$ will have the stated properties for all $\varepsilon > 0$ sufficiently small.

Proof of Theorem 3.1. — By the results of section 1 we know that M and M′ are the smooth boundaries of analytic varieties Y and Y′ respectively each of which has a holomorphic $\Delta^\times$-action extending the given S^1-action and having one isolated fixed-point, which we assume to be the point $0 \in \mathbb{C}^{n+1}$.

From section 2 we know that the intrinsic $\Delta^\times$-action on Y extends to a holomorphic $\Delta^\times$-action on a neighborthood $\mathscr{U}$ of 0 in $\mathbb{C}^{n+1}$. In fact, this extended action is equivalent, after a holomorphic change of coordinates, to the restriction of a linear $\mathbb{C}^\times$-action. Denote this action by φ_τ for $\tau \in \Delta^\times$. Observe that for $\varepsilon > 0$ sufficiently small, all of the curves $\varphi_t(z)$, for $0 < t \leqq 1$, will be transversal to the spheres $S(\rho) = \{z \in \mathscr{U} : \|z\| = \rho\}$ for $0 < \rho < \varepsilon$. (This is because in small neighborhoods of the origin, the given action is C^1-close to a linear action.) We then define the manifold

$$M(\rho) \equiv Y \cap S(\rho)$$

and prove the following.

Lemma 3.3. — *For each* ρ, $0 < \rho \leqq \varepsilon$ *there is a diffeomorphism* $f_\rho : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ *such that* $f_\rho(M) = M(\rho)$.

Proof. — For each point $z \in M$ let $t(z) = \sup\{t \in (0, 1) : \varphi_t(z) \in M(\rho)\}$. By the transversality of the curves $\varphi_t(z)$ we see that $t(z)$ is a smooth function on M. Hence the map $F_\rho : M \to M(\rho)$ defined by setting $F_\rho(z) \equiv \varphi_{t(z)}(z)$ is smooth. By the transversality above we see that F_ρ is a local diffeomorphism, and therefore a diffeomorphism.

The family of embeddings $F_{\rho, s} : M \to \mathbb{C}^{n+1}$ *given by setting*

$$F_{\rho, s}(z) \equiv \varphi_{st(z)+(1-s)}(z)$$

is an isotopy of the original embedding $F_{\rho, 0}(z) = \varphi_1(z) = z$ with the embedding $F_{\rho, 1} = F_\rho$ constructed above. The standard isotopy extension theorem (*cf.* [H]) asserts that there exists a smooth family of diffeomorphisms $f_{\rho, s} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ such that $f_{\rho, s}|_M = F_{\rho, s}$. The map $f_\rho \equiv f_{\rho, 1}$ is our desired diffeomorphism. ■

The analogous discussion applies, of course, to M′ and Y′.

Now comes the main point of the proof. For $p+q = n-1$ and $1 \leqq q \leqq n-2$ there is a natural isomorphism $H^{p, q}_{KR}(M) \xrightarrow{\approx} \mathscr{M}_0(Y)$ where $\mathscr{M}_0(Y)$ is the "moduli algebra" of Y at the point 0, i. e.

$$\mathscr{M}_0(Y) \equiv \mathbb{C}\{z_0, \ldots, z_n\} \Big/ \left(p, \frac{\partial p}{\partial z_0}, \ldots, \frac{\partial p}{\partial z_n}\right)$$

where $p \in \mathbb{C}\{z_0, \ldots, z_n\}$ is the function which defines the hypersurface Y in a neighborhood of 0. (*See* [Y].) Our main hypothesis now implies that there is an algebra isomorphism:

$$\mathcal{M}_0(Y) \overset{\approx}{\to} \mathcal{M}_0(Y').$$

It then follows from a result of J. Mather and the second author [MY], that there is a holomorphic change of coordinates $H : \mathcal{U} \overset{\approx}{\to} \mathcal{U}'$ defined on a neighborhood $\mathcal{U}$ of 0 with $H(0) = 0$, so that

$$H(\mathcal{U} \cap Y) = \mathcal{U}' \cap Y'.$$

As a consequence we know (*cf.* [M]) that for all $\rho > 0$ and $\rho' > 0$ sufficiently small, there is a diffeomorphism $h_{\rho\rho'} : B(2\rho) \overset{\approx}{\to} B(2\rho')$ such that

$$h_{\rho\rho'}[M(\rho)] = M'(\rho') \tag{3.4}$$

where $B(r) \equiv \{z \in \mathbb{C}^{n+1} : \|z\| < r\}$, $M(\rho) = Y \cap S(\rho)$ and $M'(\rho') = Y' \cap S(\rho')$. This map may be altered outside $B(\rho)$ and $B(\rho')$ to become a diffeomorphism $h_{\rho'\rho} : \mathbb{C}^{n+1} \overset{\approx}{\to} \mathbb{C}^{n+1}$ with the same property (3.4). Lemma 3.3 can now be applied to give diffeomorphisms f_ρ and $f'_{\rho'}$ with $f_\rho(M) = M(\rho)$ and $f'_{\rho'}(M') = M'(\rho')$. The concatenation $f = (f'_{\rho'})^{-1} \circ h_{\rho'\rho} \circ f_\rho$ is the desired diffeomorphism. ■

It is worth noting that our proof actually shows the following.

Corollary 3.6. — *Let* M *and* M′ *be as in Theorem* 3.1 *with* $H^{p,q}_{KR}(M) \cong H^{p,q}_{KR}(M')$. *Then as* CR-*manifolds,* M *and* M′ *are "strongly h-cobordant" in the following sense. There is a complex manifold* W *which is diffeomorphic to* $M \times [0, 1]$ *and for which there is a* CR (*i. e.,* "*holomorphic*") *diffeomorphism* $F : \partial W \to M \amalg M'$.

Proof. — Let φ_t and H be as in the proof above, and choose $t > 0$ sufficiently small that $\varphi_t(M) \subset \mathcal{U}$. Then the holomorphic map $H \circ \varphi_t$ embeds Y into interior(Y′). We set

$$W \equiv Y' - H \circ \varphi_t(Y)$$

and note that $\partial W \cong_{CR} M \amalg M'$. It remains to see that W is diffeomorphic to $M \times [0, 1]$. It is elementary that for ρ and ρ' sufficiently small and chosen so that $H(Y \cap B(\rho)) \supset Y' \cap B(\rho')$, the manifold $Y' \cap B(\rho) - H(Y \cap B(\rho))$ is diffeomorphic to $M \times [0, 1]$. Using the flow as in the proof of Lemma 3.3 we can construct diffeomorphisms $Y - B(\rho) \cong M \times [0, 1]$ and $Y' - B(\rho') \cong M' \times [0, 1]$. Hence

$$W = (Y' - B(\rho')) - [H(Y - B(\rho)) \cup H(Y \cap B(\rho))]$$

is diffeomorphic to $M \times [0, 1]$ with a collar neighborhood of one boundary component removed. ■

4. A fixed-point formula for analytic space

The first result of this section is the following.

THEOREM 4.1. — *Let* X *be a compact complex analytic space with a weakly holomorphic* S^1-*action. Then*

$$\chi(X)=\chi(F)$$

where F *is the fixed-point set of the action.*

This theorem actually holds in the more general category of compact differentiable stratified sets. (*See* Theorem 4.7 below.) However, for its interest, we shall first present a quick proof of Theorem 4.1 which is based on the resolution of singularities.

PROOF. — By the work of Hironaka [Ha] we know that there exists an equivarient resolution $\tilde{X} \xrightarrow{\pi} X$ of X. Let $S \subset X$ denote this singular set of X and let $E \equiv \pi^{-1}(S)$ be the exceptional set. Then it is well known and easy to check that

$$\chi(\tilde{X})-\chi(E)=\chi(X)-\chi(S). \tag{4.2}$$

We proceed by induction on the dimension of X. The result is clearly true for dim $X=0$. We assume that dim $X=n$ and that the results is proved in all lower dimensions.

Observe that S^1 acts analytically on the smooth manifold $\tilde{X}$ and so its fixed-point set $\tilde{F}$ is a smooth complex submanifold of $\tilde{X}$. The vector field generating the action on X is projectable to X. Hence, we have

$$\pi(\tilde{F}) \subseteqq F.$$

We shall assume for the moment that F has no (irreducible) components of top dimension (i. e., of dimension n). Now the singular set S is S^1-invariant, and therefore so is the set E. By induction we have that

$$\chi(S)=\chi(F_S) \quad \text{and} \quad \chi(E)=\chi(F_E) \tag{4.3}$$

where F_Y denotes the fixed-point set of Y. From the classical Lefschetz formula (or, say, the Atiyah-Singer formula) we know that

$$\chi(\tilde{X})=\chi(\tilde{F}).$$

Together with (4.2) and (4.3) this gives us

$$\chi(X)=\chi(\tilde{F})-\chi(F_E)+\chi(F_S). \tag{4.4}$$

We want to claim that

$$\chi(\tilde{F})-\chi(F_E)+\chi(F_S)=\chi(F). \tag{4.5}$$

Since $\tilde{X}-E \xrightarrow{\pi} X-S$ is an analytic isomorphism we see that

$$\tilde{F}-F_E \underset{\approx}{\xrightarrow{\pi}} F-F_S \tag{4.6}$$

is an isomorphism. If A and B are any two subcomplexes of a finite simplicial complex, we have that $\chi(A \cup B)=\chi(A)+\chi(B)-\chi(A \cap B)$. Taking a regular neighborhood of F_E in $\tilde{F}$, we get a triangulable set whose boundary is an odd-dimensional oriented manifold. The same holds for a regular neighborhood of F_S in F. Hence, we have

$$\chi(\tilde{F})=\chi(F_E)+\chi(\tilde{F}-F_E) \qquad \text{and} \qquad \chi(F)=\chi(F_S)+\chi(F-F_S)$$

from which it follows that

$$\begin{aligned}\chi(\tilde{F})-\chi(F_E)+\chi(F_S)\\ &=\chi(\tilde{F}-F_E)+\chi(F_S)\\ &=\chi(F-F_S)+\chi(F_S)=\chi(F)\end{aligned}$$

as desired.

Suppose now that F has a component of top dimension. Let F_0 be the union of all such components and let $X_0 \equiv \text{closure}\,(X-F_0)$. As above we have

$$\begin{aligned}\chi(X)&=\chi(X_0)+\chi(F_0-X_0)\\ &=\chi(X_0)+\chi(F-X_0)\\ &=\chi(F \cap X_0)+\chi(F-X_0)\\ &=\chi(F). \quad \blacksquare\end{aligned}$$

As noted above, Theorem 4.1 has the following generalization.

THEOREM 4.7. — *Let* X *be a compact differentiable stratified set which admits a smooth* S^1*-action with fixed-point set* F. *Then*

$$\chi(X)=\chi(F).$$

Proof. — Our first step is to choose a prime p sufficiently large that the subgroup $\mathbb{Z}_p \subset S^1$ has the same fixed-point set, i. e., so that $F=\{x \in X : g(x)=x \text{ for all } g \in \mathbb{Z}_p\}$. To see that this is possible we first note that, by standard arguments, each point $x \in X$ has a neighborhood U_x with the property that

$$G_y \subset G_x \quad \text{for all } y \in U_x$$

where by definition $G_y=\{g \in S^1 : g(y)=y\}$ is the isotropy subgroup of y. It follows that the number of distinct isotropy subgroups is finite. Hence, there exists a prime p so that $\mathbb{Z}_p \not\subset G_x$ for any $x \notin F$, and the assertion follows.

Our next step is to triangulate X so that:

(i) F is a finite subcomplex, and

(ii) f maps simplicies to simplicies, where $f: X \to X$ is a generator of the action.

To do this we need only triangulate the differentiable stratified set $X/\mathbb{Z}_p$ with F as a subcomplex.

Let $C_k(X)$ denote the group of integral k-chains for this triangulation, and write

$$C_k(X) = C_k(F) \oplus \tilde{C}_k(F) \tag{4.8}$$

where $C_k(F)$ [respectively $\tilde{C}_k(F)$] denotes the subgroup generated by the k-simplicies which are contained (respectively, not contained) in F. The induced chain map $f_* : C_k(X) \to C_k(X)$ respects the decomposition (4.8) and clearly $f_*|_{C_k(F)} = \mathrm{Id}_{C_k(F)}$. In the basis for $\tilde{C}_k(F)$ given by the k-simplicies, f_* has the form:

$$f_*|_{\tilde{C}_k(F)} \cong \begin{pmatrix} 0 & & & & * \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ * & & & & 0 \end{pmatrix}$$

To see this, it suffices to show that if for some k-simplex $\sigma \subset X$, we have that $f(\sigma) = \sigma$, then $\sigma \subset F$. Since F is a subcomplex, it will suffice to show that: $f(\sigma) = \sigma \Rightarrow F \cap (\text{interior } \sigma) \neq \varnothing$. However, since p is prime, $F \cap (\text{int } \sigma) = \varnothing$ implies that $\mathbb{Z}_p$ acts freely on $\text{int}(\sigma) \cong \mathbb{R}^k$, which in turn implies that $\chi(\mathbb{R}^k/\mathbb{Z}_p)\, p = \chi(\mathbb{R}^k) = 1$.

The result now follows immediately from the Lefschetz Fixed-point Formula. ■

5. An application to cycle spaces

By a cycle of dimension p on a compact complex analytic space X we mean a finite sum $\sum n_k V_k$ where $n_k \in \mathbb{Z}^+$ and V_k is an irreducible p-dimensional complex subvariety of X. (Thus "cycle" here means "effective analytic cycle".) The space of cycles in a fixed class has the structure of a complex analytic variety (Barlet [B]).

We shall first examine the spaces of cycles on complex projective space $\mathbb{P}^n(\mathbb{C})$. Fix positive integers p, d, and n and let $\mathscr{C}_{p,d,n} = \mathscr{C}_{p,d}$ denote the space of all cycles of dimension p and degree d in $\mathbb{P}^n(\mathbb{C})$, where $\text{degree}(\sum n_k V_k) = \sum n_k \text{degree}(V_k)$ is the homology degree of the cycle.

It is not difficult to see that each space $\mathscr{C}_{p,d,n}$ is simply connected. We shall give here a quick computation of the Euler characteristic of these spaces.

THEOREM 5.1:

$$\chi(\mathscr{C}_{p,d,n}) = \binom{\nu+d-1}{d} \qquad \text{where} \quad \nu = \binom{n+1}{p+1}$$

Note. — This theorem recaptures the well known facts that $\chi(\mathscr{C}_{n-1,\,d,\,n})=\binom{n+d}{d}$ and $\chi(\mathscr{C}_{p,\,1,\,n})=\binom{n+1}{p+1}$.

It is interesting to observe that if for each pair of integers p and n with $0\leqq p\leqq n$ we define the formal power series

$$Q_{p,\,n}(t)\equiv\sum_{d=0}^{\infty}\chi(\mathscr{C}_{p,\,d,\,n})\,t^d,$$

then Theorem 5.1 can be restated in the form

$$Q_{p,\,n}(t)=\left(\frac{1}{1-t}\right)^{\binom{n+1}{p+1}}$$

where we have adopted the convention that $\chi(\mathscr{C}_{p,\,0,\,n})=1$.

Proof of theorem 5.1. — We consider the action of the n-torus $T=T_{n+1}/T^1\subset U_{n+1}/T^1=PU_{n+1}$ on $\mathbb{P}^n(\mathbb{C})$ given by setting

$$(5.2)\qquad \Phi_t([z])=[e^{it_0}z_0,\,\ldots,\,e^{it_n}z_n]$$

where $t=(t_0,\,\ldots,\,t_n)\in T^{n+1}$ and where $[z]=[z_0,\,\ldots,\,z_n]$ are homogeneous coordinates for $\mathbb{P}^n(\mathbb{C})$. Fix $z\in\mathbb{C}^{n+1}-\{0\}$, and suppose that $z_{i_0},\,\ldots,\,z_{i_p}$ are exactly the coordinates of z which are not zero. Then one easily sees that

$$(5.3)\qquad \Phi_t([z])=[z]\quad\Leftrightarrow\quad t_{i_0}=\ldots=t_{i_p}.$$

Hence, if $T_{[z]}=\{t\in T:\Phi_t([z])=[z]\}$, then we have

$$(5.4)\qquad \dim_{\mathbb{R}}(T_{[z]})=n-p.$$

Consequently for each ordered $(p+1)$-tuple of integers $\alpha=(\alpha_0,\,\ldots,\,\alpha_p)$ with $0\leqq\alpha_0<\alpha_1<\ldots<\alpha_p\leqq n$, we consider the coordinate $(p+1)$-plane

$$\mathbb{C}_\alpha^{p+1}\equiv\{z\in\mathbb{C}^{n+1}:z_i=0\text{ if }i\neq\alpha_j\text{ for some }j\}\subset\mathbb{C}^{n+1}$$

and we denote by

$$\mathbb{P}_\alpha^p\subset\mathbb{P}^n(\mathbb{C})$$

the corresponding projective p-plane. We see immediately that

$$(5.5)\qquad \dim_{\mathbb{R}}(T_{[z]})\geqq n-p\quad\Leftrightarrow\quad[z]\in\mathbb{P}_\alpha^p\text{ for some }\alpha.$$

This fact may be viewed alternatively as follows. For each point $[z]\in\mathbb{P}^n(\mathbb{C})$ we define $\tau_{[z]}$ to be the tangent space to the orbit of T at $[z]$. Clearly we have that $\dim_{\mathbb{R}}(T_{[z]})+\dim_{\mathbb{R}}(\tau_{[z]})=n$, and so (5.5) can be rewritten as

$$(5.5)'\qquad \dim_{\mathbb{R}}(\tau_{[z]})\leqq p\quad\Leftrightarrow\quad[z]\in\mathbb{P}_\alpha^p\text{ for some }\alpha.$$

An important fact which is straightforward to verify is that the subspaces τ are all Lagrangian i.e., $\tau \perp J\tau$ at each point. In particular we have that

$$(5.6) \qquad \dim_{\mathbb{C}}(\tau_{[z]} + J\tau_{[z]}) = \dim_{\mathbb{R}}(\tau_{[z]})$$

for all $[z] \in \mathbb{P}^n(\mathbb{C})$.

From all of this we have a very clear picture of the orbit space. Consider the standard n-simplex $\Delta^n \equiv \{(x_0, \ldots, x_n) \in \mathrm{R}^{n+1} : x_j \geqq 0 \text{ for all } j \text{ and } \sum x_j = 1\}$. The following is easily proved.

PROPOSITION 5.7. — *There is a diffeomorphism of differentiable stratified sets*

$$\mathbb{P}^n(\mathbb{C})/\mathrm{T} \overset{\approx}{\to} \Delta^n$$

which associates to the orbit of $[z] \in \mathbb{P}^n(\mathbb{C})$, *the element*

$$\left(\frac{|z_0|}{|||z|||}, \ldots, \frac{|z_n|}{|||z|||}\right) \in \Delta^n$$

where $|||z||| \equiv \sum |z_j|$. *Each subspace* $\mathbb{P}^p_\alpha$ *is T-invariant, and* $\mathbb{P}^p_\alpha/\mathrm{T}$ *is mapped to the* αth *p-dimensional edge of* Δ^n.

Note. — This map is merely the "moment map" for the action given by the Kähler form.

We now come to the main proposition.

PROPOSITION 5.8. — *Let* V *be a p-dimensional analytic cycle on* $\mathbb{P}^n(\mathbb{C})$ *which is* T-*invariant. Then*

$$\mathrm{V} = \sum n_\alpha \mathbb{P}^p_\alpha$$

for integers $n_\alpha \geqq 0$.

Proof. — Let reg(V) denote the set of regular points of the (reduced) cycle. Fix $x \in \mathrm{reg}(\mathrm{V})$. Since V is T-invariant, we have $\tau_x \subset \mathrm{T}_x\mathrm{V}$, and so $\dim_{\mathbb{C}}(\tau_x + J\tau_x) \leqq p$. Applying (5.6) and (5.5)′ we conclude that

$$x \in \mathrm{reg}(\mathrm{V}) \quad \Rightarrow \quad x \in \mathbb{P}^p_\alpha \quad \text{for some } \alpha.$$

The proposition now follows easily. ■

The action of T on $\mathbb{P}^n(\mathbb{C})$ induces an action of T on $\mathscr{C}_{p,d,r}$. By Proposition 5.7 the fixed-point set of this action consists exactly of the cycles of the form $\sum n_\alpha \mathbb{P}^p_\alpha$ where $\sum n_\alpha = d$. Applying Theorem 4.1 (inductively n-times) shows that

$$\chi(\mathscr{C}_{p,d,n}) = \mathrm{card}\,\{(m_1, \ldots, m_\nu) \in \mathbb{Z}^\nu : \text{each } m_j \geqq 0 \text{ and } \sum m_j = d\}$$
$$= \binom{\nu + d - 1}{d} \quad \text{where } \nu = \binom{n+1}{p+1}. \quad ■$$

The result above suggests the following definition in the general case. Let X be a compact Kähler manifold of dimension n. For each integer p, $0 \leqq p \leqq n$, we shall define a formal sum of characters, i. e. of homomorphisms

$$H^{2p}(X; \mathbb{R})/H^{2p}(X; \mathbb{Z})_{\text{mod torsion}} \to S^1,$$

as follows. Let $\mathscr{C}$ denote the space of analytic cycles on X and for each $\lambda \in H_{2p}(X; \mathbb{Z})_{\text{mod torsion}}$ let $\mathscr{C}_\lambda$ denote the subset of cycles in $\mathscr{C}$ which represent λ. Then for $x \in H^{2p}(X; \mathbb{R})$, we define

$$Q_p(x) = \sum_\lambda \chi(\mathscr{C}_\lambda)\, e^{2\pi i \langle \lambda, x \rangle}$$

where by convention we set $\chi(\varnothing) = 0$. If we choose a basis $e_1, \ldots, e_N$ of $H_{2p}(X; \mathbb{Z})_{\text{med torsion}}$ then Q_p can be rewritten as follows. Set $\lambda = \sum \lambda_j e_j$ and for each j, set $t_j = e^{2\pi i \langle e_j, x \rangle}$. Then as a function of $t = (t_1, \ldots, t_N)$, Q_p becomes

$$Q_p(t) = \sum_\lambda \chi(\mathscr{C}_\lambda)\, t^\lambda \tag{5.9}$$

where the sum is taken over all $\lambda \in \mathbb{Z}^N$.

The method used above enables us to compute this function also in the case where $X = \mathbb{P}^n \times \mathbb{P}^m$. Here we have a canonical decomposition

$$H_{2p}(X; \mathbb{Z})_{\text{mod torsion}} = H_{2p}(X; \mathbb{Z}) = \bigoplus_{\substack{0 \leqq k \leqq n \\ 0 \leqq l \leqq m \\ k+l=p}} H_{2k}(\mathbb{P}^n) \otimes H_{2p}(\mathbb{P}^m) \tag{5.10}$$

Let $\{e_{n,l}\}_{k+l=p}$ denote the obvious basis of $H_{2p}(X; \mathbb{Z})$ with respect to this decomposition, and let $\{\chi_{k,l}\}_{k+l=p}$ denote the corresponding coordinates on $H^{2p}(X; \mathbb{R})$. Setting $t_{k,l} = e^{2\pi i x_{k,l}}$ we can express the function Q_p as in (5.9).

THEOREM 5.11. — *For* $X = \mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^m(\mathbb{C})$ *and for any integer* p, $0 \leqq p \leqq n+m$, *one has that*

$$Q_p(t) = \prod_{k+l=p} (1 - t_{k,l})^{-\binom{n+1}{k+1}\binom{m+1}{l+1}}$$

Proof. — Consider the action of the torus $T = T^n \times T^m$ on $\mathbb{P}^n \times \mathbb{P}^m$ given as the product of the actions considered above on each factor. A general class $\lambda \in H_{2p}(X; \mathbb{Z})$ can be written as $\lambda = \sum \lambda_{k,l} e_{k,l}$ (where $\lambda_{k,l} \in \mathbb{Z}$), and this class contains an analytic cycle if and only if $\lambda_{k,l} \geqq 0$ for all k, l. Arguing as above, one finds that the T-fixed cycles in λ are exactly those cycles of the form

$$c = \sum_{\substack{k+l=p \\ \alpha,\, \beta}} m_{k,\, l,\, \alpha,\, \beta}\, \mathbb{P}^k_\alpha \times \mathbb{P}^l_\beta$$

where the sum runs over all α's with $0 \leqq \alpha_0 < \alpha_1 < \ldots < \alpha_k \leqq n$ and all β's with $0 \leqq \beta_0 < \beta_1 < \ldots < \beta_2 \leqq m$; where $\mathbb{P}^k_\alpha$ and $\mathbb{P}^l_\beta$ denote the distinguished subspaces of $\mathbb{P}^n(\mathbb{C})$

and $\mathbb{P}^m(\mathbb{C})$ as before; and where

$$\sum_{\alpha,\,\beta} m_{k,l,\,\alpha,\,\beta}=\lambda_{k,l}$$

for each k and l.

From Theorem 4.1 we know that $\chi(\mathscr{C}_\lambda)=$ the number of fixed cycles in λ, and this in turn is the product $\prod \mathrm{N}_{k,l}$ where $\mathrm{N}_{k,l}=\#\{(m_{\alpha\beta})\in\mathbb{Z}^{\binom{n+1}{k+1}\binom{m+1}{l+1}}:\sum m_{\alpha,\,\beta}=\lambda_{k,\,l}\}=$ the number of distinct monomials of degree $\lambda_{k,l}$ in $\binom{n+1}{k+1}\binom{m+1}{l+1}$ variables. Hence, we have that

$$\mathrm{N}_{k,l}=\binom{\mu_{k,l}+\lambda_{k,l}-1}{\lambda_{k,l}}$$

where $\mu_{k,l}=\binom{n+1}{k+1}\binom{m+1}{l+1}$, and so

$$\chi(\mathscr{C}_\lambda)=\prod_{k+l=p}\binom{\mu_{k,l}+\lambda_{k,l}-1}{\lambda_{k,l}}.$$

It follows that

$$\begin{aligned}\mathrm{Q}_p(t)&=\sum_\lambda \chi(\mathscr{C}_\lambda)\,t^\lambda\\&=\sum_{\lambda\geqq 0}\prod_{k+l=p}\binom{\mu_{k,l}+\lambda_{k,l}-1}{\lambda_{k,l}}(t_{k,l})^{\lambda_{k,l}}\\&=\prod_{k+l=p}(1-t_{k,l})^{-\mu_{n,l}}\end{aligned}$$

For cycles of codimension one, i.e., in the case where $p=n+m-1$, the rationality of this function can be deduced as a general consequence of the rationality of generating functions associated to graded modules. In higher codimensions, this rationality is more mysterious and intriguing.

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