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FORMAL GROUPS ARISING FROM ALGEBRAIC VARIETIES

BY M. ARTIN AND B. MAZUR

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Introduction

Let X/k be a proper variety. The formal completion Pic $X^\wedge$ of the Picard group of X may be regarded as representing a problem of deformation theory about isomorphism classes of line bundles on X. If S = Spec A is an artinian local affine scheme with residue field k, the S-valued points of the formal group Pic $X^\wedge$ may be described in cohomological terms by the short exact sequence

$$0 \to \mathrm{Pic}\, X^\wedge(S) \to H^1(X \times S, \mathbf{G}_m) \to H^1(X, \mathbf{G}_m).$$

Schlessinger, in his paper [30], takes this observation as a point of departure in the study of Pic X. In particular, he establishes easily manageable criteria for pro-representability of functors of the above type, and, defining Pic $X^\wedge$ by the above exact sequence, he shows that it is pro-represented by a formal group over k.

The origin of this paper is the observation that this approach is by no means limited to studying deformation problems involving H^1. For any integer $r \geqq 0$, we can mimic the above exact sequence and define functors

$$\Phi^r: \quad \mathrm{Art}/k^0 \to (\mathrm{Ab})$$

using $H^r(X, \mathbf{G}_m)$ (*cf.* II). We show that these functors Φ^r exhibit a strong tendency to be pro-representable and, in fact, by formal Lie groups [1]. The exact hypotheses are in II, paragraph 3, where the theory is systematically developed for flat schemes X/S and Φ^r/S is studied as a formal group over the base S. For example, Φ^r/k is a formal Lie group if $h^{0,r-1} = h^{0,r+1} = 0$, hence when X/k is a smooth complete intersection of dimension $r \geqq 2$ in projective space.

The formal groups Φ^r appear, then, as higher dimensional analogues of the classical formal group Pic $X^\wedge$. In the first interesting case, $r = 2$, we shall sometimes refer to Φ^2 as $\widehat{\mathrm{B}}r$, and call it the *formal Brauer group* of X.

Two natural problems arise:

(*a*) Interpret the numerical invariants of these formal groups in terms of the algebraic geometry of X.

(*b*) Find new properties of X which are brought into focus by these formal groups and their particular properties.

The purpose of this paper is to set up some fragments of a theory to answer the above questions. We sketch some of the principal results, which we prove *under certain hypotheses*. For the exact hypotheses, the reader is referred to the body of the text.

The most obvious invariant of a formal group is its dimension. One has (§ 3):

$$\dim \Phi^r = h^{0,r} = \dim_k H^r(X, \mathcal{O}_X).$$

If k is of characteristic 0, the dimension is the only numerical invariant of a formal Lie group. However, in characteristic p there are subtler invariants contained in the Dieudonné module $D\,\Phi^r$. We establish two facts about this Dieudonné module. The first result relates $D\,\Phi^r$ to something as difficult to compute:

$$D\,\Phi^r = H^r(X, \mathscr{W}),$$

where $\mathscr{W}$ is Serre's *Witt vector sheaf* on X [31]. This provides in some measure an explanation of the fact that Serre's Witt vector cohomology may fail to be of finite type over $W = W(k)$: It is of finite type over W if and only if Φ^r is a formal group of *finite height* (i. e., has no unipotent part). More tangibly,

$$H^r(X, \mathscr{W}) \otimes_W K$$

is finite dimensional over the field of fractions K of W, whenever Φ^r is pro-representable.

(1) This fact was first noticed by Levelt (unpublished).

The second result relates $D\,\Phi^r$ to more familiar arithmetic invariants of X. Its precise statement involves some further hypotheses which may be found in chapter IV. In this introduction we shall try to apply these precise statements to the calculation of the numerical invariants of $D\,\Phi^r$. Suppose k is algebraically closed. Then the action of the semi-linear endomorphism F on the vector space $D\,\Phi^r \otimes_W K$ decomposes this vector space into eigenspaces, giving us a family of eigenvalues

$$\alpha_1, \ldots, \alpha_h \in K$$

counted with multiplicites (*see* § 1). Unlike the situation in linear algebra, these eigenvalues are not unique, for modification of the eigenvector by scalar multiplication tends to change them. Nevertheless, their p-adic ordinals $a_j = \operatorname{ord}_p \alpha_j$ are uniquely determined by the semi-linear endomorphism F. We normalize by taking $\operatorname{ord}_p p = 1$. Then the a_j are non-negative rational numbers which we may write in non-decreasing order

$$0 \leqq a_1 \leqq a_2 \leqq \ldots \leqq a_h,$$

and these rational numbers together with their multiplicities of occurrence are called the *slopes* and *multiplicities* of the Dieudonné module of Φ^r, or more succinctly, the slopes and multiplicities of Φ^r. They are numerical invariants which determine, and are determined by, the isogeny class of a "maximal quotient of Φ^r of finite height" (*see* § 1). Indeed, most of the salient invariants of this quotient are readily visible from the above set of slopes and multiplicities. For instance, its height is simply the number h. Further, Φ^r is connected and therefore there is a topologically nilpotent endomorphism V on $D\,\Phi^r \otimes_W K$ such that $FV = VF = p$. It follows that all slopes are *less than* 1.

There is another semi-linear vector space with an operator F determined by X/k. This is the crystalline cohomology group $H^r_{\text{crys}}(X/W)$.

By performing the analogous semi-linear algebra to the semi-linear endomorphism F on $H^r_{\text{crys}}(X/W) \otimes_W K$, we may obtain its set of eigenvalues

$$b_1, b_2, \ldots, b_n,$$

and setting $\beta_j = \operatorname{ord}_p b_j$ we obtain the slopes counted with multiplicities of the semi-linear F-module $H^r_{\text{crys}}(X/W) \otimes_W K$

$$0 \leqq \beta_1 \leqq \beta_2 \leqq \ldots \leqq \beta_n.$$

Here, again, we have arranged things in non-decreasing order. (One should note that when X is defined over a finite field $\mathbf{F}_q$ $(q = p^i)$, then the β_j's are indeed well-known arithmetic invariants of X/F_q, for the quantities $i\,\beta_j$ are then the p-adic orders of the eigenvalues of the Frobenius automorphism acting on the r-dimensional l-adic cohomology of X [23].)

In contrast with the α_j's, the β_j's need not be less than 1. Nevertheless, a consequence of the Corollary 3 of paragraph 4 is that, under certain hypotheses, the relationship between the α_j's and the β_j's is the best that can be hoped for. Namely, *the* α_j's *coincide with those* β_j's *which are less than* 1.

The above facts, taken together, give fairly explicit information concerning Serre's Witt vector cohomology.

It is also interesting to combine this determination of the slopes and multiplicities of Φ^r with knowledge of the dimension of Φ^r. One obtains the following inequality:

$$h^{0,r} \geqq \sum_{\beta} m_\beta (1-\beta),$$

where β ranges through the rational numbers < 1 occurring, as slopes of H^r_{crys}, and m_β denotes their multiplicity (Cor. 4, § 4). This inequality is the first of a series of inequalities known as the Katz conjecture ([26], [27]).

CONNECTIONS WITH NÉRON-SEVERI. — Let X be a smooth proper surface over k, and let ρ be the rank of the Néron-Severi group of X. In characteristic zero, one has the classical formula

$$\rho \leqq b_2 - 2h^{0,2},$$

where b_2 is the second Betti number. We prove an analogue of this formula in characteristic p involving the formal Brauer group $\widehat{\text{Br}}$ of X. Namely, suppose $\widehat{\text{Br}}$ is representable by a formal group of *finite height h*. Under some supplementary hypotheses we prove

$$\rho \leqq b_2 - 2h.$$

Since $h \geqq h^{02}$, this is *stronger* than the classical assertion, provided $h < +\infty$. On the other hand, there are many surfaces [33] in characteristic p for which the classical inequality breaks down, though the weaker inequality $p \leqq b_2$ continues to hold (Igusa [20]). The simplest example of such a surface is the product of a supersingular elliptic curve with itself, which has $\rho = b_2 = 6$, and $h^{02} = 1$. Its formal Brauer group is the additive group.

A proof of the following conjecture would complement our result, and the two could be considered a satisfactory generalization of the classical inequality to characteristic p:

CONJECTURE. — With the above notation, assume that the characteristic is not zero, and that $\widehat{\text{Br}\,X}$ is unipotent. Then $b_2 = \rho$.

Examples. — Throughout our work the case of K 3 surfaces [4] (over an algebraically closed field) in characteristic $p > 0$ has been an extremely useful guide. For these surfaces $h^{0,1} = 0$ and $h^{0,2} = 1$ and consequently the formal Brauer group is a formal one-parameter group; denote its height by h. The second Betti number of a K 3 surface is 22. The Néron-Severi group of a K 3 surface is a free abelian group (of rank ρ). Using results of the present paper, the relation between h and the eigenvalues of Frobenius (best visualized by the Newton polygon [26]) may be summarized as follows:

$h = \infty$: Equivalently, one has that $\widehat{\text{Br}} = \hat{\mathbf{G}}_a$, or that the 22 eigenvalues of Frobenius acting on 2-dimensional cohomology have ord_p equal to 1. These K 3 surfaces are called *supersingular* in [4] (compare with definition p. 199 of [23]). An elliptic K 3 surface is supersingular if and only if $\rho = 22$ [4].

$h < \infty$: Equivalently, $\hat{\mathrm{Br}}$ is a p-divisible formal group. In this case $1 \leqq h \leqq 10$, and as is proved in [4] and [35], every h in this range occurs. The 22 eigenvalues of Frobenius acting on 2-dimensional cohomology distribute themselves as follows: There are h such eigenvalues with ord_p equal to $h - 1/h$; there are h eigenvalues with ord_p equal to $h + 1/h$; the remaining $22 - 2h$ eigenvalues have ord_p equal to 1.

To treat moduli problems arising from this example, and others, it is convenient to develop the theory of the formal Brauer group over general bases. Given a parameter space T of K 3 surfaces, the function h has an upper-semi-continuous behavior on T, and may be used to define a stratification on T, studied in [4]. It is proved in [4] that the supersingular K 3's determine a stratum of relatively low dimension in "the" moduli space (²).

Two examples where the formal Brauer group $\widehat{\mathrm{Br}}$ is of multiplicative type (a finite product of $\hat{\mathbf{G}}_m$'s if the groundfield is algebraically closed) are worth mentioning:

(*a*) The Fermat surface $\mathscr{F}(d)$: $X^d + Y^d + Z^d + W^d = 0$ in characteristic p, where $p \equiv 1 \bmod d$.

(*d*) Any "sufficiently general" smooth surface in P^3.

One obtains (*a*) using the results of the present paper, together with a calculation of the Newton polygon of Fermat varieties ([23], V. 2) and similarly (*b*) is obtained using a theorem of Koblitz ([23] (II); the condition of "sufficient generality" is, however, not explicit, and therefore one doesn't obtain specific examples in hand).

THE "ENLARGED" FUNCTOR Ψ. — The formal Brauer group is, by definition, a *connected* formal group, and therefore the slopes of its Dieudonné module are constrained to be < 1. It is natural to seek an enlargement of $\hat{\mathrm{Br}}$, which, under suitable hypotheses, will be a (not necessarily connected) *p-divisible group* whose Dieudonné module slopes coincide with the slopes of $H^2_{\mathrm{crys}}(X/W) \otimes_W K$ in the closed interval $[0, 1]$, and whose connected part is $\hat{\mathrm{Br}}$.

This intention is served (under suitable hypotheses) by the functor Ψ introduced in (IV. 1). Its étale quotient $\Psi^{\text{ét}}$ is the divisible part of $H^2_{fl}(X_{\bar{k}}, \mu_{p^\infty})$ [as $\mathrm{Gal}(\bar{k}/k)$-module]. Using a result of Bloch [7] when $p > 2$, we obtain that the height of $\Psi^{\text{ét}}$ is the number of eigenvalues of slope 1 in $H^2_{\mathrm{crys}}(X/W) \otimes_W K$. Although this gives what we wish as far as numerical values are concerned, it would be better to have a direct relationship between the Dieudonné module of Ψ and the quotient of $H^2_{\mathrm{crys}}(X/W)$ comprising all eigenvalues whose slopes lie in the interval $[0, 1]$. Can this be obtained by considering a hybrid "crystalline-fppf" site, in analogy with the construction of crys-ét in paragraph 3 below?

COHOMOLOGY WHICH IS ANALYSABLE BY p-DIVISIBLE GROUPS; HODGE-TATE DÉCOMPOSITIONS. — Let $K/\mathbf{Q}$ be a finite extension, $R \subset K$ its ring of integers and X/K a proper smooth surface. Let $T(-)$ denote the Tate construction. In (IV.4) we study

(²) There are several moduli problems interesting to consider in connection with K 3's : polarized K 3's (or not), elliptic K 3's...

$H = T(H^2_{\text{ét}}(X/\overline{K}'/\mu_\infty)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ the finite dimensional $\mathbf{Q}_p$-vector space, with its natural $\text{Gal}(\overline{K}/K)$ action. The cup-product pairing induces an isomorphism between H and H*, its vector space dual, with respect to which the action of $\text{Gal}(\overline{K}/K)$ enjoys an evident compatibility. Let us say that H is *analysable by p-divisible groups* if there is a filtration

$$(\bigstar) \qquad 0 \subset W \subset V \subset H$$

by sub-$\mathbf{Q}_p$-vector spaces, stable under the action of $\text{Gal}(\overline{K}/K)$, such that:

(*a*) W is a $\text{Gal}(\overline{K}/K)$-representation "coming from a *p*-divisible group" $\mathbf{W}/R$ (i. e. $W \cong T\mathbf{W} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$).

(*b*) V is the $\text{Gal}(\overline{K}/K)$-representation coming from $\mathbf{W}^0/R$, the connected component of $\mathbf{W}$ ($V = T\mathbf{W}^0 \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$).

(*c*) The filtration ($\bigstar$) is self-dual under the cup-product pairing, in the sense that $W = V^\perp$ and (equivalently) $V = W^\perp$.

One has a rather tight description of the $\text{Gal}(\overline{K}/K)$-representation H, when it is "analysable by *p*-divisible groups". For example, using results of Tate one sees that if H is "analysable by *p*-divisible groups", then the semi-simplification of the $\text{Gal}(\overline{K}/L)$-representation H admits a Hodge-Tate decomposition with the expected "Hodge" numbers; H itself admits such a Hodge-Tate decomposition if $\mathbf{W}^0$ is of multiplicative type. It is not true that *all* surfaces (even those with good reduction in characteristic *p*) will have "analysable" 2-dimensional cohomology. Indeed, it is plausible that for any X/R a proper smooth surface such that $\widehat{\text{Br}}/R$ is pro-representable by a formal group of *infinite* height, the 2-dimensional cohomology of X/K is *not* analysable. We show however, that if $p > 2$, and X/R is a proper smooth surface such that $\text{Pic}^\tau(X/R)$ is smooth and Br X/R is of *finite* height, then the 2-dimensional cohomology of X/K is analysable. These hypotheses hold, for example, when X/K is a K 3 surface which admits a good nonsupersingular reduction to characteristic $p > 2$, or when X/K is *any* lifting of the Fermat surface $\mathscr{F}(d)$ ($p \equiv 1 \bmod d$). In the latter case one obtains that the 2-dimensional cohomology of X/K admits a Hodge-Tate decomposition.

QUESTIONS FOR FURTHER STUDY:

(*a*) It is clear that as long as our theory is set in its present frame, it is doomed to be dependent upon a steady rain of hypotheses, and cannot be totally general. One problem rests in our insisting that the functors Φ^r be pro-representable. The should be viewed as a dispensable crutch used to convince the reader that one is dealing with a manageable object, and to suggest the appropriate directions of study. The arithmetic content of the theory (e. g., the inequality quoted above) should be independent of any such hypothesis. Moreover, the most satisfactory theory would deal with some object in a derived category finer than the simple collection of Φ^r's for all *r*. Such an extension will not be merely technical, for it involves a systematic extension of the elegant theory of Cartier on which much of our work is based.

(*b*) Our groups Φ^r might be suggestively denoted $\Phi^{0,r}$, since they are related to the Hodge cohomology of bidegree (0, *r*), and they recapture only the part of *r*-dimensional cohomology given by slopes in the interval $[0, 1]$. One might hope to find a bigraded system of groups $\Phi^{i,j}$ $(i+j = r)$ where $\Phi^{i,j}$ is somehow related to the Hodge cohomology of bidegree (i, j) and whose Dieudonné module yields information about the part of r-dimensional cohomology given by slope in the interval $(i, i+1)$ [3].

(*c*) *Convergence questions, and the notion of a link between crystalline cohomology in characteristic p and étale cohomology in characteristic* 0.

Let X/W be a smooth proper scheme, and suppose the system of deformation cohomology groups

$$\Phi^r(X \times_W W_n, G_m), \qquad n \geqq 0,$$

yields a smooth formal group which we shall denote Φ^r over $V = \mathrm{Spf}(W)$. Suppose further that Φ^r is of *finite height*. Then Φ^r yields a p-divisible (connected) group scheme $G^r/\mathrm{Spec}\, W$, following the equivalence of categories given in [34]. Associated to the p-divisible group $G^r/\mathrm{Spec}\, W$ one has the $\mathrm{Gal}(\overline{K}/K)$-module $TG^r \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, where $TG^r = T\Phi$ denotes the Tate module. One has also the $\mathrm{Gal}(\overline{K}/K)$-module coming from étale cohomology at the geometric generic point

$$H^r(X \times_W \overline{K}, T\boldsymbol{\mu}_{p^\infty}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

QUESTION. — Can one define, in the above situation, a natural injective homomorphism of $\mathrm{Gal}(\overline{K}/K)$-modules

$$TG^r \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \xrightarrow{\lambda} H^r(X \times_W \overline{K}, T\boldsymbol{\mu}_{p^\infty}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p?$$

(*d*) It would be interesting to prove that $H^2(X \times_W \overline{K}, T\boldsymbol{\mu}_{p^\infty}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is a $\mathrm{Gal}(\overline{K}/K)$-module of Hodge-Tate type, for a reasonably large class of surfaces. In this direction see IV, § 2, Remark 4 below.

I. — Formal Groups

1. CARTIER MODULES. — To begin, let us set some terminology for this section : k, a perfect field of characteristic p, often taken to be algebraically closed in the key propositions below;

W, the Witt vectors of k;

K, the field of fractions of W;

$\varphi : k \to k$, the p-th power map $x \to x^p$ referred to as the *Frobenius automorphism of* k;

$\varphi : W \to W$, the lifting of the p-th power map, referred to as the *Frobenius automorphism of* W.

$\varphi : K \to K$, the map induced by the Frobenius automorphism of W, on its field of fractions, called the *Frobenius automorphism of* K.

[3] This program has been carried out now by Bloch [6], 4.5, [7] under the hypothesis that $p > \dim X$.

DÉFINITION. — A *Cartier module* is a pair (M, f) where M is a free W-module of finite rank and $f : M \to M$ is an endomorphism which is compatible with φ, in the sense that

$$f(a.m) = \varphi(a) f(m)$$

for all $a \in W$ and $m \in M$. (In other words f is a φ-*linear* endomorphism.)

Cartier modules form a category where morphisms are defined in the evident way. We shall abbreviate our notation by referring to the "Cartier module M" when the φ-linear endomorphism f which goes along with M admits no possible confusion of identity.

If M is a Cartier module, let $V = M \otimes_W K$. Then V is a finite dimensional vector space over K with a φ-linear endomorphism f, such that f preserves a W-lattice in V (f preserves M).

Let $\mathscr{A} = K[T]$ be the noncommutative polynomial ring in one variable T over K where the commutation law is given by

$$\varphi(a).T = T.a \quad \text{for all} \quad a \in K.$$

The vector space V above may be regarded as a left $\mathscr{A}$-module by letting T act as f:

$$T.r = f(r), \qquad v \in V.$$

For integers r, s, with $s \geqq 1$, let $U_{r,s}$ denote the left $\mathscr{A}$-module

$$U_{r,s} \simeq \mathscr{A}/\mathscr{A}.(T^s - p^r).$$

We refer to $U_{r,s}$ as the *canonical* $\mathscr{A}$-module of pure *slope* r/s and *multiplicity* s. One can check that $U_{r,s}$ is a K-vector space of dimension s.

In the case $r \geqq 0$, the action of T on $U_{r,s}$ preserves the W-lattice

$$W[T]/W[T].(T^s - p^r) \subseteq U_{r,s}.$$

This is an inclusion since

$$W[T].(T^s - p^r) = W[T] \cap \mathscr{A}.(T^s - p^r).$$

If $r < 0$, then T preserves *no* W-lattice in $U_{r,s}$.

One has the following relation for any integers r, s, with $s \geqq 1$ $m \geqq 1$:

$$U_{mr,ms} \simeq (U_{r,s})^m.$$

Moreover, $U_{r,s}$ is a *simple* $\mathscr{A}$-module if and only if $(r, s) = 1$.

PROPOSITION (Dieudonné, Manin [25]). — *Let k be algebraically closed. Let* V *be a finite dimensional vector space over* K *admitting a φ-linear endomorphism* T. *Then* V *may be regarded as a left $\mathscr{A}$-module via the action of* T, *and for a unique choice of integers r_i, s_i, with $s_i \geqq 1$ such that $r_1/s_1 < r_2/s_2 < \ldots < r_t/s_t$, the left $\mathscr{A}$-module* V *may be expressed uniquely as a direct sum*

$$V = \bigotimes_{i=1}^{t} V_{r_i/s_i}, \tag{1.1}$$

where V_{r_i/s_i} is an $\mathscr{A}$-submodule of V, *noncanonically isomorphic to the $\mathscr{A}$-module U_{r_i,s_i}.*

The action of T on V preserves a W-lattice if and only if the integers r_i which occur are all non-negative.

We refer to (1.1) as the canonical *slope decomposition* of V, and we say that V has *slopes* r_i/s_i with *multiplicity* s_i $(i = 1, \ldots, t)$. It is clear from the above proposition that the numerical data consisting of the slopes of V given with their multiplicities determine V, up to noncanonical isomorphism. This numerical data is most conveniently represented by a "Newton polygon" (*see* [23], [25] and [26].) We refer to $V_{r_i/s_i} \subset V$ as the *part of* V *of pure slope* r_i/s_i.

Continuing with the hypothesis that k is algebraically closed, let M be a Cartier module. By the *slopes* and *multiplicities* of M we shall mean the slopes of $V = M \otimes_W K$ with their multiplicities. If α is a non-negative rational number, let $M_\alpha \subset M$ denote the sub-Cartier module defined by

$$M_\alpha = M \cap V_\alpha \subset V,$$

where $V_\alpha \subset V$ is the part of V of pure slope α.

By the construction of M_α we have:

(i) M_α is a *saturated* submodule of M in the sense that if $x \in M$ and $px \in M_\alpha$ then $x \in M_\alpha$.

(ii) $V_\alpha = M_\alpha \otimes_W K$, and M_α has pure slope α.

(iii) If $M' \subset M$ is a sub-Cartier module of pure slope α, then $M' \subset M_\alpha$.

Thus, M_α deserves the name: *the part of* M *of pure slope* α.

If α_i $(i = 1, \ldots, t)$ are distinct non-negative rational numbers, then the natural map

$$\bigoplus_i M_{\alpha_i} \to M \tag{1.2}$$

is an injection. Moreover, if the α_i run through all the slopes of M, the cokernel of (1.2) is a W-torsion module. We then refer to (1.2) as the *slope decomposition* of M.

If $\alpha = \alpha_{i_0}$ is a slope of M, let N_α denote the module obtained from $M/(\bigoplus_{i \neq i_0} M_{\alpha_i})$ by annihilating its W-torsion. This is a Cartier module of pure slope α, and if $M \xrightarrow{h} M'$ is any map of Cartier modules such that M' has pure slope α, it must factor through N_α. We refer to N_α at the *maximal quotient of* M *of pure slope* α. Clearly $N_\alpha \otimes_W K \simeq V_\alpha$, and the natural map

$$M_\alpha \xrightarrow[h_\alpha]{} N_\alpha \tag{1.3}$$

is an injection whose cokernel is W-torsion. Thus by (1.2), the map

$$M \to \prod_{i=1}^{t} N_{\alpha_i} \tag{1.4}$$

is also an injection whose cokernel is W-torsion.

One final notion: We say that two Cartier modules M, M′ are *equivalent* ($M \equiv M'$) if $M \otimes_m K = V$ is isomorphic as $\mathscr{A}$-module to $M' \otimes_W K = V'$. This is the same as asserting that M and M′ have the same slopes and multiplicities. It is also the same as asking that there exist a morphism $h : M \to M'$ which is injective and whose cokernel is W-torsion. Thus the slope decompositions (1.2) and (1.4) are equivalences of Cartier modules, and consequently any Cartier module is equivalent to a direct sum of Cartier modules of pure slope.

The classification of Cartier modules up to *isomorphism* and not just equivalence is undoubtedly a subtle matter involving further numerical invariants, such as the lengths of the cokernels of the morphisms h_α of (1.3) (compare [25]).

Examples:

(1) Let Γ be a formal Lie group of finite height h over k. Let M denote its Dieudonné module. For definiteness we take M to be the module of *typical curves* of Γ as defined in Cartier's theory (*cf.* § 3).

Since Γ is of finite height h, the module M is free of rank h over W. There are two operators F and V on M with the following properties:

(i) F is φ-linear; V is φ^{-1}-linear;

(ii) V is topologically nilpotent as an endomorphism of M;

(iii) $FV = VF = p$.

Therefore (M, F) is a Cartier module. One may retrieve the operator V from the Cartier module (M, F) using the commutation relations (iii) and the fact that M has no p-torsion. The existence of such a topologically nilpotent operator V insures that the slopes of the Cartier module (M, F) are rational numbers in the half-open interval $[0, 1)$. Conversely, any Cartier module whose slopes are in the interval $[0, 1)$ is equivalent to a Dieudonné module. This can be shown as follows: Adjoin formally an operator V with the property $FV = VF = p$, and check that the module obtained in this way is isogeneous to the old one.

Or, one may take $M' \subseteq M \otimes \mathbf{Q}$ to be : $M' = \sum_{n \geq 0} p^n F^{-n} M$ and check that M' is stable under F and under $V = p . F^{-1}$ and, moreover, M' is equivalent to M.

If Γ, Γ' are two formal Lie groups of finite height whose Dieudonné modules are M, M′ respectively, then M is equivalent to M′ if and only if Γ and Γ' are isogenous.

Let Γ be an arbitrary (finite dimensional) formal Lie group over k, and consider multiplication by p^n in Γ. One checks immediately that the image $\bar{\Gamma}$ of p^n is independent of n for large n, and that $\bar{\Gamma}$ is a formal Lie group of finite height. The Dieudonné module of Γ is equivalent to the quotient of the Dieudonné module of Γ, modulo W-torsion.

If Γ, Γ' are arbitrary (finite dimensional) formal groups over k, with Dieudonné modules M, M′ respectively, we say that Γ and Γ' are *equivalent* if $M \equiv M'$ (i. e. if $M \otimes_W K = V$ is isomorphic to $M' \otimes_W K = V'$ as $\mathscr{A}$-modules). This is the same as asking that $\bar{\Gamma}$ and $\bar{\Gamma}'$ be isogenous.

(2) Cartier modules coming from crystalline cohomology. Let X/k be a proper and smooth scheme. Then the crystalline cohomology group [5] $H^r_{\text{crys}}(X/W)$ is a W-module of finite type endowed with a φ-linear operator F induced by the Frobenius morphism.

To recall the Frobenius operator, let $W^{(\varphi)}$ denote W regarded as W-module *via* $\varphi : W \to W$. If X is a W-scheme write $X^{(p)}/W$ for the pull back of X/W *via* φ. Then Frobenius is a morphism from X to $X^{(p)}$, inducing a homomorphism,

$$H^r_{\mathrm{crys}}(X/W) \underset{W}{\otimes} W^{(\varphi)} \cong H^r_{\mathrm{crys}}(X^{(p)}/W) \to H^r_{\mathrm{crys}}(X/W),$$

which may be viewed as a φ-linear operator on $H^r_{\mathrm{crys}}(X/W)$. If $H^r_{\mathrm{crys}}(X/W)$ is torsion-free over W, the pair (H_{crys}, F) is a Cartier module.

In slightly more special circumstances, one has an interesting but partial picture of the Cartier module H^r_{crys}. Namely, let $\tilde{X}/W$ be a lifting of X/k to a smooth projective scheme, such that all the W-modules $H^q(\tilde{X}, \Omega^p_{\tilde{X}/W})$ are free. Then [26], [27] the H^r_{crys} are free W-modules as well, and the homomorphism $F : H^r_{\mathrm{crys}} \to H^r_{\mathrm{crys}}$ may be put in the form of a diagonal matrix with entries:

$$[\overbrace{p^0, p^0, \ldots, p^0}^{h^{0,r}}, \overbrace{p^1, p^1, \ldots, p^1}^{h^{1,r-1}}, \ldots, \overbrace{p^r, p^r, \ldots, p^r}^{h^{r,0}}]$$

by making the appropriate *independent* choices of W-basis for domain and range of F. Here

$$h^{p,q} = \mathrm{rank}_W(H^q(\tilde{X}, \Omega^p_{\tilde{X}/W})) = \dim_k(H^q(X, \Omega^p_{X/k})).$$

It follows from this description that the slopes of the Cartier modules H^r_{crys}, if this group is non-zero, lie in the interval $[a, b]$ where

a = smallest non-negative integer such that $h^{a,r-a} \neq 0$.

b = largest integer such that $h^{b,r-b} \neq 0$.

One also obtains from this a strong requirement concerning the geometric position of the "Newton polygon" of the Cartier module H^r_{crys}: the conjecture of Katz (*see* [26], [27]).

An open area of questions concerning the structure of the Cartier module H^r_{crys} lies in the direction of a study of its finer numerical invariants. In particular, what are the k-lengths of the cokernels of the homomorphisms h_α of (1.3)?

We conclude this section by listing some further interesting structures possessed by the category of Cartier modules:

THE SHIFT OPERATOR. — If M stands for the Cartier module (M, f), let $M[k]$ stand for the Cartier module $(M, p^k f)$, for any non-negative integer k. If α runs through the slopes of M, then $\alpha + k$ runs through the slopes of $M[k]$ and with the same multiplicities. The rule

$$M \to M[k]$$

is a functor from the category of Cartier modules to itself—the k-fold *shift*.

Suppose M is a Cartier module all of whose slopes α are greater than or equal to a fixed positive integer k. It is not necessarily true that M is isomorphic to $N[k]$ for some Cartier module N. It is true, however, that M is *equivalent* to some $N[k]$. Here is an efficient way of constructing such an N:

Regard M as a module over the noncommutative polynomial ring $W[T]$, where T acts as f. Consider

$$W[T] \subset W[p^{-k}T] \subset \mathscr{A}.$$

Form the $W[p^{-k}T]$-module

$$N = \operatorname{image}(W[p^{-k}T] \otimes_{W[T]} M \subset V)$$

and define the φ-linear operator $f : N \to N$ to be given by the action of $p^{-k}T$ on N. To show that (N, f) is a Cartier module, it suffices to show that N is contained in a W-lattice in V. This one can do by using the hypothesis

$$(1.5) \qquad \alpha \geqq k$$

for all slopes α of M, and by reduction of the problem to the canonical modules $U_{r,s}$ of pure slope α.

Under hypothesis (1.5) for the Cartier module M and the non-negative integer k, we denote the constructed Cartier module (N, f) by the symbol $M[-k]$. Then

$$M \equiv M[-k][k].$$

As Grothendieck remarked, we may use the shift operator to associate to any Cartier module M whose slopes lie in the interval $[0, r)$ a sequence

$$\Gamma_0, \Gamma_1, \ldots, \Gamma_{r-1}$$

of formal groups up to equivalence as follows:

Let $M^{[j, j+1)} \subset M$ denote the sum of the parts of M of pure slope α, where α ranges through the rational numbers in the interval $[j, j+1)$. Then

$$M^{[j, j+1)}[-j] = L_j$$

is a Cartier module whose slopes are concentrated in the interval $[0, 1)$. It follows that L_j is equivalent to the Dieudonné module of some formal group Γ_j.

2. Cartier groups. — Here we shall study formal Lie groups G over the ring of Witt vectors. By a *Cartier group* we shall mean a pair (G, v) where G is a finite-dimensional commutative formal Lie group over W and $v : G \to G$ is a homomorphism of formal Lie groups compatible with the automorphism φ^{-1} of the base ring W, which has the following additional property:

The restriction to the residue field $v_0 : G_0 \to G_0$ is the Verschiebung of the formal group G_0/k [10, p. 511].

As with Cartier modules, we shall refer to a Cartier group by the letter G if there is no possible confusion concerning the identity of its « lifting of Verschiebung'' v. Cartier groups form a category in an evident manner.

Let TG denote the tangent space of a Cartier group G over W. It is a free W-module of finite rank. The homomorphism $v : G^{(\varphi)} \to G$ induces a W-homomorphism

$$f : (TG)^{(\varphi)} = T(G^{(\varphi)}) \underset{Tv}{\rightarrow} TG.$$

Which we view as a φ-linear endomorphism of TG.

Thus "tangent space over W" may be viewed as a functor

$$(\text{Cartier groups}) \xrightarrow{T} (\text{Cartier modules}),$$
$$G \mapsto TG.$$

A beautiful result, and one that is essential to our theory, is the following (*see* [9], [10], [11] and [24]).

THEOREM OF CARTIER. — *Let k be a perfect field. The functor* T *is an equivalence of categories.*

Cartier establishes this theorem by constructing explicity an essential inverse S to the functor T. The particular properties of this inverse will undoubtedly be useful in further elaborations of our theory, but for the present, the above assertion will suffice.

PRELIMINARY REMARKS:

(1) If G is a Cartier group, and $M = TG$ is its associated Cartier module, then the equivalence class of the formal group G_0/k is determined by the equivalence class of the Cartier module M.

Proof. — Let G, G′ be two Cartier groups, and M, M′ their associated Cartier modules. Suppose M and M′ are equivalent. We shall show that G_0, G'_0 are equivalent. Note that if M and M′ are equivalent, there are morphisms of Cartier modules

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend left, "p^\nu"] & M' \arrow[r] \arrow[rr, bend right, "p^\nu"'] & M \arrow[r] & M' \end{tikzcd}$$

such that the indicated compositions are multiplication by p^ν for some suitable ν. By virtue of the Theorem of Cartier, one gets an analogous diagram in the category of Cartier groups

$$\begin{tikzcd} G \arrow[r] \arrow[rr, bend left, "p^\nu"] & G' \arrow[r] \arrow[rr, bend right, "p^\nu"'] & G \arrow[r] & G' \end{tikzcd}$$

This establishes the equivalent of the formal groups G_0/k and G'_0/k.

(2) (vector groups). If P is a free W-module of finite rank, define a formal group Vec (P) by the following law: For any artinian ring A over the ring W of Witt vectors, of k set

$$\text{(2.1)} \qquad \mathrm{Vec}(\mathrm{P})(\mathrm{A}) = \ker(\mathrm{P} \otimes \mathrm{A} \to \mathrm{P} \otimes \mathrm{A}_{\mathrm{red}}),$$

where the right-hand side is taken with its additive group structure.

Clearly Vec (P) is isomorphic to a finite direct sum of copies of the formal completion of $\mathbf{G}_a$. Its tangent space over W is canonically isomorphic to the W-module P. The morphisms $\mathrm{Vec}(\mathrm{P})^{(\varphi)} \to \mathrm{Vec}(\mathrm{P})$ of formal groups are in natural one-one correspondence with the φ-linear homomorphisms of the W-module P.

Explicitly, if $g : \mathrm{P}^{(\varphi)} \to \mathrm{P}$ is a φ-linear homomorphism, then (2.1) permits us to define an endomorphism

$$\mathrm{Vec}(g) : \quad \mathrm{P}^{(\varphi)} = \mathrm{P}$$

(defined on A-valued points by $g \otimes \mathrm{id} : \mathrm{P}^{(\varphi)} \otimes \mathrm{A} \to \mathrm{P} \otimes \mathrm{A}$).

For which pairs (P, g) is (Vec (P), Vec (g)) a Cartier group? Clearly the missing requirement is that Vec (g) be a lifting of the Verschiebung endomorphism of $\mathrm{Vec}(\mathrm{P})_0$. But since the Verschiebung endomorphism of $\mathbf{G}_a$ is identically zero, it is also zero on $\mathrm{Vec}(\mathrm{P})_0$. Thus we need

$$\mathrm{Vec}(g)_0 = 0,$$

or, translated in terms of g, we must have that

$$g \equiv 0 \qquad (\mathrm{mod}\, p).$$

COROLLARY (2.2). — *Let* (P, g) *be a Cartier module such that*

$$g \equiv 0 \qquad (\mathrm{mod}\, p).$$

Then (Vec (P), Vec (g)) *is a Cartier group whose image under the functor* T *is isomorphic canonically to* (P, g). *We call such a Cartier group a* vector Cartier group.

COROLLARY (2.3). — *Let* G *be a Cartier group such that the slopes of the Cartier module* M = TG *are all greater than or equal to one. Then* G_0/k *is a unipotent formal group.*

Proof. — Since M ≡ M [−1] [1] (*see* § 1), M is equivalent to a Cartier module (P, g) such that $g \equiv 0$ (mod p). Since the question of unipotence of G_0/k depends only on the equivalence class of G_0/k by Remark 1 above, we may replace M by the Cartier module (P, g). But then, by Corollary (2.2), G = Vec (P), and we are done.

(3) (Groups of Finite Height). Let Γ/k be a formal Lie group of finite height. Let G/W be the formal completion of its universal extension as developed in [28] and [29]. The Verschiebung endomorphism of Γ induces, by functoriality, an endomorphism v of G, which makes (G, v) a Cartier group (this is asserted in [11]). The functor T sends (G, v) to (M, f) where M is the Dieudonné module of Γ and f is its Frobenius operator.

FUNDAMENTAL LEMMA (2.4). — *Let k be algebraically closed. Let G be a Cartier group and $M = TG$ its associated Cartier module. Let*

$$M_{[0,1)} \subset M$$

be the sum of the parts of M of pure slope less than one. Let DG_0 denote the Dieudonné module of typical curves of G_0. Then $DG_0 \otimes_W K$ and $M_{[0,1)} \otimes_W K$ are isomorphic $\mathscr{A}$-modules.

In other terms, $M_{[0,1)}$ is equivalent to the Dieudonné module of a maximal finite-height quotient of G_0.

Proof. — Consider the short exact sequence of Cartier modules

$$0 \to M_{[0,1)} \to M \to B \to 0. \tag{2.5}$$

By the Theorem of Cartier there is a short exact sequence of Cartier groups

$$0 \to H \to G \to Q \to 0 \tag{2.6}$$

whose image under the functor T is a short exact sequence isomorphic to (2.5).

Applying the Dieudonné module functor of typical curves to (2.6), one gets

$$0 \to DH_0 \to DG_0 \to DQ_0 \to 0.$$

Since B is a Cartier module all of whose slopes are greater than or equal to 1, Q_0 is unipotent by Corollary (2.3), and consequently DQ_0 is annihilated by a power of p. Thus

$$DH_0 \otimes_W K \xrightarrow{\sim} DG_0 \otimes_W K.$$

But now, since $M_{[0,1)}$ is equivalent to the Dieudonné module of some formal group of finite height Γ/k, by Remark 3 above, H_0 is equivalent to the universal extension E of Γ. But the universal extension of Γ is an extension of Γ by a *vector group*. Therefore

$$M_{[0,1)} \otimes_W K \simeq DE \otimes_W K = DH_0 \otimes_W K = DG_0 \otimes_W K.$$

This proves Lemma (2.4).

Example. — Let X/k be a scheme proper and smooth such that $H^r_{crys}(X/W)$ is torsion-free over W. Then H^r_{crys} is a Cartier module. By the Theorem of Cartier, there is a Cartier group Φ^r_{crys} such that applying T to Φ^r_{crys} gives a Cartier module isomorphic to H^r_{crys}. The fundamental lemma then implies:

COROLLARY (2.7). — *The part of H^r_{crys} of slope less than 1 is equivalent to the Dieudonné module of a maximal finite-height quotient of $\Phi_{crys,0}$.*

3. CARTIER'S THEORY OF CURVES. — We follow Cartier's notes [9] (*see* also Lazard [24]). Cartier studies smooth formal groups over arbitrary bases. For us it will suffice to consider smooth formal groups E over smooth k-schemes X, where k is a perfect field of characteristic p.

Regard E as a sheaf for the Zariski topology on X, and define the sheaves of abelian groups $C_n E$ by taking its sections on open affines $U = \operatorname{Spec} B \subset X$ to be:

$$C_n E(U) = \ker(E(B[t]/t^n) \to E(B)).$$

By definition, $C_1 E = \tan E$, the *Zariski tangent space* (*sheaf*) *to* E. The sheaves $(C_n)_{n \geqq 0}$ form a projective system, and each $C_n E$ admits a finite composition series of subsheaves of abelian groups each factor of which is isomorphic to tan E.

Let CE denote the projective limit of the system $(C_n E)_{n \geqq 0}$. Then CE is referred to as the *sheaf of curves on* E.

On CE one has the standard operators:

(*a*) For any open $U \subset X$ and any local section $c \in \Gamma(U, \mathcal{O}_X)$ one has an operator

$$[c]: \quad CE|_U \to CE|_U$$

obtained (in the special case where U = Spec B) by composition with the endomorphism

$$B[t] \to B[t],$$
$$t \mapsto ct.$$

(*b*) For any integer $m \geqq 1$, one has

$$V_m: \quad CE \to CE$$

induced by $t \mapsto t^m$.

(*c*) For any integer $m \geqq 1$, one has the Frobenius operator

$$F_m: \quad CE \to CE$$

defined explicitly in [9]. One way of thinking of these operators is as follows: One extends k by forming $k[\xi]/(\xi^m - 1)$. Then F_m is the unique operator such that $V_m F_m = \sum_{i=0}^{m-1} [\xi_i]$. Let I ($p$) denote the integers relatively prime to p. Since I (p) is invertible over X, the projection operators defined in [9] break CE up into the product of I(p) copies of the sheaf TCE, the *sheaf of typical curves of* E. The sheaf TCE is defined to be the intersection of the kernels of F_m for $m > 1$, $m \in I(p)$. By DE, or the (*Cartier*)-*Dieudonné module of* E we mean the abelian group TCE endowed with the operators $F = F_p$, $V = V_p$, and regarded as $W = W(k)$ module in a certain natural way (*see* [9]).

In the special case where $X = k$, and E is a smooth formal group over k, DE is indeed isomorphic to the "classical" Dieudonné module of E [11]. There seems to be no published reference for this fact, and Messing has provided us with a proof of it, which he intends to publish shortly.

II. — Infinitesimal properties of cohomology

1. DEFORMATION COHOMOLOGY. — We work for the most part with the étale topology, or more precisely, with the big étale site on a scheme (or algebraic space) S. Let E be a sheaf on S which in our applications will be $\mathbf{G}_m$ or $\mathbf{G}_a$. Then we define a functor $\hat{E}$ by

$$\hat{E}(Z) = \ker(E(Z) \to E(Z_{red})).$$

We call $\hat{E}$ the *formal completion* of E (along its zero section). The fact that passage to the associated reduced scheme is compatible with étale maps implies that $\hat{E}$ is a sheaf. It is a left exact functor of E. A sheaf E is called *discrete* if $\hat{E} = 0$.

Let us denote by $E \mid Z$ the restriction of E to the small étale site on Z. We will usually identify this small site with that of Z' if $Z \subset Z'$ is an infinitesimal extension, i. e., is a closed immersion defined by a nilpotent ideal, and thus we may consider the "restriction" map

$$E \mid Z' \to E \mid Z \tag{1.1}$$

of sheaves on this small site.

A map $E \to F$ of sheaves is called formally smooth if the natural map

$$E \mid Z' \to E \mid Z \times_{F \mid Z} F \mid Z'$$

is surjective for every infinitesimal extension $Z \subset Z'$, and E is *formally smooth* if (1.1) is always surjective. It is immediately seen that

$$\text{The map } \hat{E} \to E \text{ is formally smooth.} \tag{1.2}$$

The notion of formal completion extends naturally to the relative case. Let $f : X \to S$ be a map of schemes or algebraic spaces, and let E be a sheaf on X. Then on the small étale site on X, we define a *relative completion* $\tilde{E}_S$ by the rule

$$\tilde{E}_S(X') = \ker(E(X') \to E(X' \times_S S_{red})) \tag{1.3}$$

for any X' étale over X. The *deformation cohomology* of X/S, with coefficients in E, is the sheaf

$$\Phi^q(X/S, E) \underset{\text{def } n}{=} R^q f_* \tilde{E}_S, \tag{1.4}$$

where $R^q f_*$ is taken respect to the étale topology ([4]).

It is defined on the small site of S, and by pull-back on the small site of any S' over S. Hence its definition naturally extends to give a sheaf on the big site of S. Note that $\Phi^q(X/S, E)$ is zero on any reduced S. In other words, this sheaf is its own formal completion. When no confusion will arise, we may use the notation $\Phi^q(X, E)$ or $\Phi^q(E)$ for it.

We have

$$\Phi^0(X/S, E) = \ker(E(X) \to E(X \times_S S_{red})) = f_* \tilde{E}, \tag{1.5}$$

and there is a natural map

$$\Phi^q(X/S, E) \to R^q f_* E. \tag{1.6}$$

Main examples. — When $E = \mathbf{G}_m$, $S = \operatorname{Spec} k$ (k an algebraically closed field), and X/k is proper and smooth, then $\Phi^q(X/k, \mathbf{G}_m)$ defines a functor on Artin local k-algebras $A = k + I$ (I = kernel of $A \to A/m = k$):

$$\Phi^q(X/k, \mathbf{G}_m)(A) = H^q(X, 1 + I \otimes_k \mathcal{O}_X),$$

([4]) We may also refer to this as $\Phi^q f_* E$.

where $1+I \otimes_k \mathcal{O}_X$ is the sheaf of "principal units relative to I over X" contained in $(A \otimes_k \mathcal{O}_X)^*$. As will be shown under more general circumstances (2.6 below) the cohomology may be computed for the Zariski, étale or fppf sites and one obtains the same answer, since $1+I \otimes_k \mathcal{O}_X$ can be constructed as a successive extension of sheaves of abelian groups, which are coherent $\mathcal{O}_X$-modules.

PROPOSITION (1.7):

(i) $\Phi^q(X/S), \hat{E}) \approx \Phi^q(X/S, E)$;

(ii) *if* X *is smooth over* S, *then* $\Phi^q(X/S, E) \approx R^q f_* \hat{E}$;

(iii) *if* E *is formally smooth, then the map* $\Phi^q(X/S, E) \to R^q f_* E$ *is formally smooth, and moreover, the tangent spaces to these functors at any point, if defined, are equal.*

Proof. — The first assertion is clear. If X is smooth over S, then $X \times_S S_{red} = X_{red}$. Hence $\tilde{E}_S = \hat{E} \mid X$, and so $\Phi^q(X/S, E) = R^q f_* \tilde{E}_S = R^q f_* \hat{E}$, which proves (ii). Suppose E formally smooth, and let $Z \subset Z'$ be an infinitesimal extension of S-schemes. Denote by $X_Z \subset X_{Z'}$ the product with X, and let $X_0 = X_S \times_S Z_{red}$. Then we have a diagram

$$\begin{array}{ccccccccc} 0 \to & \tilde{E}_{Z'} & \to & E \mid X_{Z'} & \to & E \mid X_0 & \to 0 \\ & \downarrow{\scriptstyle\tilde{\alpha}} & & \downarrow{\scriptstyle\alpha} & & \| & \\ 0 \to & \tilde{E}_Z & \to & E \mid X_Z & \to & E \mid X_0 & \to 0. \end{array}$$

Thus the kernel and cokernel of $\tilde{\alpha}$ and α are isomorphic, and so the hypercohomology of the complexes $\tilde{\alpha}$ and α is the same, yielding a diagram

$$\begin{array}{ccccccc} R^q f_* \tilde{\alpha} & \to & R^q f_* \tilde{E} \mid Z' & \to & R^q f_* \tilde{E} \mid Z & \to & R^{q+1} f_* \tilde{\alpha} \\ \downarrow\wr & & \downarrow & & \downarrow & & \downarrow\wr \\ R^q f_* \alpha & \to & R^q f_* E \mid Z' & \to & R^q f_* E \mid Z & \to & R^{q+1} f_* \alpha. \end{array}$$

This diagram shows that the map $\Phi^q(X/S, E) = R^q f^* \tilde{E}_S \to R^q f^* E$ is formally smooth, as required. The last assertion of the proposition also follows from this diagram.

Here are four questions we may ask about the deformation functors $\Phi = \Phi^q(X/S, E)$.

(I) *Is* Φ *formally smooth ?*

(II) *Is* Φ *representable by a formal group over* S ?

Here, formal group will mean in the extended sense. If the answer to (II) is yes, Φ will always be a *connected* formal group. If the answer to (I) and (II) is yes, then Φ will "be" a formal Lie group.

(III) *What is the Zariski tangent space of* Φ?

Here, we shall mean either: at any point of S, or under suitable conditions, as a coherent sheaf over S.

(IV) *What is the Dieudonné module of* Φ?

Concerning question (II), we have

PROPOSITION (1.8). — *Let* $f: X \to S$ *be a flat proper map and let* E *be a formally smooth sheaf on* X *such that* $\hat{E}$ *is represented by a (necessarily smooth) formal group scheme on* X.

If $R^{q-1} f_* E$ *is formally smooth, then* $\Phi^q (X/S, E)$ *is represented by a formal group scheme on* S.

As mentioned in the introduction, it is desirable to work with a general global base S so as to be able to consider stratifications induced on moduli spaces.

We defer the proof to Section 3 below and conclude this section with the following remarks:

Remarks (1.9):

(i) with the assumptions of (1.8), it follows that the sheaf $\Phi^q (X/S, E)$ is a sheaf for the flat topology;

(ii) it may happen that $R^q f_* E$ is itself a representable functor. In this case (1.7) shows that $\Phi^q (X/S, E)$ is the formal completion of $R^q f_* E$ along its zero section.

2. Review of Schlessinger's theory. — Let R be a local ring with residue field k, and denote by $\mathscr{C}$ the category of artinian affine local R-schemes with residue field k. In this section, we consider the restrictions of the functors $\Phi^q (X, E)$ to $\mathscr{C}$. If $S \in \mathscr{C}$, a subscript S will denote pull-back from Spec R to S. As before, we work with the étale topology.

By a *deformation couple* $S \subset S'$ we mean a closed immersion on the category $\mathscr{C}$ given by a homomorphism of R-algebras of the following sort:

(2.1) $$0 \to I \to A \to A' \to 0,$$

where $m_A I = 0$. The ideal I may then be regarded as a k-vector space, and we make the further hypothesis that I is of k-dimension one. We call I the ideal of definition of the deformation couple.

Let X be a flat scheme or algebraic space over R, and let E be a smooth group scheme, or a smooth formal group scheme, on X. Given a deformation couple (1.1) as above, one has the usual exact *deformation sequence* of sheaves on X_k:

(2.2) $$0 \to I \otimes_k e_0 \to E_{S'} \to E_S \to 0,$$

where e_0 is the sheaf of sections of the Lie algebra of E_k. This abelian sheaf is actually a locally free $\mathcal{O}_{X_k}$-module of finite rank on X_k. Of course, $I \otimes_k e_0 \approx e_0$ since I is one-dimensional.

The sequence (1.2) fits into a diagram

(2.3) $$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & I \otimes e_0 \arrow[r] \arrow[d, equal] & \tilde{E}_{S'} \arrow[r] \arrow[d] & \tilde{E}_S \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I \otimes e_0 \arrow[r] & E_{S'} \arrow[r] \arrow[d] & E_S \arrow[r] \arrow[d] & 0 \\ & & E_k \arrow[r, equal] \arrow[d] & E_k \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

The exact cohomology sequences of the rows of this diagram show:

COROLLARY (2.4). — *The Zariski tangent spaces to* $\Phi^q(X, E)$ *and* $R^q f_* E$ *are both isomorphic to* $H^r(X_0, e_0)$.

COROLLARY (2.5). — $\Phi^q(X, E)$ *and* $R^q f_* E$ *are formally smooth if* $H^{r+1}(X, e_0) = 0$.

Since e_0 is coherent, induction on the length of the nilradical of S shows:

COROLLARY (2.6). — *If* X *is a scheme, the sheaves* $\Phi^q(X, E)$ *may be computed as higher direct images for the Zariski topology:* $\Phi^q(X, E) = R^q f_{zar*} \tilde{E}_S$.

By definition, a square of deformation couples is a cocartesian square in the category $\mathscr{C}$:

$$\begin{array}{ccc} S_2' & \supset & S_2 \\ \uparrow & & \uparrow \\ S_1' & \supset & S_1 \end{array} \tag{2.7}$$

where $S_1' \supset S_1$ is a deformation couple. The word cocartesian means that

$$S_2' \cong S_1' \amalg_{S_1} S_2,$$

or in terms of rings, that $A_2' \approx A_1' \times_{A_1} A_2$. It follows that $S_2' \supset S_2$ is also a deformation couple and that the natural map $I_2 \to I_1$ of the ideals of definition is an isomorphism of k-vector spaces.

By definition, a functor $\Phi : \mathscr{C}^0 \to$ (Sets) will be said to satisfy the *Mayer-Vietoris* condition if for every square of deformation couples (2.7), the natural map

$$\Phi(S_2') \to \Phi(S_1') \times_{\Phi(S_1)} \Phi(S_2) \tag{2.8}$$

is a bijection.

LEMMA (2.9). — *Let* E *be a smooth group scheme or formal group, as above. Then the map* (2.8) *is surjective if* $\Phi = \Phi^q(X/S, E)$. *It is bijective if* $\Phi^{q-1}(X/S, E)$ *is formally smooth, or if* $R^{q-1} f_* E$ *is formally smooth along its zero section.*

Proof. — The surjectivity of (2.8) follows from the diagram

$$\begin{array}{ccccccc} H^q(X_0, e_0) & \to & \Phi^q(X_2', E) & \to & \Phi^q(X_2, E) & \to & H^{q+1}(X_0, e_0) \\ \| & & \downarrow & & \downarrow & & \| \\ H^q(X_0, e_0) & \to & \Phi^q(X_1', E) & \to & \Phi^q(X_1, E) & \to & H^{q+1}(X_0, e_0) \end{array} \tag{2.1}$$

obtained from the sequence (2.2). If Φ^{q-1} is smooth, then the left-hand arrows of (2.10) are injective, and (2.8) is bijective. Since the map $\Phi^{q-1}(X/S, E) \to R^{q-1} f_* E$ is smooth [(1.7) (iii)], Φ^{q-1} will be smooth if $R^{q-1} f_*$ is.

COROLLARY (2.11). — *A sufficient condition for* $\Phi^q(E)$ *to be representable by a formal, group (not necessarily smooth) over* R *is that* $\Phi^{q-1}(E)$ *or* $R^{q-1} f_* E$ *be formally smooth and* $H^q(X_0, e_0)$ *be finite dimensional over* k.

Proof. — By Schlessinger's criterion [30], if Φ^q satisfies the Mayer-Vietoris condition and if its Zariski tangent space over k is finite dimensional, then Φ^q is pro-representable.

The Corollary follows from Lemma (2.9) and the fact the cohomology of a coherent sheaf over a proper scheme is finite dimensional.

Combining the above results, we have:

COROLLARY (2.12). — *If* $H^{q+1}(X_0, e_0) = 0$ *and* $\Phi^{q-1}(E)$ *is formally smooth, then* $\Phi^q(E)$ *is pro-represented by a formal Lie group.*

PROPOSITION (2.13). — *Suppose* X *smooth and let* E *be a formal smooth group over* X. *If* $H^r(X, e_0)$ *is a finite dimensional vector space over* k (e. g., *if* X/k *is proper*), *then the Dieudonné module of typical curves of* $\Phi^r(X, E)$ *may be computed as follows*

$$D\Phi^r(X, E) = H^r(X, DE).$$

Proof. — We use two properties of the sheaves $\mathbf{C}_n E$, both stemming from the fact that $\mathbf{C}_n E$ has a finite composition series whose successive quotients are isomorphic to tan E:

(*a*) $H^r(U, \mathbf{C}_n E) = 0$ for $r > 0$, and any affine open $U \subset X$;

(*b*) $H^r(X, \mathbf{C}_n E)$ are W-modules of finite length.

Consequently the Mittag-Leffler conditions of [19] Ch. 0., III *a*, 13.2.3 are satisfied and one has

$$C(\varphi^r) = \varprojlim_n H^r(X, \mathbf{C}_n E) = H^r(X, \mathbf{C}E).$$

This isomorphism is compatible with the decomposition of both sides into the product of $I(p)$ copies of typical curves. This proves the proposition.

3. REPRESENTABILITY OVER GENERAL BASES S. — In the previous paragraph we have worked over artinian bases, and have obtained a proof of Proposition (1.8) in this case (Cor. 2.11). We shall now prove the full Proposition (1.8) by combining the infinitesimal techniques of Schlessinger with the approximation techniques of [2] and [3].

To prove Proposition (1.8), we may replace E by $\hat{E}$, using (1.2) and (1.6). Let e denote the Lie algebra of $E = \hat{E}$, which is a vector bundle on X. If $Z \subset Z'$ is an extension of S-schemes defined by a square-zero ideal $I \subset \mathcal{O}_{Z'}$, then there is an exact sequence analogous to (2.3):

$$0 \to e \otimes I \to \tilde{E}\,|_{X_{Z'}} \to \tilde{E}\,|_{X_Z} \to 0, \tag{3.1}$$

where we view the term on the left as a coherent sheaf on X_Z. Since $\Phi^{q-1}(E)$ is formally smooth, this leads to an exact cohomology sequence

$$0 \to R^q f_*(e \otimes I) \to \Phi^q(E)\,|\,Z' \to \Phi^q(E)\,|\,Z \to R^{q+1} f_*(e \otimes I). \tag{3.2}$$

Again since $\Phi^{q-1}(E)$ is formally smooth, $R^{q-1} f_* e$ commutes with base change on S, and $R^q f_* e \otimes M$ is a right-exact functor of the $\mathcal{O}_S$-module M. This is because e is the first-order term of E, i. e., $e \otimes M \approx \tilde{E}\,|\,X_{Z'}$ when $Z' = \operatorname{Spec} \mathcal{O}_Z \otimes M$. Therefore [17] (Chap. III, 7), $R^q f_*(e \otimes M)$ is left exact, and is of the form

$$R^q f_*(e \otimes M) = \ker(V \otimes M \to V' \otimes M) \tag{3.3}$$

for some map $V \to V'$ of locally free $\mathcal{O}_S$-modules (tensor products being over $\mathcal{O}_S$).

LEMMA (3.4). — *Let α' be a section of $\Phi^q(E)$ on Z'. Then the condition $\alpha' = 0$ is represented by a closed subscheme of Z'.*

Proof. — We use induction on Z' to reduce to the situation in which $Z \subset Z'$ is defined by a square-zero ideal I, and $\alpha' = 0$ is represented by a closed set $C = V(J)$ above Z.

We first show that if $T' \to Z'$ is any scheme on which $\alpha' = 0$, then T' lies over the subscheme $Z^* = V(J^2)$. Replacing Z by Z^* it will then follow that it suffices to treat the case $C = Z$.

Consider, therefore, $T' \to Z'$ such that $\alpha' = 0$ on T'. Then $T = \operatorname{Spec} \mathcal{O}_T/I.\mathcal{O}_T$ lies over Z, and hence over C:

$$I.\mathcal{O}_{T'} = J.\mathcal{O}_{T'}.$$

Thus $J.\mathcal{O}_{T'}$ is a square-zero ideal, as required.

Now when $C = Z$, our element α' lies in $R^q f_* (e \otimes I)$, by (3.2). So, using (3.3), we can view α' as a section of $V \otimes I$. If T' is a scheme lying over Z', then pull-back to T' changes $V \otimes I$ to $V \otimes I\, \mathcal{O}_{T'}$. Therefore we may view $V \otimes I$ functorially as a submodule of $V \otimes \mathcal{O}_{Z'}$, hence $\alpha' \in V \otimes \mathcal{O}_{Z'}$. Since V is locally free, it is clear that the condition $\alpha' = 0$ is closed on Z'. This completes the proof of the lemma.

Now to represent $\Phi^q(E)$ by a formal group on S, it suffices to represent by algebraic spaces each of the subfunctors $R_n \subset \Phi^q(E)$ defined as follows: Let α be a section of $R^q f_* G$ on some scheme Z. Let J_α denote the ideal such that $V(J_\alpha)$ represents the condition $\alpha = 0$. Then

$$(3.5) \qquad \alpha \in R_n \quad \text{iff} \quad J_\alpha^n = 0.$$

By Lemma (3.4), the map $R_n \to \Phi^q(E)$ is represented by closed immersions. It is clear that $\bigcup R_n = \Phi^q(E)$, and that this union will give $\Phi^q(E)$ the required structure of formal group.

To prove R_n representable, we want to apply the criterion of [3, 5.4]. The main point is to check that Schlessinger's conditions hold, and to find a reasonable obstruction theory. We denote $\Phi^q(E)$ by Φ, and take $n \geqq 2$.

Consider a surjection $A' \to A$ of infinitesimal extensions of a reduced $\mathcal{O}_S$-algebra A_0, whose kernel M is an A_0-module.

LEMMA (3.6). — *Assume that $\alpha \in R_n(A)$ and that some lifting α' of α to $\Phi(A')$ lies in $R_n(A')$. Then they all do, provided $n \geqq 2$.*

Proof. — Let $J = J_\alpha$, and let $\bar{A} = A/J$, $\bar{A}' = A'/J_{\alpha'}$. Lemma (3.4) implies that $\bar{A} = \bar{A}' \otimes_{A'} A$. By right exactness of tensor product, the map $A' \to A$ induces a surjection $J_{\alpha'} \to J$. Thus, denoting by J' the inverse image of J in A', we have

$$J' = J_{\alpha'} + M \cap J' = J_{\alpha'} + N,$$

where $N = M \cap J'$. Since $N \subset M$ is an A_0-module, $J_{\alpha'} N = N^2 = 0$. Therefore $J'^n = J_{\alpha'}^n$, if $n \geqq 2$. This shows that the vanishing of $J_{\alpha'}^n$ is independent of the lifting α', and proves Lemma (3.6).

We now check Schlessinger's conditions using notation of [3]. The exact sequence (3.2) shows that (S 1′ 2) holds for Φ and S 1′ for R_n follows formally from this, together with the fact (3.4) that $R_n \to \Phi$ is represented by closed immersions. By Lemma (3.6), the functors R_n and Φ have the same tangent spaces $D_{a_0}(M) = R^q f_* e \otimes M$. Standard arguments show that the conditions S 2 and (4.1) of [3] hold for this D.

To obtain an obstruction theory, we consider the A_0-module $\mathrm{Ext}^1_{\mathcal{O}_S}(L_{A/\mathcal{O}}, M) = E$ classifying infinitesimal extensions A′ of A with kernel M (*cf.* Illusie [21]). Consider the condition $J'^n = 0$ on A′ [notation as in the proof of Lemma (3.6)]. The set of extensions with $J'^n = 0$ is easily seen to form a submodule V of E. We set $\bar{E} = E/V$. Given a class $\alpha \in R_n(A)$ and an extension $A' \to A$, there is an obstruction o in

$$\mathcal{O} = \bar{E} \times R^{q+1} f_* e \otimes M$$

to lifting α to $R_n(A')$. Namely, the obstruction to lifting to $\alpha' \in \Phi(A')$ lies in $R^{q+1} f_* e \otimes M$, and the element of $\bar{E}$ determined by the extension A′ vanishes if and only if $\alpha' \in R_n(A')$, by (3.5). Thus $(\mathcal{O}, o)$ serves as an obstruction theory for R_n. Conditions [3] (4.1) for $\mathcal{O}$ are easily checked using generic flatness.

The remaining conditions of [3] (5.4) to be verified are (2) and (4). But (4) is vacuous since R_n is trivial on reduced rings, and (2) reduced to [3] [4.1, (ii)] by induction on the nilradical of $\hat{A}$.

4. The multiplicative group. — We now apply the above theory to the sheaf $E = \mathbf{G}_m$, and to a flat proper map $f : X \to S$. Recall that the Lie algebra of $\mathbf{G}_m$ is $\mathbf{G}_a$. Assuming that the map f is cohomologically flat in dimension zero [16], $R^1 f_* \mathbf{G}_m = \mathrm{Pic}\, X/S$ is represented by an algebraic space whose formal completion along the zero-section is $\Phi^1(\mathbf{G}_m)$. Combining (1.8) and (1.7) (iii), we obtain:

Corollary (4.1). — *Suppose* Pic X/S *is formally smooth along its zero section. Then* $\Phi^2(\mathbf{G}_m)$ *is represented by a formal group scheme on* S.

We call $\Phi^2(\mathbf{G}_m)$ the *formal Brauer group* of X/S ($\Phi^2 = \widehat{\mathrm{Br}}\, X/S$). If X is smooth over a field k we just write $\widehat{\mathrm{Br}}\, X$ for its formal Brauer group.

The Zariski tangent space of $\Phi^r(\mathbf{G}_m)$ at a point $s \in S$ is isomorphic with $H^r(X_s, \mathcal{O}_{X_s})$ and hence is of dimension

$$h^{0,r}(s) = \dim_{k(s)} H^r(X_s, \mathcal{O}_{X_s}).$$

The functor $\Phi^r(\mathbf{G}_m)$ is formally smooth at s if $h^{0,r+1}(s) = 0$. In particular, the formal Brauer group of a surface is always formally smooth.

Proposition (1.8) provides us with a general criterion for representability of $\Phi^r(\mathbf{G}_m)$ for all r which implies the following useful:

Corollary (4.2). — *Let* $S = \mathrm{Spec}\, k$, *and suppose that* $h^{0,r-1} = 0$. *Then* $\Phi^r(\mathbf{G}_m)$ *is representable.*

Proof. — For, in this case one shows by induction on the length of a test object $S \in \mathcal{C}$ that $\Phi^{r-1}(\mathbf{G}_m)$ is zero, and hence formally smooth.

SPECIAL CASE. — Let X/k be a complete intersection of dimension $r \geqq 2$ in projective space. Then $\Phi^r(\mathbf{G}_m)$ is pro-representable by a formal Lie group over k.

For, in this case: $h^{0,r-1} = h^{0,r+1} = 0$.

Finally, suppose $R = k$ is a *perfect* field. Suppose X/k is proper.

COROLLARY (4.3). — *Let* $\mathscr{W}$ *denote the sheaf of Witt vectors over* X, *as defined in Serre's paper* [31]. *Then*

$$D\Phi^r(X, \hat{\mathbf{G}}_m) = H^r(X, \mathscr{W}).$$

Proof. — This follows immediately from the identification

$$D\mathbf{G}_m = \mathscr{W}$$

and proposition (2.14).

COROLLARY (4.4). — *Suppose that* Φ^{r-1} *is formally smooth. Then* $H^r(X, \mathscr{W}) \otimes_W K$ *is a finite dimensional vector space. Moreover, suppose* Φ^r *is a formal Lie group. Then* $H^r(X, \mathscr{W})$ *is of finite type over* W *if and only if* Φ^r *is of finite height* (i. e., *has no unipotent part*).

III. — Crystalline and de Rham Cohomology

1. FORMAL GROUPS ASSOCIATED TO DE RHAM COHOMOLOGY. — Let Σ denote either the étale or the Zariski site. Let X/R be a smooth proper scheme, where R is a local ring as in II, paragraph 2. There are two related complexes of abelian sheaves on X that have been the object of study:

The additive (*or ordinary*) *de Rham complex:*

$$\Omega^{\bullet}_{X/R} : \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/R} \xrightarrow{d} \Omega^2_{X/R} \xrightarrow{d} \dots$$

The multiplicative de Rham complex (*cf.* [28]):

$$\Omega^{\times\bullet}_{X/R} : \mathcal{O}^{\times}_X \xrightarrow{d.\log} \Omega^1_{X/R} \xrightarrow{d} \Omega^2_{X/R} \xrightarrow{d} \dots$$

By definition, the *additive* or *multiplicative* de Rham cohomology of X/R is the hypert cohomology of the additive or multiplicative de Rham complex of X/R. We adopt the following purely evocative notation:

$$H^r_{DR-\Sigma}(X/R, \mathbf{G}_a) = H^r_{\Sigma}(X, \Omega^{\bullet}_{X/R}),$$

$$H^r_{DR-\Sigma}(X/R, \mathbf{G}_m) = H^r_{\Sigma}(X, \Omega^{\times\bullet}_{X/R}).$$

We also denote by $\Phi^r_{DR-\Sigma}(X/R, \mathbf{G}_m)$ the functor whose value on $S \in \mathscr{C}$ (*see* section 2) is

$$H^r_{\Sigma}(X, \tilde{\Omega}^{\times\bullet}_{X/R}(S)),$$

where $\tilde{\Omega}^{\times\bullet}_{X/R}(S)$ is the total complex associated to the double complex

$$\Omega^{\times\bullet}_{X\times_R S/S} \to \Omega^{\times\bullet}_{X\times_R S_0/S_0}.$$

Here $S_0 = S_{red} = \operatorname{Spec} k$, and the term $\Omega^{\times r}_{X \times_R S/S}$ of the above double complex is given bidegree $(0, r)$ while the term $\Omega^{\times r}_{X \times_E S_0/S_0}$ is given bidegree $(1, r)$.

If we let E denote ambiguously $\mathbf{G}_a$ or $\mathbf{G}_m$ we may then refer to $H^r_{DR}(X/R, E)$ which will be additive or multiplicative de Rham cohomology depending upon the value of the symbol E.

Now let $S \subset S'$ be a deformation couple whose ideal of definition is I. Let $S = \operatorname{Spec} A$, $S' = \operatorname{Spec} A'$. For any $S = \operatorname{Spec} A$ in $\mathscr{C}$ we have:

$$\Omega^{\bullet}_{X \times_R S/S} = \Omega^{\bullet}_{X/R} \otimes_R A$$

and therefore if we tensor the short exact sequence of R-modules

$$0 \to I \to A' \to A \to 0$$

by $\Omega^{\bullet}_{X/R}$, we obtain the additive de Rham deformation sequence

$$0 \to I \otimes_k \Omega^{\bullet}_{X_0/k} \to \Omega^{\bullet}_{X \times_R S'/S'} \to \Omega^{\bullet}_{X \times_R S/S} \to 0. \tag{1.1 add}$$

One also forms the multiplicative de Rham deformation sequence

$$0 \to I \otimes_k \Omega^{\bullet}_{X_0/k} \to \Omega^{\times \bullet}_{X \times_R S'/S'} \to \Omega^{\times \bullet}_{X \times_R S/S} \to 0 \tag{1.1 mult}$$

which, in degrees 0 and 1, looks like:

$$\begin{tikzcd}
0 \arrow[r] & I \otimes_k \mathcal{O}_{X_0} \arrow[r] \arrow[d, "d"] & \mathcal{O}^{\times}_{X \times_R S'/S'} \arrow[r] \arrow[d, "d\log"] & \mathcal{O}^{\times}_{X \times_R S/S} \arrow[r] \arrow[d, "d\log"] & 0 \\
0 \arrow[r] & I \otimes_k \Omega^1_{X_0/k} \arrow[r] & \Omega^1_{X \times_R S'/S'} \arrow[r] & \Omega^1_{X \times_R S/S} \arrow[r] & 0
\end{tikzcd}$$

and which coincides with the additive sequence in degrees > 1. Consequently it, too, is exact.

Let us now introduce a third complex which will play a role in our analysis of both additive and multiplicative de Rham cohomology:

$$J^{\bullet}: \quad 0 \to \Omega^1_{X/R} \xrightarrow{d} \Omega^2_{X/R} \xrightarrow{d} \cdots$$

It fits into the exact sequence of complexes:

$$0 \to J^{\bullet} \to \Omega^{\bullet}_{X/R} \to \mathcal{O}_X \to 0, \tag{1.2 add}$$

where the last term $\mathcal{O}_X$ is taken to be the complex concentrated in dimension 0, and

$$0 \to J^{\bullet} \to \Omega^{\times \bullet}_{X/R} \to \mathcal{O}^{\times}_X \to 0, \tag{1.2 mult}$$

where again the last term is taken to be a complex concentrated in dimension 0.

For completeness, one can include the deformation sequence for $J^{\bullet}$:

$$0 \to I \otimes_k J_{X_0/k} \to J^{\bullet}_{X \times_R S'/S'} \to J^{\bullet}_{X \times_R S/S} \to 0, \tag{1.3}$$

which sits comfortably in both additive and the multiplicative de Rham deformation sequences.

Recall. – We may regard all complexes and exact sequences of abelian sheaves so far discussed as complexes and exact sequences of sheaves for the site Σ over X, where Σ is the étale or the Zariski site. By rights Σ should occur everywhere as subscript, indicating which site is being used. Although one has

$$\mathrm{H}^r_{\mathrm{DR/\acute{e}t}}(\mathrm{X/R}, \mathbf{G}_a) \simeq \mathrm{H}^r_{\mathrm{DR/zar}}(\mathrm{X/R}, \mathbf{G}_a),$$
$$\mathrm{H}^r_{\acute{e}t}(\mathrm{X/R}, \mathrm{J}^{\cdot}) \simeq \mathrm{H}^r_{\mathrm{zar}}(\mathrm{X/R}, \mathrm{J}^{\cdot}),$$

it is not always true that $\mathrm{H}^r_{\mathrm{DR-\acute{e}t}}(\mathrm{X/R}, \mathbf{G}_m)$ and $\mathrm{H}^r_{\mathrm{DR-zar}}(\mathrm{X/R}, \mathbf{G}_m)$ are equal.

The exact sequences listed above put us in the same formal framework as paragraph 3, and one obtains analogous results, some of which we summarize below. As usual in this section, let X/R be proper and smooth.

PROPOSITION (1.4). – *The tangent space of* $\Phi^r_{\mathrm{DR}}(\mathrm{X/R}, \mathbf{G}_m)$ over R *is given by* $\mathrm{H}^r_{\mathrm{DR}}(\mathrm{X/R}, \mathbf{G}_a)$. *If* $\Phi^{r-1}_{\mathrm{DR}}(\mathrm{X/R}, \mathbf{G}_m)$ *is formally smooth, then* $\Phi^r_{\mathrm{DR}}(\mathrm{X/R}, \mathbf{G}_m)$ *is representable by a formal group over* R (*in the extended sense*). *If* $\mathrm{H}^{r+1}_{\mathrm{DR}}(\mathrm{X}_0/k) = 0$, *then* $\Phi^r_{\mathrm{DR}}(\mathrm{X/R}, \mathbf{G}_m)$ *is formally smooth.*

If we wish to compare the functors $\Phi^r_{\mathrm{DR}}(\mathrm{X/R}, \mathbf{G}_m)$ and $\Phi^r(\mathrm{X}, \mathbf{G}_m)$, we must introduce $\Phi^r(\mathrm{X/R}, \mathrm{J}^{\cdot})$, the functor on $\mathscr{C}$, whose value is given by the r-dimensional hypercohomology of the total complex associated to the double complex

$$[\mathrm{J}^{\cdot}_{\mathrm{X}} \times_{\mathrm{R}} \mathrm{S/S} \to \mathrm{J}^{\cdot}_{\mathrm{X}} \times_{\mathrm{R}} \mathrm{S}_0/\mathrm{S}_0].$$

One then has the long exact sequence of functors on $\mathscr{C}$:

$$\ldots \to \Phi^r(\mathrm{X/R}, \mathrm{J}^{\cdot}) \to \Phi^r_{\mathrm{DR}}(\mathrm{X/R}, \mathbf{G}_m) \to \Phi^r(\mathrm{X}, \mathbf{G}_m) \to \Phi^{r+1}(\mathrm{X/R}, \mathrm{J}^{\cdot}) \to \ldots$$

The functor $\Phi^r(\mathrm{X/R}, \mathrm{J}^{\cdot})$ is exceptionally well-behaved. We shall related it to the uncompleted functor $\mathbf{R}^r f_* \mathrm{J}^{\cdot}$, the higher direct image in hypercohomology. This is the sheaf on R associated to the presheaf

$$\mathrm{S} \to \mathbf{H}^r(\mathrm{X} \times_{\mathrm{R}} \mathrm{S/S}, \mathrm{J}^{\cdot}).$$

More generally, let X be proper and smooth over R. Let $\mathscr{F}$ be a coherent sheaf over X which is flat as an R-module, or let it be an finite complex of flat R-module sheaves over X, each term of which is coherent. Consider the functor $\mathbf{R}^r f_* \mathscr{F}$. One says that *formation of* $\mathbf{H}^r(\ ,\mathscr{F})$ *commutes with base change* in $\mathscr{C}$ if the natural map

$$\mathbf{H}^r(\mathrm{X}, \mathscr{F}) \otimes_{\mathrm{R}} \mathrm{A} \to \mathbf{R}^r f_* \mathrm{F}(\mathrm{S})$$

is an isomorphism for every S = Spec A in $\mathscr{C}$. We have the evident:

LEMMA (1.5). – *Suppose formation of* $\mathbf{H}^{r-1}(\ ,\mathscr{F})$ *commutes with base change. Then these conditions are equivalent:*

(*a*) $\mathbf{R}^r f^* \mathscr{F}$ *is formally smooth, for all r;*

(*b*) $\mathbf{H}^r(\mathrm{X}, \mathscr{F})$ *is a free* R-*module of finite rank, and formation of* $\mathbf{H}^r(\ ,\mathscr{F})$ *commutes with base change in* $\mathscr{C}$, *for all r. Moreover, if these conditions hold, then* $\mathbf{R}^r f_* \mathscr{F}$ *is represented by the vector group over* R *whose associated* R-*module is* $\mathbf{H}^r(\mathrm{X}, \mathscr{F})$.

Proof. – Clearly (*b*) implies (*a*) and the final assertion. To show that (*a*) implies (*b*) use the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_R I \arrow[r] & H^r(X \times_R S'/S', \mathcal{F}) \arrow[r] & H^r(X \times_R S/S, \mathcal{F}) \arrow[r] & 0 \\
& M \otimes_R I \arrow[r] \arrow[u, "="] & M \otimes_R A' \arrow[r] \arrow[u, "\alpha'"] & M \otimes_R A \arrow[r] \arrow[u, "\alpha"] & 0
\end{tikzcd}$$

where $M = \mathbf{H}^r(X, \mathcal{F})$ and $S = \operatorname{Spec} A$, $S' = \operatorname{Spec} A'$, $S \subset S'$ being a deformation couple with ideal of definition I.

An easy diagram-chase shows that α is an isomorphism if and only if α' is. Starting with $S' = \operatorname{Spec} R$, and by descending induction on the length of the maximal ideal, one shows that

$$\alpha: \quad M \otimes_R k \to \mathbf{H}^r(X_0/k, \mathcal{F})$$

is an isomorphism. By ascending induction on the length of the maximal ideal, one shows that

$$\alpha: \quad M \otimes_R A \to \mathbf{H}^r(X \times_R S/S, \mathcal{F})$$

is an isomorphism for any $S = \operatorname{Spec} A$ in the category $\mathcal{C}$. Thus the bottom line of the above diagram is isomorphic to the top line, for any deformation couple $S \subset S'$. It follows that formation of $\mathbf{H}^r(\ , \mathcal{F})$ commutes with base change. It remains to show that M is free:

LEMMA (1.6). – *Let* R *be an artinian local ring, and* M *a module of finite type over* R, *such that*

$$0 \to M \otimes_R I \to M \otimes_R A' \to M \otimes_R A \to 0$$

is exact for every deformation couple $S = \operatorname{Spec} A \subseteq S' = \operatorname{Spec} A'$ *over* R. *Then* M *is free over* k.

Proof. – Let $m \subseteq R$ be the maximal ideal. We shall prove the lemma by induction on the length of m over k. It is evident if this length is zero. Consider an ideal $I \subseteq m$ of length 1, and form $R_0 = R/I$. Then $M_0 = M \otimes_R R_0$ is free over R_0 by the inductive hypothesis, for M_0/R_0 satisfies the same hypotheses that M/R does. Thus we have:

(*a*) I is a nilpotent ideal in the radical of R.

(*b*) $M_0 = M \otimes R/I$ is free over R/I.

(*c*) $I \otimes M \to M$ is injective.

By Proposition 5 of [8] (Chap. II, § 3), M is free over R.

COROLLARY (1.7). – *Let* X/R *be smooth and proper. Suppose that the* R*-modules* $\mathbf{H}^r(X/R, J^{\cdot})$ *are free, and that formation of* $\mathbf{H}^{r-1}(\ , \mathcal{O}^{\cdot})$ *and* $\mathbf{H}^r(\ , J^{\cdot})$ *commutes with base change in* $\mathcal{C}$. *Then* $\mathbf{R}^r f_* J^{\cdot}$ *is represented by a finite dimensional vector group over* R *whose associated module is* $\mathbf{H}^r(X/R. J^{\cdot})$ *and we have the short exact sequence of functors on* $\mathcal{C}$:

$$\ldots \xrightarrow[\partial_r]{} \widehat{\mathbf{R}^r f_* J^{\cdot}} \to \Phi^r_{DR}(X/R, \mathbf{G}_m) \to \Phi^r(X, \mathbf{G}_m) \xrightarrow[\partial_{r+1}]{} \widehat{\mathbf{R}^{r+1} f_* J^{\cdot}} \to \ldots$$

Let us call the morphism $\partial_{r+1} : \Phi^r(X, \mathbf{G}_m) \to \widehat{\mathbf{R}^{r+1} f_* \mathbf{J}^\cdot}$ the DR-*Chern map*. When is this map zero? It is clearly zero if $\Phi^r(X, \mathbf{G}_m)$ is p-divisible or if $\widehat{\mathbf{R}^{r+1} f_* \mathbf{J}^\cdot}$ vanishes, which will happen frequently in our applications. The following lemma, due to Messing, provides a more interesting case where this map vanishes.

LEMMA (1.8). — *Let* $\tilde{X}/W$ *be a smooth proper scheme. Set* $R_n = W_{n+1}$, *and let* $X_n = \tilde{X} \times_W R_n$. *Suppose that for a fixed* r,

(i) $\Phi^r(X_n/R_n, \mathbf{G}_m)$ is smooth and representable for all n;

(ii) formation of $\mathbf{H}^{r+1}(\ , \mathbf{J}^\cdot)$ commutes with arbitrary artinian base changes of W, and $\mathbf{H}^{r+1}(\tilde{X}/W, \mathbf{J}^\cdot)$ is flat over W;

(iii) either the Hodge Spectral sequence for X_n/R_n degenerates for all n, or at least the maps:

$$H^r(X_n, \mathcal{O}_{X_n}) \to \mathbf{H}^{r+1}(X_n/R_n, \mathbf{J}^\cdot)$$

are all zero. Then for R any local ring as in II, paragraph 2 the DR-Chern map ∂_{r+1} vanishes, for $\tilde{X} \otimes_W R/R$.

Proof. — We prove this using the following general fact:

Let $f : A \to B$ be a morphism of smooth formal groups over $\Lambda = \mathrm{Spf}(W)$. Let $\tan_n$ denote the tangent space functor over W_n. Suppose

$$\tan_n(f) : \tan_n(A) \to \tan_n(B)$$

is zero for all n. Then f is zero.

The lemma follows quite simply: By (i) and (ii) the domain and range of the DR-Chern map may both be considered smooth formal groups over Λ; and by (iii) $\tan_n(\partial_{r+1}) = 0$ for all n.

COROLLARY (1.9). — *The long exact sequence above splices to the following short exact sequence of* formally smooth *groups*

$$0 \to \widehat{\mathbf{R}^r f_* \mathbf{J}^\cdot} \to \Phi^r_{\mathrm{DR}}(X/R, \mathbf{G}_m) \to \Phi^r(X, \mathbf{G}_m) \to 0$$

in the following cases:

(*a*) X *is a smooth complete intersection in projective space over* R, *of dimension* r;

(*b*) *take* $r = 2$, *and suppose that* X/R *is a smooth proper surface such that* $H^0(X \times_R S, \Omega^1) = H^1(X \times_R S, \mathcal{O}_{X \times_R S}) = 0$ *for all* S *in* $\mathscr{C}$ (*e. g. a* K 3 *surface*);

(*c*) *take* $r = 2$, *and suppose that* X/R *is a smooth proper surface such that:*

(i) $\mathbf{H}^1(\ , \mathbf{J}^\cdot)$ *and* $\mathbf{H}^2(\ , \mathbf{J}^\cdot)$ *commute with base change in* $\mathscr{C}$,

(ii) $\mathbf{H}^2(X/R, \mathbf{J}^\cdot)$ *is flat over* R,

(iii) $\mathrm{Pic}_{X/R}$ *is formally smooth,*

(iv) $\Phi^2(X, \mathbf{G}_m)$ *is p-divisible.*

Proof. — Set $L^r = \widehat{R^r f_* J^{\cdot}}$.

(*a*) First, we may assume that $r \geqq 2$. Secondly, X/R lifts to a smooth complete intersection $\tilde{X}/W$. The functor Φ_r is representable by a formal smooth group because $h^{0,r-1} = h^{0,r+1} = 0$. Finally, $\tilde{X}/W$ satisfies the conditions of lemma 1.8 by [16] (Exp. XI, theorem 1.5).

It then follows that the sequence splices, and L^r is formally smooth. Consequently Φ^r_{DR} is also formally smooth.

(*b*) By Serre duality, we also have $H^2(\Omega^1) = H^1(\Omega^2) = 0$. It follows that $\widehat{\mathrm{Pic}}$, L^1, L^3 are all zero. Consequently L^2 is formally smooth, and the long exact sequence splits. Since $h^{0,3}$ vanishes, Φ^2 is formally smooth, and therefore so is Φ^2_{DR}.

(*c*) Follows in a straightforward way.

COROLLARY (1.10). — *Suppose* $R = k$, *and both functors* Φ^r_{DR} *and* Φ^r *are representable by formal Lie groups. Then* $\Phi^r_{DR} \equiv \Phi^r$ (*modulo unipotent groups*).

2. THE CRYSTALLINE NATURE $\Phi^r_{DR}(X/R, \mathbf{G}_m)$. — We shall put ourselves in the context of the theory of Berthelot. For this, we draw extensively from the results and terminology of his thesis [5].

The results of this section are due to Berthelot, and we are quoting, with mild modifications, a letter he sent to us on August 22, 1973.

As usual we shall have to make a choice of basis site Σ which can be either the small étale or Zariski site. However, we shall consider other sites depending on Σ. For example, if X/R is smooth, we consider Σ-Crys (X/R). Its objects are triples (U, T, δ), where $U \to X$ is an "open set" of the site Σ, T is a scheme over R containing U as a closed subscheme, and whose ideal of definition is endowed with the divided power structure δ. A family of maps $(U_j, T_j, \delta_j) \to (U, T, \delta)$ is a covering family if and only if $(T_j \to T)$ is a covering family for Σ. If X_0/k is smooth, and the maximal ideal of R is given a divided power structure γ, then we will also consider the site Σ-Crys $(X_0/(R, \gamma))$. Its objects are triples (U, T, δ) where $U \to X_0$ is an "open set" of the site Σ, T is an R-scheme containing U as a subscheme whose ideal of definition is endowed with the divided power structure γ. Covering families are as before.

From now on we fix a site, and drop the symbol Σ to simplify the typography. Thus we shall refer to the site Crys (X/R), etc.

Let k be a perfect field of characteristic p, and let $R = W_{n+1}(k) = W_{n+1}$ for some $n \geqq 0$. Assume, as usual, that X/R is smooth. For any S in $\mathscr{C}$, let $\tilde{\mathscr{O}}^{\times}_{\text{crys}}(S)$ denote the complex of length 2 on $\text{Crys}(X \times_R S/S)$ given by

$$\tilde{\mathscr{O}}^{\times}_{\text{crys}}(S)(U, T, \delta) = [\mathscr{O}^{\times}(T) \to \mathscr{O}^{\times}(U \times_R k)].$$

Let $\Phi^r_{\text{crys}}(X/R, \mathbf{G}_m)(S)$ be the r-dimensional hyper-cohomology over $\text{Crys}(X \times_R S/S)$ of the complex $\tilde{\mathscr{O}}^{\times}_{\text{crys}}(S)$.

PROPOSITION (2.1) (Berthelot). — *There is an isomorphism of functors*

$$\Phi^r_{DR}(X/R, \mathbf{G}_m) \cong \Phi^r_{\text{crys}}(X/R, \mathbf{G}_m).$$

Proof. — Let L denote the "linearization functor" [5, Ch. V. 3] which associates to a differential operator on $X \times_R S$ relative to S, an $\mathcal{O}_X \times_R S/S$ linear homomorphism of sheaves on Crys($X \times_R S/S$). Denote by $L(\Omega^{\times}_{X \times_R S/S})$ the complex

$$L(\mathcal{O}^{\times}_{X \times_R S/S}) \xrightarrow{d\,\mathrm{Log}} L(\Omega^{1}_{X \times_R S/S}) \xrightarrow{L(d)} L(\Omega^{2}_{X \times_R S/S}) \to \ldots,$$

and by $L(\tilde{\Omega}^{\times}(S))$ the total complex associated to the double complex

$$L(\Omega^{\times}_{X \times_R S/S}) \to L(\Omega^{\times}_{X_0/k}),$$

where the term of degree r in the constituent complexes is given the bidegree $(0, r)$ and $(1, r)$ respectively.

By the Poincaré lemma [5] (V. 2) $L(\Omega^{\times}_{X \times_R S/S})$ is acyclic in degrees $\geqq 2$, in degree 0 its cohomology is $\mathcal{O}^{\times}_{(X \times_R S/S)_{\mathrm{crys}}}$, and in degree 1 all one can say is that its cohomology is some sheaf ε_S which would vanish if there were an exponential map [5]. But since such a map is not available to us on the site Crys ($X \times_R S/S$), we cannot suppose that it is zero. What is true, however, is the following:

LEMMA (2.2). — *The complex* $[\tilde{\varepsilon}_S \to \varepsilon_k]$ *has trivial hypercohomology in all dimensions over* Crys ($X \times_R S/S$).

Proof of proposition (2.1), *granted lemma* (2.2). — Let u denote the projection of the crystalline topos $(X \times_R S/S)_{\mathrm{crys}}$ onto the Σ-topos $X \times_R S$. The calculation of cohomology by "Čech-Alexander complexes" (*cf.* [5], V, Cor. 2.2.4) shows that

$$R\,u_*(L(\Omega^{\times}_{X \times_R S/S})) \cong \Omega^{\times}_{X \times_R S/S},$$

and consequently:

$$R\,u_*(L(\tilde{\Omega}^{\times}(S))) \cong \tilde{\Omega}^{\times}(S).$$

We obtain the following cohomological calculations:

$$H^*((X \times_R S/S)_{\mathrm{crys}}, L(\Omega^{\times}_{X \times_R S/S})) \cong H^*(X \times_R S/S, \Omega^{\times}_{X \times_R S/S}),$$

$$H^*((X \times_R S/S)_{\mathrm{crys}}, L(\tilde{\Omega}^{\times}(S))) \cong H^*(X \times_R S/S, \tilde{\Omega}^{\times}(S)).$$

By virtue of the last isomorphism and the above discussion, we have the following long exact sequence of hypercohomology:

$$\to H^r((X \times_R S/S)_{\mathrm{crys}}, \tilde{\mathcal{O}}^{\times}_{\mathrm{crys}}) \to H^r(X \times_R S, \tilde{\Omega}^{\times}(S)) \to H^{r-1}(X \times_R S, \varepsilon(S)) \to,$$

and the lemma therefore establishes the proposition.

Proof of lemma (2.2). — It suffices to prove that $R^i u_*(\varepsilon_S) \to R^i u_*(\varepsilon_{S_0})$ is an isomorphism for all i. But the $R^i u_*$ are the cohomology sheaves of the "Čech-Alexander" complex relative to the smooth imbedding given respectively by X and X_0. What we shall show is that there is an isomorphism of the Čech-Alexander complexes themselves: $\check{C}A^{\bullet}_{X \times_R S/S}(\varepsilon_S) \xrightarrow{\sim} \check{C}A^{\bullet}_{X_0}(\varepsilon_{S_0})$. This assertion is local, and we may therefore suppose that X is étale over affine m-space, and therefore the complex $L(\Omega^{\times}_{X\times_R S/S})$ can be written over any object (U, T, δ) of the crystalline site as follows:

$$K^{\cdot}(\mathcal{O}_T) = \mathcal{O}_T\langle\tau_1, \ldots, \tau_n\rangle \xrightarrow{d\log} \mathcal{O}_T\langle\tau_1, \ldots, \tau_n\rangle \otimes \Omega^1 \xrightarrow{d} \mathcal{O}_T\langle\tau_1, \ldots, \tau_n\rangle \otimes \Omega^2 \to \ldots,$$

where $A\langle\tau_1, \ldots, \tau_n\rangle$ is the divided power series ring over A in the indeterminates $\tau_1, \ldots, \tau_n$ (*cf.* [5], Chap. I) and d is the $\mathcal{O}_T$-linear derivation with respect to the indeterminates $\tau_1, \ldots, \tau_n$. It suffices, then, to prove the following assertion: Let B, B′ be two rings such that $0 \to I \to B' \to B \to 0$ where I is an ideal of square zero and let $N^{\cdot}$ denote the complex which is the kernel of the surjective homomorphism $K^{\cdot}(B) \to K^{\cdot}(B)$. Then $N^{\cdot}$ is acyclic in dimensions greater than zero. But $N^{\cdot}$ is the complex:

$$1+I.B\langle\tau_1, \ldots, \tau_n\rangle \xrightarrow{d\log} I.B\langle\tau_1, \ldots, \tau_n\rangle \otimes \Omega^1 \xrightarrow{d} I.B\langle\tau_1, \ldots, \tau_n\rangle \otimes \Omega^2 \to \ldots$$

Now this complex is acyclic in dimensions ≥ 2 by the Poincaré lemma [5] (V.2). In dimension 1, the Poincaré lemma tells us that for every ω such that $d(\omega) = 0$, there is an element $f \in I.B\langle\tau_1, \ldots, \tau_n\rangle$ such that $\omega = df$. Since I is of square zero, $g = \exp(f)$ exists and $dg/g = \omega$.

Remarks. — 1. We emphasize again that we do not necessarily have an isomorphism between the groups $H^r_{DR}(X\times_R S/S, G_m)$ and $H^r_{Crys}(X\times_R S/S, G_m)$ due to the nonexistence of an exponential map. We do, however, have an exponential map in the site *Nil. Crys.*$-\Sigma$ which is the analogous variant of Berthelot's nilpotent crystalline site. Consequently,

$$H^r_{DR}(X\times_R S/S, G_m) \xrightarrow{\sim} H^r_{Nil.\,Crys.}(X\times_R S/S, G_m).$$

The only problem with the site Nil. Crys. is that the divided power structure on the ideal 2 of the Witt vectors of a perfect field of characteristic 2 is not nilpotent.

2. The main advantage of the crystalline interpretation for $\Phi^r_{DR}(X/R, G_m)$ is that it is functorially dependent only on X_0/R. Explicitly, consider γ, the standard divided power structure on the ideal $p.W_{n+1}(k) \subset W_{n+1}(k)$. If $S = \operatorname{Spec} A$ is in $\mathcal{C}$, let $\bar{S} = \operatorname{Spec}(A/p.A)$. Working over the site $\operatorname{Crys}(X_0 \times_k \bar{S}/(S, (p), \gamma))$, let $\Phi^r_{Crys}(X_0/R, G_m)(S)$ be the r-dimensional hypercohomology of the complex $\mathcal{O}^{\times}_{Crys}(S)$ of length two, defined at the beginning of this paragraph.

Proposition (2.3). — $\Phi^r_{DR}(X/R, G_m) \cong \Phi^r_{Crys}(X_0/R, G_m)$.

3. The above functor $\Phi^r_{Crys}(X_0/R, G_m) : \mathcal{C} \to (Ab)$ is available to us even if no smooth lifting X/R of X_0 to R exists. To establish our theory in the case where X_0 does not lift to the Witt vectors of k, it would undoubtedly be necessary to work with this functor directly. We have not done this.

3. The Frobenius endomorphism of $\Phi^r_{crys}(X_0/R, G_m)$. — Continuing with $R = W_{n+1}$ consider the Frobenius morphism $F : X_0 \to X_0^{(p)}$.

When $\Phi^r_{crys}(X_0/R, G_m)$ is representable by a formal group over Spec R, the above morphism induces a morphism of formal groups $v : \Phi^r_{crys}(X_0/R, G_m)^{(\varphi)} \to \Phi^r_{crys}(X_0/R, G_m)$.

LEMMA (3.1). — *When $\Phi^r_{\mathrm{crys}}(X_0/R, \mathbf{G}_m)$ is pro-representable by a formal lie group Φ, the operator v is a lifting of the canonical Verschiebung morphism $\Phi^{(\varphi)} \to \Phi_0$ ($\Phi_0 = \Phi \times_R k$).*

Proof. — Since the assertion is about Φ_0, we may suppose $R = k$ and consequently we have $\Phi^r_{\mathrm{crys}}(X_0/k, \mathbf{G}_m) = \Phi^r_{\mathrm{DR}}(X_0/k, \mathbf{G}_m)$. Denote by Ver the canonical Verschiebung of Φ_0. We must show $v - \mathrm{Ver} = 0$. It suffices to show that $\mathrm{Frob.}(v - \mathrm{Ver}) = 0$, for the kernel of the Frobenius is a finite subgroup scheme of Φ_0 and there are no nonconstant morphisms from a smooth formal group to a finite group scheme over k. Consequently we are reduced to establishing $\mathrm{Frob.}v = p$.

But what is the composition of v with Frob ? The reader will easily obtain from the definitions that the endomorphism $\mathrm{Frob.}v$ on $H^r_{\mathrm{DR}}(X_0 \times_k T_0, \mathbf{G}_m)$ is induced from the Frobenius morphism for the scheme $X_0 \times_k T_0$. Thus, denoting the scheme $X_0 \times_k T_0$ by the letter Z we are reduced to showing that the Frobenius morphism for Z/Spec k nduces multiplication by p on the multiplicative de Rham complex of Z:

$$\begin{array}{ccccccc}
\mathcal{O}_Z^\times & \xrightarrow{d\log} & \Omega^1_{Z/k} & \xrightarrow{d} & \Omega^2_{Z/k} & \xrightarrow{d} & \dots \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
\mathcal{O}_Z^\times & \longrightarrow & \Omega^1_{Z/k} & \longrightarrow & \Omega^2_{Z/k} & \longrightarrow & \dots
\end{array}$$

One finds that Frob acting on $\mathcal{O}_Z^\times$ is multiplication by p, but on $\Omega^i_{Z/k}$ ($i \geqq 1$) it is identically zero. Since multiplication by p on $\Omega^i_{Z/k}$ ($i \geqq 1$) is identically zero, we have indeed that the Frobenius endomorphism is simply multiplication by p on the multiplicative de Rham complex. A consequence of the above discussion is the following:

Let X/W be a smooth proper scheme. For any n, set $X_n = X \times_{\mathrm{Spec}\, W} \mathrm{Spec}\, W_{n+1}$. Fix a nonnegative integer r, and consider

$$\Phi_n = \Phi^r_{\mathrm{DR}}(X_n/R_n, \mathbf{G}_m) = \Phi^r_{\mathrm{crys}}(X_0/R_n, \mathbf{G}_m),$$

where $R_n = W_{n+1}$. Suppose that the Φ_n are representable as formal Lie groups over R_n for every n. Let $v_n : \Phi_n^{(\varphi)} \to \Phi_n$ denote the morphism which was denoted v previously. Then the system of formal groups and morphisms (Φ_n, v_n) are compatible with restriction and give rise to a formal group Φ over $\Lambda = \mathrm{Spf}(W)$, together with a morphism $v : \Phi^{(\varphi)} \to \Phi$.

PROPOSITION (3.2). — *Under the hypothesis that the Φ_n are formal Lie groups over R_n for all, n, the pair (Φ, v) is a Cartier group whose associated Cartier module is $H^r_{\mathrm{Crys}}(X_0/W, \mathbf{G}_a) = H^r_{\mathrm{Crys}}(X_0/W)$ whose associated endomorphism f is the one induced by Frobenius.*

Proof. — This follows from the previous lemma and discussion.

COROLLARY 3.3. — *Let X/W be proper and smooth. Suppose, for a given r, that $\Phi^r(X_0, \mathbf{G}_m)$ is a formal Lie group, and that $\Phi^r_{\mathrm{DR}}(X_n/R_n, \mathbf{G}_m)$ are formal Lie groups over R_n for all $n \geqq 0$. Then*

$$H^r(X_0, \mathscr{W}) = D\Phi^r(X_0, \mathbf{G}_m) \equiv H^r_{\mathrm{Crys}}(X_0/W)_{[0,1)} \quad (^5).$$

---

(5) The isomorphism $H^r(X_0, \mathscr{W}) \otimes K \cong H^r_{\mathrm{Crys}}(X_0/W)_{[0,1)} \otimes K$ has also been proved by Bloch [7] (*see* also [6] 4.5) when $\dim X_0 > \mathrm{char}\, k$.

Proof. — This merely combines corollary (I, 2.27), lemma (II, 2.14) and proposition (3.2) above.

COROLLARY 3.4. — *Continuing with the hypotheses and notation of the previous corollary, let*

$$0 \leqq a_0/b_0 < a_1/b_1 < \ldots < a_t/b_t < 1$$

be the slopes of the Cartier module $\mathrm{H}^r_{\mathrm{crys}}(\mathrm{X}_0/\mathrm{W})_{[0,1)}$ *written in such a way that* a_i/b_i *has multiplicity* b_i. *Let* $h^{0,r} = \dim_k \mathrm{H}^r(\mathrm{X}_0, \mathcal{O}_{\mathrm{X}_0})$. *Then*

$$\sum_{i=0}^{t} b_i - a_i \leqq h^{0,r},$$

where one has equality if and only if $\Phi^r(\mathrm{X}_0, \mathbf{G}_m)$ *is of finite height, or equivalently, if and only if* $\mathrm{H}^r(\mathrm{X}_0, \mathscr{W})$ *is of finite type over* W.

Proof. — If we let Γ be a formal group of finite height whose Dieudonné module is equivalent to $\mathrm{H}^r_{\mathrm{crys}}(\mathrm{X}_0/\mathrm{W})_{[0,1)}$ then one easily computes the dimension of Γ to be $\Sigma(b_i - a_i)$. The inequality follows from the fact that $\dim \Phi_r = h^{0,r}$, as does the rest of the corollary.

Remarks:

1. The above inequality is the first of a series of inequalities involving the Hodge numbers and the slopes of $\mathrm{H}^r_{\mathrm{crys}}(\mathrm{X}_0/\mathrm{W})$, known as the *Katz conjecture* ([26], [27]).

2. The hypotheses of these corollaries are fulfilled when X/W is a smooth proper scheme of any of the types *a*, *b*, *c*, of corollary (1.9).

IV. — The formal Brauer group of a surface as link between crystalline cohomology in characteristic *p* and étale cohomology in characteristic zero

1. REPLACING $\mathbf{G}_m$ BY $\boldsymbol{\mu}_{p^\infty}$. — To introduce the "enlarged functor" Ψ (*cf.* introduction) we shall consider the cohomology of $\boldsymbol{\mu}_{p^\infty}$ rather than $\mathbf{G}_m$. Since the group schemes $\boldsymbol{\mu}_{p^n}$ are not smooth, the étale site is insufficient for cohomological calculations, and we must work in the fppf site. Since $\boldsymbol{\mu}_n \to [\mathbf{G}_m \xrightarrow{n} \mathbf{G}_m]$ is a quasi-isomorphism of complexes of fppf sheaves, and since $\mathbf{G}_m$ is smooth, we will use a theorem of Grothendieck to obtain an isomorphism

$$\mathrm{H}^i_{\mathrm{fl}}(-, \boldsymbol{\mu}_n) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{\acute{e}t}}(-, [\mathbf{G}_m \xrightarrow{n} \mathbf{G}_m])$$

enabling us to work with the étale site. Passing to the inductive limit gives an expression for $\mathrm{H}^i_{\mathrm{fl}}(-, \boldsymbol{\mu}_{p^\infty})$ in terms of étale hypercohomology. By means of this, we will show (Prop. 1.5) that the *fppf*-deformation cohomology of $\boldsymbol{\mu}_{p^\infty}$ is isomorphic to the *étale* deformation cohomology of $\mathbf{G}_m$, at least over bases where *p* is nilpotent. This will enable us to identify the connected component of Ψ with $\widehat{Br}$ [*see* introduction, and (1.8) below].

The flat topology will be indicated by the subscript fl.

We assume that our spaces X are defined (not necessarily of finite type) over Spec $\mathbf{Z}/p^\nu$ for some ν, or that we are working with formal objects over Spec $\hat{\mathbf{Z}}_p$.

Let G be a sheaf on X. We can consider the sheaf $G_{\mathbf{Q}} = G \otimes \mathbf{Q}$, which is the direct limit of G, mapping to itself by multiplication by n. Define complexes of sheaves $G[n]$, $n \leqq \infty$, by

$$(1.1)\qquad \begin{cases} G[n] = [G \xrightarrow{n} G] & \text{if } n < \infty, \\ G[\infty] = \varinjlim_{n} G[n] = [G \to G_{\mathbf{Q}}]. \end{cases}$$

Suppose that G is formally smooth on X, and that $\hat{G}$ is represented by a (smooth) formal group on X, with Lie algebra $\mathfrak{g}$. Let $X_1 \subset X'_1$ be an infinitesimal extension of X-schemes defined by a square-zero ideal J, so that we have an exact sequence

$$(1.2)\qquad 0 \to \mathfrak{g} \otimes J \to G \,|\, X'_1 \to G \,|\, X_1 \to 0.$$

Then $\mathfrak{g} \otimes J$ is p-torsion, hence tensoring by $\mathbf{Q}$ gives

$$G_{\mathbf{Q}} \,|\, X'_1 \xrightarrow{\sim} G_{\mathbf{Q}} \,|\, X_1.$$

In other words, $G_{\mathbf{Q}}$ is discrete. Correspondingly, there is an exact triangle of complexes

$$(1.3)\qquad \mathfrak{g} \otimes J \to G[\infty] \,|\, X'_1 \to G[\infty] \,|\, X_1.$$

Let $f : X \to S$ be a proper flat map, and let G be as above, on X. Then if $Z \subset Z'$ is an infinitesimal extension of S-schemes defined by a square-zero ideal I, we obtain from (1.3) an exact sequence

$$(1.4)\qquad \ldots \to R^q f_* \mathfrak{g} \otimes I \,|\, Z \xrightarrow{\iota} R^q f_* G[\infty] \,|\, Z' \to R^q f_* G[\infty] \,|\, Z \to \ldots$$

PROPOSITION 1.5. — *With the above notation, assume that* $R^{q-1} f_* G$ *is formally smooth along every* torsion *section. Then*

(i) $R^{q-1} f_* G[\infty]$ *is formally smooth, and hence the map* ι *of* (1.4) *is injective:*

(ii) $\Phi^q f_* \hat{G} \approx \Phi^q f_* G[\infty]$, *and this sheaf is represented by a formal group scheme on* S*;*

(iii) *let* α_0 *be a section of* $R^q f_* G[\infty]$ *over a point* $s_0 \in S$. *Then* $R_q f^* G[\infty]$ *is pro-representable, along the section* α_0, *say by* $\hat{y}$, *and* $R^q f_* G$ *operates simply and transitively on* $\hat{y}$.

Proof. — (i) Consider the exact sequence

$$R^{q-2} f_* G_{\mathbf{Q}} \to R^{q-1} f_* G[\infty] \to R^{q-1} f_* G \to R^{q-1} f_* G_{\mathbf{Q}}.$$

The extreme terms are $\mathbf{Q}$-modules and are discrete (II, 1.4), while $R^{q-1} f_* G[\infty]$ is torsion. It follows from the assumption ($R^{q-1} f_* G$ is formally smooth along every torsion section) that $R^{q-1} f_* G[\infty] \,|\, Z' \to R^{q-1} f_* G[\infty] \,|\, Z$ is surjective for every infinitesimal extension $Z \subset Z'$, as required.

(ii) The exact sequence (1.2) shows that $\hat{G}$ is a torsion sheaf. Hence there is a natural map $\hat{G} \to G[\infty]$, inducing $\Phi^q f_* \hat{G} \to \Phi^q f_* G[\infty]$. This map is an isomorphi because of (i) and [II, 1.7 (iii)].

Assertion (iii) is clear from Schlessinger's theory.

Remark 1.6. — The important point in this discussion was the fact that $\mathfrak{g} \otimes J$ is p-torsion. So, it would work equally well if $G[\infty]$ were replaced by

$$\varinjlim_{\nu} G[p^{\nu}] = G \to G \otimes \mathbf{Z}[1/p].$$

Consider the case that G is represented by a smooth algebraic space on X which is divisible, i. e., such that $G \xrightarrow{n} G$ is a flat epimorphism. Denote its kernel by G_n, and let $G_{\infty} = \bigcup_n G_n$. The group G_n ($n \leqq \infty$) will be a flat algebraic space, but it will usually not be smooth. By a theorem of Grothendieck ([15], Appendice) the étale and the flat cohomology for G are isomorphic, and so the same is true of the complexes $G[n]$. On the other hand, the complex $G[n]$ is isomorphic, in the derived category of sheaves for the flat topology, to the sheaf G_n concentrated in dimension zero. Thus we have

COROLLARY 1.7. — *Let* G *be represented by a smooth divisible algebraic group on* X. *Then for all* $n \leqq \infty$ *we have:*

(i) $H^q_{\text{ét}}(X, G[n]) \xrightarrow{\sim} H^q_{\text{fl}}(X, G_n)$;

(ii) *if* $f : X \to S$ *is a proper map, then* $R^q_{\text{ét}} f_* G[n]$ *is the étale sheaf associated to the presheaf* $\mathscr{R}$ *defined by* $\mathscr{R}(S') = H^q_{\text{fl}}(X_{S'}, G_n)$.

A technical point. — One can not quite identify $R^q_{\text{ét}} f_* G[n]$ with $R^q_{\text{fl}} f_* G$, for we don't know whether $\mathscr{R}^q_{\text{ét}} f_* G[n]$ is a sheaf for the flat topology.

We apply the above formalism as follows. Let R be a discrete valuation ring with perfect residue field of characteristic p. Let X/R be a proper smooth surface and let Ψ denote the subsheaf of

$$R^2_{\text{ét}} f_* \mathbf{G}_m[p^{\infty}] = \varinjlim_{n} R^2_{\text{ét}} f_* \mathbf{G}_m[p^n] = \underline{R},$$

which to any Artinian local R-algebra A with residue field $\bar{k}$ associates the subgroup $\Psi(A)$ of those sections of $\underline{R}(A)$ which map to the divisible part of $H^2_{\text{fl}}(X_{\bar{k}}, \boldsymbol{\mu}_{p^{\infty}})$:

$$0 \to \Psi(A) \to H^2_{\text{fl}}(X_A, \boldsymbol{\mu}_{p^{\infty}}) \to \operatorname{Div} H^2_{\text{fl}}(X_{\bar{k}}, \boldsymbol{\mu}_{p^{\infty}}).$$

PROPOSITION 1.8. — *Suppose* (Hypothesis A) $\operatorname{Pic}^{\tau}(X/R)$ *is smooth and the formal Brauer group* $\widehat{Br}$ *is of* finite *height* h. *The functor* Ψ *is* (*the restriction to the category of artinian local* R*-algebras of the functor associated to a p-divisible group over* R. *Denoting this* (*necessarily unique*) *p-divisible group by the same letter* Ψ, *we have* $\Psi^0 = \widehat{Br}$ *and the étale part of* Ψ *may be identified with the divisible part of* $H^2_{\text{fl}}(X_{\bar{k}}, \boldsymbol{\mu}_{p^{\infty}})$ *as* $\operatorname{Gal}(\bar{k}/k)$ *module.*

Proof. — First suppose that k is algebraically closed. Since X/R is a surface, applying (1.4) one sees that Ψ is formally smooth. Applying (1.5) and hypothesis A, one sees that for every $\xi \in \operatorname{Div} H^2_{\mathrm{fl}}(X_{\bar{k}}, \mu_{p^\infty})$ the functor Ψ_ξ defined by

$$\Psi_\xi(A) = \{ x \in \Psi(A) \mid x \to \xi \},$$

is pro-representable as a torsor over $\widehat{\mathrm{B}r}$ (where Div means "the divisible part of"). The proposition then follows for k algebraically closed, since $\operatorname{Div} H^2_{\mathrm{fl}}(X_{\bar{k}}, \mu_{p^\infty})$ is a p-divisible (abstract) group of finite corank. It is easy to descend to the case where k is perfect:

For a pair of positive integers m, n, let $\Psi_{m,n}$ be the sheaf kernel of p^m in Ψ, taken over the base $R_n = R/p^n R$. We must show that $\Psi_{m,n}$ is representable by a finite flat group scheme over R_n. But the pullback of $\Psi_{m,n}$ to $\bar{R}_n = R_n \otimes_{W(k)} W(\bar{k})$ is so representable. Since we are given $\Psi_{m,n}$ as sheaf for the big étale site over R_n, we have the requisite "descent data" for the étale extension $\bar{R}_n/R_n$.

PROPOSITION 1.9. — *Let* $p > 2$, *and suppose* X/R *is a smooth proper surface as above, satisfying hypothesis* A. *Then height* $\widehat{\mathrm{B}r} = \operatorname{rank}_{\mathbf{Z}_p} H^2_{\mathrm{crys}}(X_0/W)_{[0,1)}$ *and*

$$\text{height } \psi = \operatorname{rank}_{\mathbf{Z}_p} H^2_{\mathrm{crys}}(X_0/W)_{[0,1]}.$$

Proof. — The first equality is just Corollary (3.3) and its footnote. The second comes from the first, proposition (1.8), and the theorem of Bloch [7] which when $p > 2$ interprets the number of eigenvalues of Frobenius of slope 1 in terms of flat cohomology

$$\operatorname{rank}_{\mathbf{Z}_p} H^2_{\mathrm{crys}}(X_0/W)_{[1,1]} = \operatorname{corank}_{\mathbf{Z}_p} H^2_{\mathrm{fl}}(X_{\bar{k}}, \mu_{p^\infty})$$

(*see* also [6] 4.5).

Relation to blow-up. — If X/R satisfies hypothesis A, and $f : X' \to X$ is the blow-up of X along a section over R, then $R^q f_* \mathcal{O}_{X'} = 0$ for $q > 0$, which implies that $\operatorname{Pic}^\tau X'/R = \operatorname{Pic}^\tau X/R$ and $\widehat{\mathrm{B}r}\, X' = \widehat{\mathrm{B}r}\, X$. It follows that X/R satisfies hypothesis A if and only if X'/R satisfies hypothesis A. One has the following relation for the p-divisible groups Ψ over R, associated to X and X':

$$(1.10) \qquad 0 \to \Psi_X \to \Psi_{X'} \to \mathbf{G}_p/\mathbf{Z}_p \to 0.$$

2. THE RELATIONSHIP BETWEEN Ψ AND ÉTALE COHOMOLOGY IN CHARACTERISTIC 0. — We shall work in a mixed characteristic local situation. Best for our purposes is to let R denote a discrete valuation ring whose residue field k is algebraically closed of characteristic p and whose field of fractions K is of characteristic zero. Let S = Spec R, and let X/S be a smooth proper surface. Let the subscript 0 denote reduction to k.

Let $\bar{R}$ denote the absolute integral closure of R, i. e., the integral closure in the algebraic closure $\bar{K}$ of K. Let $\bar{S} = \operatorname{Spec} \bar{R}$, and let the subscript $\bar{\eta}$ denote passage to the geometric general fiber (i. e. the fiber over $\bar{\eta}$). If G is a group, G (n) denotes the kernel of multiplication by p^n on G. Let TG denote the projective system $\varprojlim_n G(n)$.

Consider the $\mathbf{Z}_p$-module (of finite rank $= b_2$) : $H = T(H^2_{\mathrm{\acute{e}t}}(X_{\bar{\eta}}, \mu_{p^\infty}))$. This is a $\operatorname{Gal}(\bar{K}/K)$-module and $H \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ possesses a nondegenerate quadratic form given by cup product on the surface $X_{\bar{\eta}}$.

PROPOSITION 2.1 (CONVERGENCE). — *Under hypothesis* A (Pic^{τ} X/S *is formally smooth and* $\widehat{\mathrm{Br}}$ *is of finite height*):

(*a*) *the natural map*

$$\mathrm{TH}^2_{\mathrm{fl}}(X_{\bar{S}}, \mu_{p^\infty}) \to \mathrm{T}(\Psi(\bar{S}))$$

is an isomorphism.

(*b*) *the natural map*

$$\mathrm{TH}^2_{\mathrm{fl}}(X_{\bar{S}}, \mu_{p^\infty}) \to \mathrm{TH}^2_{\mathrm{fl}}(X_{\bar{\eta}}, \mu_{p^\infty})$$

is injective.

Preparation for the proof. — Our object is to reduce (*a*) and (*b*) to assertions concerning *one*-dimensional cohomology, and then, by treating a one-dimensional cohomology class as a torsor, we shall have strong geometric tools at our disposal.

Since the assertions of our proposition are statements over $\bar{S}$, we are free to replace X/R by X′/R′ where R′ is (say) any finite discrete valuation ring extension of R, and using (1.10), we may also replace X by the blow-up of X along a finite union of disjoint sections over S.

Choose a family of Lefschetz pencils ([18], SGA 7, II, Pinceaux de Lefschetz: théorème d'existence) over the family X/S. Any family of pencils which specializes to give a Lefschetz pencil over k will automatically give a Lefschetz pencil over η since the fibre dimension is odd. Replace X by an appropriate blow-up along disjoint sections of S, to obtain a fibration

$$X \xrightarrow{f} Y = \mathbf{P}^1_S$$

whose only singularities are quadratic degeneracies above finitely many disjoint sections $P_1, \ldots, P_l$ of Y over S. Put $P = \bigcup_i P_i$.

Let us write Pic X/Y $= A \oplus \mathbf{Z}$, where A is the fibration of generalized jacobians of X/Y. This decomposition is canonical if we fix a section, say θ, of X/Y. Let the subscript j denote reduction modulo p^{j+1}. The Leray spectral sequence for f shows that the group $H^1_{\mathrm{fl}}(Y_j, A_n)$ is canonically identified with the subgroup of $H^2_{\mathrm{fl}}(X_j, \mu_n)$ of elements whose restrictions to the fibres $(X_j)_y$ and the section θ_j are trivial. Since the fibre and section are algebraic classes, we may for our purposes work with $H^1_{\mathrm{fl}}(Y_j, A_n)$ as well as $H^2_{\mathrm{fl}}(X_j, \mu_n)$.

To prove (*a*) and (*b*) of (2.1) it will suffice to prove the following two lemmas. The second we state slightly more generally than needed.

LEMMA 2.2. — *The restriction map* $H^1_{\mathrm{fl}}(\bar{Y}, A_n) \to H^1_{\mathrm{fl}}(Y_{\bar{\eta}}, A_n)$ *is injective for every* n.

LEMMA 2.3. — *Let* R *be a complete local ring with maximal ideal* $\mathfrak{m}$ *and* S = Spec R. *Let* $f: Y \to S$ *be a proper map, and let* A *be a quasi-finite, flat, separated group scheme over* Y. *Assume that* A *is finite over* Y *at all points of the closed fibre* Y_0 *except for a finite set* $p_1, \ldots, p_l \in Y_0$. *Denote by a subscript j truncation modulo* $\mathfrak{m}^{j+1}$. *Then, the canonical map*

$$H^1_{\mathrm{fl}}(Y, A) \to \varprojlim_j H^1_{\mathrm{fl}}(Y_j, A)$$

is surjective. If f *is flat, it is bijective.*

We shall also show:

LEMMA 2.4. — *The map* $H^1(Y, A_n) \to H^1(Y_0, A_n)$ *is surjective, and its kernel is isomorphic to the group* $\bar{B}_n$ *of sections of order n of* $\widehat{Br}\, X/S$ *over* $\bar{S}$.

We defer the proofs of these lemmas to the next section and terminate the present one by relating cup-product over Y to intersection pairing on the Néron-Severi group of X.

LEMMA 2.5. — *There is, on* Y, *a canonical pairing of sheaves* $A_n \otimes A_n \to \mu_n$, *induced by the autoduality* map $A \to \mathscr{E}xt^1(A, G_m)$ *of the jacobians.*

Proof. — According to [1], there is an isomorphism

$$A \to \mathscr{E}xt^1(A, G_m) \tag{2.6}$$

if these sheaves are restricted to the small smooth site on Y, the map being induced by the classical pairing of idèles. Since A is smooth, the universal class on A defines a morphism (2.6) for the big flat site, though it is no longer an isomorphism above the locus P. Multiplication by n yields a map $A_n \to \mathscr{H}om(A_n, G_m) \approx \mathscr{H}om(A_n, \mu_n)$, hence a pairing $A_n \otimes A_n \to \mu_n$, as required.

Thus we can consider the cup product pairing

$$H^1_{fl}(Y, A_n) \otimes H^1_{fl}(Y, A_n) \to H^2_{fl}(Y, \mu_n). \tag{2.7}$$

It is compatible with restriction to the fibres of Y over S.

LEMMA 2.8. — *Let* N *denote the Néron-Severi group of* X_0, *and let* $N' \subset N$ *be the subgroup of divisor classes orthogonal to the fibres* X_{0_y} *and section* θ_0. *The intersection pairing on* N′ *is compatible with cup product on* $H^1_{fl}(Y_0, A_n)$, i. e., *the diagram*

$$\begin{array}{ccc} N' \otimes N' & \to & \mathbf{Z} \\ \downarrow & & \downarrow \\ H^1 \otimes H^1 & \to & \mathbf{Z}/n = H^2_{fl}(Y_0, \mu_n) \end{array}$$

commutes.

Proof. — The autoduality map (2.6) gives rise to a Yoneda pairing

$$H^1(Y_0, A) \otimes H^1(Y_0, A) \to H^1(Y_0, G_m) = \mathbf{Z}. \tag{2.7}$$

Since the pairing on A_n is induced from autoduality, the pairings (2.7) and (2.9) are compatible (up to sign), and so it suffices to relate (2.9) to intersection theory on X_0, *via* the surjection $H^0(Y_0, A) \to N'$. This relation is clear from the definition of the autoduality pairing: If D is a divisor and $\bar{\alpha}$ is an idèle class on X, then an idèle class $\langle \alpha, D \rangle$ on Y is defined as $\alpha((D))$, where α is an idèle representing $\bar{\alpha}$ and with support disjoint from D. It is clear that if $\bar{\alpha}$ represents the divisor class of C, then the degree of $\langle \bar{a}, D \rangle$ on Y is (C.D). That is the required compatibility.

3. PROOFS OF LEMMAS (2.2), (2.3) AND (2.4).

Proof of Lemma 2.2. — Let R′ be a discrete valuation ring finite over R, with field of fractions K′, and let $Y_{R'} = \operatorname{Spec} \times_S Y$, etc. By an easy limit argument, it suffices to show the map $H^1(Y_{R'}, A_n) \to H^1(Y_{K'}, A_n)$ injective for every such R'. We may take the case that $R' = R$.

Let a class $\alpha \in H^1_{fl}(Y, A_n)$ be given, which is zero on the generic fibre Y_K. We represent it by a torsor T under the quasi-finite group scheme A_n. Then T has a section Γ over Y_K; let $\overline{\Gamma}$ be its Zariski closure in T. We claim that $\overline{\Gamma}$ is a section of T over Y. This will show that T is trivial, as required. Since Y is normal and A is quasi-finite, it follows from Zariski's Main Theorem that the map $\overline{\Gamma} \to Y$ is an open immersion. We have only to show that it is surjective.

The local structure of A_n is as follows: At any point $y \in Y$ whose fibre X_y is smooth, A_n is a finite group scheme. Hence A_n is finite over $U = Y - P$, and so $\overline{\Gamma}$ is, too, i. e., $\overline{\Gamma}$ covers U. It also covers the generic point of each P_v.

Let p_v be the closed point of P_v, and work locally at p_v for the étale topology. By Hensel's lemma, there is a finite subgroup $FA_n \subset A_n$ whose stalk at p_v is the same as that of A_n:

$$(3.1) \qquad 0 \to FA_n \to A_n \to G \to 0 \qquad \text{(locally at } p_v).$$

The sheaf G is a quasi-finite flat group scheme which is trivial at p_v. Hence its zero section is open and closed and so G is étale. Since its stalk at p_v is trivial, G has trivial cohomology on the henselization, and therefore any torsor T under A_n is induced from a canonical torser FT under TA_n,

$$T \cong FT \times_{FA_n} A_n.$$

Thus FA_n acts on T, and T/FA_n may be canonically identified with G. The inverse image in T of the zero-section of G is the finite subscheme $FT \subset T$.

Now the fibration X/Y has a node above both points of P_v, and so the restriction of A_n to p_v is a finite group scheme of constant rank n^{2g-1}. It follows that G is zero on P_v. Therefore the image of $\overline{\Gamma}$ in G must be zero. Thus $\overline{\Gamma}$ lies in $F\overline{\Gamma}$, and so $\overline{\Gamma}$ is finite over Y at p_v, i. e., it covers p_v.

Proof of 2.3. — We represent the cohomology classes by torsors. If A were finite over Y, we could apply Grothendieck's existence theorem [19] to get the surjectivity immediately. Since this is not assumed, we proceed by using suitable compactifications.

Let P be the locus of points of Y above which A is not finite. This is a closed set which decomposes by Hensel's lemma into closed sets P_v containing p_v. Each P_v is finite over S, and so is P.

If $\pi : W \to Y$ is a quasi-finite, separated map which is finite outside of P, it can be embedded as schematically dense open set into some space $\overline{W}$ finite over Y, which we call a *compactification* of W. The choice of $\overline{W}$ is not unique, but is given by any finite subalgebra $\mathcal{O}_{\overline{W}}$ of $\pi_* \mathcal{O}_W$ which is large enough. The set of such subalgebras is filtered by inclusion, its union being the integral closure of $\mathcal{O}_Y$ in $\pi_* \mathcal{O}_W$. It is clear that $\mathcal{O}_{\overline{W}}$ is determined locally for the étale topology, and hence by descent, that any collection of such subalgebras which is given locally in étale neighborhoods of the p_v defines a subalgebra globally.

The group law on A, and the action of A on a torsor T, will not usually extend to compactifications. But any map $W \to W'$ of quasi-finite schemes will extend to $\overline{W} \to \overline{W'}$ if $\overline{W}$ is chosen "big" enough. So, the group law will extend to a map

$$\overline{A \times A} \to \overline{A},$$

where the left term is some compactification of $A \times_Y A$, and so on.

Now let $\{ T_v \}$ be a family of torsors under A on $\{ Y_v \}$. To prove surjectivity we have to find a torsor T on Y inducing $\{ T_v \}$. Consider the problem locally along $P = \amalg P_j$. By Hensel's lemma, we have an exact sequence (3.1) with A_n replaced by A, and where again FA is finite flat, while G is an étale group scheme having stalk zero at each p_j. So, G is completely acyclic (locally), and hence every torsor T under A is induced in a canonical way from a torsor Z under FA. This means that our torsors T_v are induced locally by a compatible family Z_v of torsors under F.

Let $\hat{Y}$ denote the formal m-adic completion of the semi-local henselian scheme Spec $(\prod_j \tilde{\mathcal{O}}_{y, p_j})$, so that $\hat{Y}_v$ is the localization of Y_v at $p = \bigcup p_j$. Each torsor Z_v is finite and flat over $\hat{Y}_v$. So we obtain an F-torsor $\hat{Z}$ on $\hat{Y}$ as $\hat{Z} = \operatorname{Spec} (\varprojlim \mathcal{O}_{Z_v})$. Let $\hat{T}$ be the induced A-torsor. We choose compactifications extending the action on $\hat{T}$:

$$\overline{\hat{A}} \xleftarrow{\bar{p}_1} \overline{A \times \hat{T}} \overset{\bar{p}_2}{\underset{\bar{a}}{\rightrightarrows}} \overline{\hat{T}}. \tag{3.3}$$

This gives us local compactifications at p for each T_v, and hence global ones:

$$\overline{A} \xleftarrow{\bar{p}_1} \overline{A \times_y T} \overset{\bar{p}_2}{\underset{\bar{a}}{\rightrightarrows}} \overline{T}, \tag{3.4}$$

forming a compatible sequence.

Grothendieck's existence theorem [19] (Chap. III) applies to this formal diagram, and shows that (3.4) is induced from a diagram of schemes

$$\overline{U} \leftarrow \overline{V} \rightrightarrows \overline{W}$$

finite over Y. Since the left side $\overline{\hat{A}}$ of (3.3) was an arbitrary compactification, we may induce it from a chosen one $\overline{A}$ of A, and then $\overline{U} = \overline{A}$. Next, the open set $\hat{T} \subset \overline{\hat{T}}$ is the complement of a closed set C finite over P, and hence over S. Therefore it is induced by an isomorphic closed subscheme of $\overline{W}$, and we set $W = \overline{W} - C$. An open set $V \subset \overline{V}$ is defined similarly. Then $V_v \approx A_v \times W_v$ for each v, and also $\hat{V} \approx \hat{A} \times \hat{W}$. Therefore $V \approx A \times W$ by flat descent [16]. This defines an action of A on W, which makes W into the required A-torsor.

It remains to prove injectivity of the map of (3.2) when f is flat. Let T be an A-torsor on Y. Then we can view the functor $f_* \mathcal{H}om_Y(., T)$ as a closed subfunctor of $\mathcal{H}ilb_S(T)$ [16]. Hence it is represented over S by some algebraic space Z. Now since T is quasi-finite over Y, one sees easily that the fibres of Z over S are all of finite type.

Therefore Z is of finite type. Suppose T_v trivial for every v. The trivializations are given by points of Z with values in S_v. By [2], the existence of these points for all v implies the existence of a section over S. Thus T is trivial.

We now proceed with the proof of Lemma (2.4). Since the fibres of X/Y have ordinary double points at most, the jacobian family A is divisible. Therefore (1.7) we may work with $H^1_{ét}(Y, A[n])$ instead of $H^1_{fl}(Y, A_n)$. By construction,

$$\operatorname{Pic} X/S \approx \operatorname{Pic} Y/S \otimes \pi_* \operatorname{Pic} X/Y \approx \mathbf{Z}^2 + \pi_* A.$$

Since $\operatorname{Pic}^\tau X/S$ is smooth, $\pi_* A$ is smooth along every torsion section. Therefore $R^1 \pi^* A[\infty]$ is pro-represented by $R^1 \pi_* A$, which is immediately seen to be $\hat{B}r\, X/S = \hat{B}$. By assumption, $\hat{B}_0$ is a p-divisible group. So $\hat{B}$ is, too.

Let $\alpha_0 \in H^1(Y_0, A[\infty])$ be a section of order n. Let $\hat{Z}$ denote the hull of $R^1 \pi_* A[\infty]$ at α_0 [30]. This is a formally smooth formal scheme which is a torsor under $\hat{B}$. The torsor can be trivialized by a choice of section of $\hat{Z}/S$.

Multiplication by n in $R^1 \pi_* A[\infty]$ yields a map $\hat{Z} \xrightarrow{\varphi} \hat{Z}^{(n)}$, where $\hat{Z}^{(n)}$ the denotes hull of $R^1 \pi^* A[\infty]$ at the point $n\,\alpha_0$, which is canonically isomorphic to the n-fold torsor under $\hat{B}$. Since $n\,\alpha_0 = 0$, $\hat{Z}^{(n)} \approx \hat{B}$ canonically, and the inverse image $\varphi^{-1}(\theta)$ of the zero section θ in $\hat{B}$ is the locus of formal deformations of α_0 of order n.

Now since $\hat{Z}$ is a trivial torsor the map φ is isomorphic to multiplication by n in $\hat{B}$, which is an isogeny. So $\varphi^{-1}(\theta)$ is a finite flat covering of $\theta \approx S$. Therefore there is a finite extension $S' \to S$ and a point Γ of $\hat{Z}$ with values in S', which maps to zero in $\hat{B}$, *via* φ. This point gives a formal lifting $\hat{\alpha} = \{\alpha_v\}$ of α_0 to $Y' = S' \times_S Y$, of order n. We now check that this formal lifting is represented by a class of order n in $H^1(Y', A[\infty])$.

Consider the exact sequence

$$(3.5) \qquad 0 \to \pi_* A[\infty]/n \to R^1 \pi_* A[n] \to (R^1 \pi_* A[\infty])_n \to 0.$$

Since S'_v is strictly local, $H^0(S'_v, R^1 \pi_* A[n]) = H^1(Y'_v, A[n])$. We know the structure of $\pi_* A[\infty]$; it is the torsion subsheaf of $\operatorname{Pic}^\tau X/S$, and is formally smooth. The sheaf $\pi_* A[n]$ is represented by the scheme $(\operatorname{Pic}^\tau(X/S))_n$, and so its sections over $\{S'_v\}$ satisfy the Mittag-Leffler condition. The exact sequence

$$(3.6) \qquad 0 \to \pi_* A[n] \to \pi_* A[\infty] \to \pi_* A[\infty] \to \pi_* A[\infty]/n \to 0,$$

now shows that $\pi^* A[\infty]/n$ is formally smooth; moreover the inverse system formed by the sections of (3.6) over $\{S'_v\}$ has an exact limit.

Let us call $\varepsilon = \varepsilon(\mathscr{F})$ the map

$$H^0(S', \mathscr{F}) \xrightarrow{\varepsilon} \varprojlim_v H^0(S'_v, \mathscr{F}).$$

Then ε is a bijection for the first three terms of (3.6), hence for the last one, which is the left-hand term of (3.5). Proposition (3.2) and (1.7) imply that ε is bijective for the middle term of (3.5). Thus it is bijective for the right-hand term too, as was claimed. We have therefore.

LEMMA 3.7. — *Given a class* $\alpha_0 \in H^1(Y_0, A[\infty])$ *of order n, there is a finite ramified cover* $S' \to S$ *so that* α_0 *extends to a class* α *of order n in* $H^1(Y', A[\infty])$. *The set of such* α *is a torsor under the group* $(\hat{B}(S'))_n$ *of sections of order n of* $\hat{B}$ *over* S'.

Passing to the limit over S' gives us an exact sequence

(3.8) $$0 \to (\bar{B})_n \to H^1(\bar{Y}, A[\infty])_n \to H^1(Y_0, A[\infty])_n \to 0,$$

where $\bar{B} = \hat{B}(\bar{S})$. Now it is easy to identify the group

$$H^0(\bar{Y}, A[\infty])/n = H^0(\bar{S}, \pi_* A[\infty]/n).$$

For, the group of torsion sections of $\mathrm{Pic}^0 X/S$ over $\bar{S}$ is divisible, and so the group in question is just the group of section of $T_n = (\mathrm{Pic}^\tau X/S)/n\,(\mathrm{Pic}^\tau X/S)$. This is a finite smooth group scheme. So

$$H^0(\bar{S}, \pi_* A[\infty]/n = T(\bar{S}) = T(S_0) = H^0(S_0, \pi_* A[\infty]/n).$$

Lemma (2.4) now follows from the sequences (3.5) and (3.8).

4. ANALYSABILITY BY p-DIVISIBLE GROUPS. — Let $K/\mathbf{Q}_p$ be a finite field extension, $R \subset K$ its ring of integers, and $k = R/\mathbf{m}$ its residue field. Let X/K be a proper smooth surface and $H = TH^2_{\text{ét}}(X/k', \boldsymbol{\mu}_{p^\infty}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ the $\mathrm{Gal}(\bar{K}/K)$ module associated to its 2-dimensional étale cohomology. The cup-product pairing induces an autoduality (,) on the $\mathbf{Q}_p$-vector space H with respect to which we have the following compatibility formula for the action of $\mathrm{Gal}(\bar{K}/K)$: $(gx, gy) = (x, y)$.

DÉFINITION. — We shall say that the $\mathrm{Gal}(\bar{K}/K)$ representation H is *analysable by p-divisible groups* if there is a filtration

(★) $$0 \subset V \subset W \subset H$$

stable under the action of $\mathrm{Gal}(\bar{K}/K)$ such that:

(*a*) W is a $\mathrm{Gal}(\bar{K}/K)$ representation "coming from a p-divisible group" $\mathbf{W}$ over R in the sense that $W \cong T\mathbf{W} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$;

(*b*) V is the $\mathrm{Gal}(\bar{K}/K)$ representation coming from the connected part $\mathbf{W}^0$ of the p-divisible group $\mathbf{W}/R$ $(V = T\mathbf{W}^0 \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$;

(*c*) the filtration (★) is self-dual with respect to the auto-duality of H. This means that $W^\perp = V$, and $V^\perp = W$.

Remarks:

1. If such a filtration (★) satisfying (*a*), (*b*) and (*c*) exists, then it is unique. This may be seen using the following fact: If U is a $\mathrm{Gal}(\bar{K}/K)$ representation which comes from a p-divisible over R, and its dual U^* also comes from a p-divisible group over R, then U is an unramified $\mathrm{Gal}(\bar{K}/K)$ representation.

Also, by Tate's theorem [34], V and W are unique.

We may identify W and V as the largest "p-divisible subrepresentation", and "connected p-divisible subrepresentation" in H, in the following sense. If $W' \subset H$ is a $\mathbf{Q}_p$-sub-stable under Gal $(\bar{K}/K)$ which comes from a p-divisible group over R, then $W' \subset W$; if $V' \subset H$ is a $\mathbf{Q}_p$-subspace stable under Gal (K/K) which comes from a connected p-divisible group over R, then $V' \subset V$.

2. Given a filtration ($\bigstar$) of H which satisfies (a) and (b) above, then it satisfies (c) if (and only if) $\dim H = \dim V + \dim W$. For, using the fact quoted in remark 1 above, an (a), (b) one has that V and W are orthogonal with respect to the auto-duality of H. Thus $V \subset W^{\perp}$ and $W \subset V^{\perp}$, and to check that the inclusions are equalities one is reduced to counting dimensions.

3. If H is analysable by p-divisible groups, then its (Gal $(\bar{K}/K)$) semi-simplification possesses a Hodge-Tate decomposition ([32]), [34]). For since W comes from a p-divisible group, it possesses a Hodge-Tate decomposition by Corollary 2, paragraph 4 of [34] and H/W is the dual of V and therefore also has one. If W^0 is a multiplicative type p-divisible group (i. e. the dual of an étale p-divisible group) then one can say more: H itself admits a Hodge-Tate decomposition. This follows from a theorem of Tate (§ 3, Th. 2 [34]) which implies that any extension of Gal $(\bar{K}/K)$ modules

$$0 \to C(\chi) \to \mathscr{E} \to C(\chi') \to 0$$

splits, if $\chi \neq \chi'$ (notation as in [34]).

4. It is not true that H is analysable for all surfaces X/K, (even those admitting a good reduction in characteristic p). Consider an elliptic curve E/K with complex multiplication possessing good, supersingular reduction in characteristic p. We may suppose, further, that K contains the field of complex multiplication of E. Take $X = E \times E$. One has good control of the Gal $(\bar{K}/K)$ representation on H, using the theory of complex multiplication. In particular, one can find a $\mathbf{Q}_p$-subspace of H of dimension 2, irreducible under the action of Gal $(\bar{K}/K)$, such that neither it, nor its dual comes from a p-divisible group over R. Thus H is not "analysable by p-divisible groups".

THEOREM 4.1. — *Suppose $p > 2$. Let* X/R *be a proper smooth surface satisfying hypothesis* A (Pic$^\tau$ (X/R) *is smooth, and* $\widehat{Br}$ *is of finite height*). *Then its* 2-*dimensional cohomology* H *is analysable.*

Proof. — We use (1.8) and (2.1). Define the filtration ($\bigstar$) by setting W = image of of $TH^2_{fl}(X_{\bar{S}}, \boldsymbol{\mu}_{p^\infty}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ in H under the natural mapping [which is injective by (2.1) (b)]. By (2.1) (a) W is a Gal $(\bar{K}/K)$ subrepresentation coming from the p-divisible group Ψ. The subspace $V \subset W$ is then the subrepresentation associated to the connected part of Ψ which is $\widehat{Br}$ (1.8). By remark 2, our theorem would follow from the equality $b_2 = \text{height}(\widehat{Br}) + \text{height}(\Psi)$. But,

(i) $\quad \text{height}(\widehat{Br}) = \text{rank}_{W(k)} H^2_{crys}(X_0/W(k))_{[0,1)} = \text{rank}_{W(k)} H^2_{crys}(X_0/W(k))_{(1,2]}$

using (1.*c*). The second equality above comes from Poincaré duality for crystalline cohomology [5] (Chap. VII).

(ii) $$\text{height}(\Psi) = \text{rank}_{W(k)} H^2_{\text{crys}}(X_0/W(k))_{[0,1]}$$

using (1.9).

Therefore

$$\text{height}(\widehat{\text{Br}}) + \text{height}(\Psi) = \text{rank}_{W(k)} H^2_{\text{crys}}(X_0/W(k))$$

and our equality follows from the fact that the étale and the crystalline (second) Betti numbers agree [23] ([5], VII, § 3).

As mentioned in the introduction, we may take as examples of application of our theorem a K 3 surface over R admitting good, nonsupersingular reduction to the residue field k, supposed of odd characteristic or any proper surface over R whose reduction to k is the Fermat surface of degree d (where $d \equiv 1 \bmod p = \text{char}\, k$). In the latter case the formal Brauer group is of multiplicative type, and therefore the H associated to that surface admits a Hodge-Tate decomposition.

Note also that if X/R is a proper smooth surface satisfying hypothesis A, by virtue of the embedding of the Néron-Severi group of X_0 in $TH^2_{fl}(X_0, \mu_{p^\infty})$ one has that ρ, the rank of the Néron-Severi group is majorized by the height of $\Psi^{\text{ét}}$. If the characteristic of k is odd one then immediately deduces the inequality: $\rho \leqq b_2 - 2h$.

In characteristic 2 one may conclude the same inequality if one imposes the appropriate hypothesis so as to be able to apply (3.3).

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