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ON THE INTERSECTION MATRIX

OF A HYPERSURFACE

By Nicholas M. KATZ.

INTRODUCTION. — Let (R, m) be a complete discrete valuation ring whose residue field has q elements, and whose fraction field K has characteristic zero. Let X be a projective and smooth hypersurface of dimension n and degree d over R, in " general position " (*see* section 1 for a precise definition). Dwork [3] has constructed a finite-dimensional K-vector spece W^s, and an endomorphism α of W^s, which determines the zeta function [10] of the special fibre $\overline{X}$ as follows :

$$(0.1) \qquad Z_{\overline{X}}(t) = \frac{\det\left(1 - t\frac{\alpha}{q} \;\middle|\; W^s\right)^{(-1)^{n+1}}}{\prod_{i=0}^{n}(1 - q^i t)}.$$

The verity of the then-conjectural functional equation [10] for $Z_{\overline{X}}$,

$$(0.2) \qquad Z_{\overline{X}}\left(\frac{1}{q^n t}\right) = \pm t^{\chi} q^{\frac{n\chi}{2}} Z_{\overline{X}}(t)$$

[here χ is minus the degree of $Z_{\overline{X}}(t)$ as rational function] is equivalent, by (0.1), to the fact that the eigenvalues of $\frac{\alpha}{q}$ are mapped bijectively to themselves by $x \mapsto q^n/x$. In order to verify this, Dwork constructed a non-degenerate bilinear pairing

$$W^s \times W^s \to K$$

with respect to which $q^{-\frac{n}{2}-1} \alpha$ was an isometry ([4], p. 266 and th. 9.2).

The space W^s, together with its pairing, is of a purely algebraic nature, and may be constructed for every hypersurface X, projective, smooth

74.

and in general position over an arbitrary field K of characteristic zero. There is an isomorphism, Θ [7], between W^S and $\mathrm{Prim}^n(X/K)$, the subspace of $H^n_{DR}(X/K)$ consisting of primitive (in the sense of Hodge theory) classes. W. Cassellman conjectured that Dwork's pairing should correspond via Θ to the cup-product pairing.

Oun purpose is to prove that this is essentially the case, i. e., that there exists a non-zero rational number, c, depending only on the dimension, n, and the degree, d, of the hypersurface X, such that the following diagram is commutative :

$$
\begin{array}{ccc}
W^S \times W^S & \xrightarrow{\;c \times (\text{Dwork's pairing})\;} & K \\
{\scriptstyle \Theta \times \Theta}\Big\downarrow & & \wr\Big\downarrow \\
\mathrm{Prim}^n(X/K) \times \mathrm{Prim}^n(X/K) & \xrightarrow{\;\text{cup-product}\;} & H^{2n}_{DR}(X/K)
\end{array}
$$

(The commutativity of this diagram is the only known property of the constant c.)

The basic idea is this : the two constructions, W^S and Prim^n, globalize to give locally free sheaves on T, the classifying space for nonsingular hypersurfaces of dimension n and degree d in general position. These sheaves are both endowed with integrable connections. The isomorphism Θ globalizes to an isomorphism of these sheaves, which is compatible with the connections. The desired theorem is then the commutativity of the following diagram of sheaves on T :

$$
\begin{array}{ccc}
W^S \times W^S & \xrightarrow{\;c \times (\text{Dwork's pairing})\;} & \mathcal{O}_T \\
{\scriptstyle \Theta \times \Theta}\Big\downarrow & & \wr\Big\downarrow \\
\mathrm{Prim}^n(X/T) \times \mathrm{Prim}^n(X/T) & \xrightarrow{\;\text{cup-product}\;} & H^{2n}_{DR}(X/T)
\end{array}
$$

Two reductions are now possible. First, it suffices to prove that the above diagram commutes after the base-change which replaces T by the *formal* neighborhood of a point $s \in T$ (i. e. the completion of the local ring at s). Second, both pairings respect the connection (i. e. the " product formula " holds) in the formal neighborhood of the point in T which corresponds to the diagonal hypersurface $\sum_{i=1}^{n+2} X_i^d = 0$; because the connection is *integrable*, both pairings are then determined by their *value* at this point.

As explicit formulas for Dwork's pairing at this point are available, the problem is to compute the intersection matrix of the hypersurface $\sum_{i=1}^{n+2} X_i^d = 0$, with respect to a *given* basis of its De Rham cohomology. The shape of this matrix is dictated by the presence of a large group of

automorphisms. We succeed in the *precise* determination only by recourse to a deformation-theoretic method, which consists in differentiating (with respect to the parameters) certain relations in the cohomology ring of the general hypersurface.

1. SOME DE RHAM COHOMOLOGY. — Fix positive integers n and d, and an ordered set of homogeneous coordinates $X_1, \ldots, X_{n+2}$ for the projective space $\mathbf{P} = \mathbf{P}^{n+1}$. Let S be an arbitrary scheme of characteristic zero (i. e. over $\mathbf{Q}$). Let X be a nonsingular hypersurface of dimension n and degree d, over S, in general position with respect to the coordinates axes. This means X is defined by the vanishing of a homogeneous form $F(X_1, \ldots, X_{n+2})$ of degree d, and that, denoting by $\mathcal{O}_S[X]$ the sheaf of graded algebras on S given by $\mathcal{O}_S[X_1, \ldots, X_{n+1}] = \bigoplus_{k \in \mathbf{Z}} \mathrm{pr}_{2*}(\mathcal{O}_{\mathbf{P} \times S}(k))$, the Koszul complex defined by the $X_i \dfrac{\partial F}{\partial X_i}$

(1.0) $\qquad o \leftarrow \mathcal{O}_S[X] \leftarrow \bigoplus_{1 \leq i \leq n+2} \mathcal{O}_S[X] \leftarrow \bigoplus_{1 \leq i \leq j \leq n+2} \mathcal{O}_S[X] \leftarrow$

is exact.

Let us explicitly write the defining equation $\sum_w A_w X^w = o$, the sum taken over the $\dbinom{n+d+1}{d}$ monomials X^w of degree d in $X_1, \ldots, X_{n+2}$. The coefficients A_w " are " a morphism from S to the projective space over $\mathbf{Q}$ of $\dbinom{n+d+1}{d} - 1$ dimensions. This projective space is thus the classifying space for (flat) families of hypersurfaces of degree d in the " coordinatized " projective space $\mathbf{P}^{n+1}$ over $\mathbf{Q}$. It is easily seen (theory of the resultant) that the classifying space for those families which are nonsingular and in general position is a non-empty Zariski open-subset T of $\mathbf{P}^\star_\mathbf{Q}$, $\star = \dbinom{n+d+1}{d} - 1$.

Over T we dispose of the universal non-singular hypersurface in general position, $X \rightarrow T$, its complement $\mathbf{P}_T - X \rightarrow T$, and the constant projective space $\mathbf{P}_T \rightarrow T$.

Recall that for any smooth morphism $f : A \rightarrow B$, the relative De Rham cohomology sheaves on B, written simply $H^q_{DR}(A/B)$, are, by definition, the $\mathbf{R}^q f_*(\Omega^\cdot_{A/B})$, and that [6], when B is (the spectrum of) the complex numbers $\mathbf{C}$, $H^\cdot_{DR}(A/B) \simeq H^\cdot(A^{an}, \mathbf{C})$, the second member being the " ordinary " cohomology groups of the assosciated complex manifold A^{an}. Recall also that the sheaf $H^q_{DR}(A/B)$ is said to commute with base change if, for any morphism $g : S \rightarrow B$, the homomorphism

$$g^\star H^q_{DR}(A/B) \rightarrow H^q_{DR}(A \times_B S/S)$$

is an isomorphism.

PROPOSITION 1. — *The sheaves* $H^i_{DR}(X/T)$ *and* $H^i_{DR}(\mathbf{P}_T - X/T)$ *are locally free, and commute with base change.*

Proof. — We introduce the *coherent* complex $\Omega^{\cdot}_{\mathbf{P}/T}(\log X)$ on $\mathbf{P}_T$, whose local sections are the differentials of the form $\omega + \tau \wedge \dfrac{df}{f}$, where ω and τ are local sections of $\Omega^{\cdot}_{\mathbf{P}/T}$ and f is a local equation for X (we write $\mathbf{P}/T$ for $\mathbf{P}_T/T$). Equivalently, $\Omega^{\cdot}_{\mathbf{P}/T}(\log X)$ may be described as the tensor product

$$l(X)^{-1} \otimes_{\mathcal{O}_{\mathbf{P}}} (\text{kernel of } \Omega^{\cdot}_{\mathbf{P}/T} \to \Omega^{\cdot}_{X/T}),$$

where $l(X)$ is the sheaf of ideals defining X on $\mathbf{P}_T$. Because the inclusion $\Omega^{\cdot}_{\mathbf{P}/T}(\log X) \to \Omega^{\cdot}_{\mathbf{P}-X/T}$ is a quasi-isomorphism (i. e., induces an isomorphism of cohomology sheaves), the induced map of sheaves on T, $\mathbf{R}^i f_*(\Omega^{\cdot}_{\mathbf{P}/T}(\log X)) \to \mathbf{R}^i f_*(\Omega^{\cdot}_{\mathbf{P}-X/T})$ is an isomorphism. Thus from the exact sequence of complexes on $\mathbf{P}_T$,

(1.1) $$0 \to \Omega^{\cdot}_{\mathbf{P}/T} \to \Omega^{\cdot}_{\mathbf{P}/T}(\log X) \to \Omega^{\cdot-1}_{X/T} \to 0$$

$\Big($the last map, the residue, is defined locally by extracting the coefficient of $\dfrac{df}{f}$ and restricting to $X\Big)$ we obtain the long cohomology sequence

(1.2) $$\to H^{i-2}_{DR}(X/T) \to H^i_{DR}(\mathbf{P}/T) \to H^i_{DR}(\mathbf{P}-X/T) \to H^{i-1}_{DR}(X/T) \to$$

We will apply the base-changing theorems of ([2], § 7), this being permissible because we are dealing with T-flat coherent complexes, T-linear differentials, and projective morphisms. Let $s \in T$ be any point, $K(s)$ its residue field, $\mathbf{P}_s$ and X_s the corresponding fibres, which are varieties over $K(s)$. We have a corresponding exact sequence of $K(s)$-vector space (by repeating the above construction over $K(s)$)

$$\to H^{i-2}_{DR}(X_s) \to H^i_{DR}(\mathbf{P}_s) \to H^i_{DR}(\mathbf{P}_s - X_s) \to H^{i-1}_{DR}(X_s) \to$$

and by the Lefschetz theorems [8], $H^i_{DR}(\mathbf{P}_s - X_s)$ is zero unless $i = 0$ or $i = n+1$. Thus ([2], 7.8.3 and 7.8.4) the sheaves $H^i_{DR}(\mathbf{P}-X/T)$ are locally free and commute with base change, because these sheaves vanish save for $i = 0$ and $i = n+1$. Next we remark that each

$$H^i_{DR}(\mathbf{P}/T) = H^i_{DR}(\mathbf{P}_{\mathbf{Q}} \times_{\mathbf{Q}} T) \simeq H^i_{DR}(\mathbf{P}_{\mathbf{Q}}/\mathbf{Q}) \otimes_{\mathbf{Q}} \mathcal{O}_T$$

is certainly locally free and commutes with base change, and thus the map $H^i_{DR}(\mathbf{P}/T) \to H^i_{DR}(\mathbf{P}-X/T)$ is zero for $i \geq 1$, this being so at each $K(s)$.

Referring to the exact sequence (1.2), we then see that

$$H^i_{DR}(X/T) \overset{\sim}{\longrightarrow} H^{i+2}_{DR}(\mathbf{P}/T) \qquad \text{for } i \neq n.$$

Hence, by ([2], 7.8.3), $H_{DR}^n(X/T)$ is also locally free, and commutes with base change.

<div align="right">Q. E. D.</div>

Recall now that a local section of $H_{DR}^n(X/T)$ is called *primitive* [8] if its cup-product with the cohomology class of a hyperplane section is zero.

COROLLARY. — *The residue mapping* $H_{DR}^{n+1}(\mathbf{P} - X/T) \to H_{DR}^n(X/T)$ *establishes an isomorphism between* $H_{DR}^{n+1}(\mathbf{P} - X/T)$ *and* $\mathrm{Prim}^n(X/T)$, *the sheaf of primitive cohomology classes in* $H_{DR}^n(X/T)$. *In particular,* $\mathrm{Prim}^n(X/T)$ *is locally free and commutes with base change.*

Proof. — From the exact sequence, we have

$$0 \to H_{DR}^{n+1}(\mathbf{P} - X/T) \to H_{DR}^n(X/T) \to H_{DR}^{n+2}(\mathbf{P}/T) \to 0.$$

Since the restriction mapping $H_{DR}^{n+2}(\mathbf{P}/T) \to H_{DR}^{n+2}(X/T)$ is an isomorphism, we may rewrite the above sequence

$$0 \to H_{DR}^{n+1}(\mathbf{P} - X/T) \to H_{DR}^n(X/T) \to H_{DR}^{n+2}(X/T) \to 0,$$

where the last map is given by cup-product with the " class of a hyperplane section ", whence the corollary, by definition of primitive.

<div align="right">Q. E. D.</div>

We conclude this section by recalling that all the De Rham sheaves are equipped with an integrable connection, that of Gauss-Manin [9], with which all the maps in (1.2) are compatible. The compatibility is most easily seen as follows. For any smooth $g : Y \to T$, the Gauss-Manin connection is obtained as the differential d_1 in the spectral sequence of the filtered object $\Omega_{Y/\mathbf{Q}}^\cdot$ with respect to the functor $\mathbf{R}^\cdot g_*$, the filtration being defined by

$$F^i(\Omega_{Y/\mathbf{Q}}^\cdot) = \mathrm{image}\,(\Omega_{Y/\mathbf{Q}}^{\cdot-i} \otimes g^*(\Omega_{T/\mathbf{Q}}^i) \to \Omega_{Y/\mathbf{Q}}^\cdot).$$

These considerations apply *mutatis mutandis* to the complex $\Omega_{\mathbf{P} \times T/\mathbf{Q}}^\cdot(\log X)$, and provide the $\mathbf{R}^q f_*(\Omega_{\mathbf{P}/T}^\cdot(\log X))$ with an integrable connection. The only point to be checked is that, under the filtration

$$F^i(\Omega_{\mathbf{P} \times T/\mathbf{Q}}^\cdot(\log X)) = \mathrm{image}\,\Omega_{\mathbf{P} \times T/\mathbf{Q}}^{\cdot-i}(\log X) \otimes g^*(\Omega_{T/\mathbf{Q}}^i) \to \Omega_{\mathbf{P} \times T/\mathbf{Q}}^\cdot(\log X)$$

the associated graded objects are given by

$$\mathrm{gr}^i(\Omega_{\mathbf{P} \times T/\mathbf{Q}}^\cdot(\log X)) = F^i/F^{i+1} = \Omega_{\mathbf{P}/T}^{\cdot-i}(\log X) \otimes g^*(\Omega_{T/\mathbf{Q}}^i).$$

As the quasi-isomorphism $\Omega_{\mathbf{P}/T}^\cdot(\log X) \to \Omega_{\mathbf{P}-X/T}^\cdot$ is the gr^0 of the map $\Omega_{\mathbf{P} \times T/\mathbf{Q}}^\cdot(\log X) \to \Omega_{\mathbf{P} \times T-X/\mathbf{Q}}^\cdot$, the connection on $\mathbf{R}^q f_*(\Omega_{\mathbf{P}/T}^\cdot(\log X))$ corresponds, via the isomorphism $\mathbf{R}^q f_*(\Omega_{\mathbf{P}/T}^\cdot(\log X)) \to H_{DR}^q(\mathbf{P} - X/T)$, to the

Gauss-Manin connection on $H_{DR}^q(\mathbf{P} - X/T)$. Finally, the exact sequence (1.1) is the gr^0 of the exact sequence of filtered (as above) complexes

$$0 \to \Omega_{\mathbf{P} \times T/\mathbf{Q}}^{\cdot} \to \Omega_{\mathbf{P} \times T/\mathbf{Q}}^{\cdot}(\log X) \to \Omega_{X/\mathbf{Q}}^{\cdot -1} \to 0,$$

whence the compatibility of the maps in (1.2) with the connections.

2. The constructions of Dwork. — We begin by introducing certain modules over T.

$\mathcal{L}$ the free module with base those monomials $X_1^{w_1} \ldots X_{n+2}^{w_{n+2}}$, or simply X^{w}, with all $w_i \geq 0$, and $\sum_{i=1}^{n+2} w_i$ divisible by d. One defines $w_0 = \frac{1}{d} \sum_{i=1}^{n+2} w_i$.

$\mathcal{L}^S$, the submodule with base those of the above monomials which in addition have all $w_i \geq 1$.

$$W^S = \mathcal{L}^S / \mathcal{L}^S \cap \left(\sum_{i=1}^{n+2} D_i \mathcal{L} \right), \text{ where } D_i \text{ is the T-linear mapping}$$

$$D_i : \quad \mathcal{L} \to \mathcal{L}, \quad D_i(X^{w}) = w_i X^{w} + X_i \frac{\partial F}{\partial X_i} X^{w}.$$

Proposition 2. — *The module* W^S *is locally free, and commutes with base change.*

Proof. — By the exactness of the Koszul complex for X *and* its intersections with coordinate hyperplanes, we have

$$\mathcal{L}^S \cap \sum D_i \mathcal{L} = \sum D_i \mathcal{L}^{S-i},$$

where $\mathcal{L}^{S-i}$ is the span of those X^{w} having $w_j > 0$, $\forall j \neq i$. Thus W^S as the cokernel of a map between free modules, commutes with base change. By ([3]. lemma 3.19), for each point $s \in T$, the corresponding vectorspace $W_{K(s)}^S$ over the residue field $K(s)$ of s has constant dimension $d^{-1}[(d-1)^{n+2} + (-1)^n(d-1)]$, whence W^S is locally free.

Q. E. D.

We recall that W^S is provided with an integrable connection [9], deduced by passage to quotients from an integrable connection on $\mathcal{L}$. It is defined by having $D \in \mathrm{Der}_{\mathbf{Q}}(\mathcal{O}_T, \mathcal{O}_T)$ act through $\sigma_D : \mathcal{L} \to \mathcal{L}$ by

$$\sigma_D(t X^{w}) = D(t) X^{w} + F^D X^{w},$$

where $t \in \mathcal{O}_T$, and F^D denotes the result of applying D to the coefficients of F.

3. The comparison theorem. — There is a (T-linear) map

$$\mathcal{R} : \quad \mathcal{L}^S \to H_{DR}^{n+1}(\mathbf{P} - X/T)$$

given by

$$\mathcal{R}: \quad X^{w} \to (-1)^{w_0-1} (w_0-1)! \frac{X^{w}}{F^{w_0}} \frac{d(X_1/X_{n+2})}{X_1/X_{n+2}} \wedge \cdots \wedge \frac{d(X_{n+1}/X_{n+2})}{X_{n+1}/X_{n+2}}.$$

It is quickly verified that, because all $w_i \geq 1$, the above differential form is regular on all of $\mathbf{P} - X$, and thus defines a section of $H_{DR}^{n+1}(\mathbf{P} - X/T)$.

The map $\Theta : \mathcal{L}^S \to \mathrm{Prim}^n(X/T)$ is defined to be the composition $\mathcal{L}^S \xrightarrow{\mathcal{R}} H_{DR}^{n+1}(\mathbf{P} - X/T) \xrightarrow{\sim} \mathrm{Prim}^n(X/T)$.

THEOREM 3. — *The maps $\mathcal{R}$ and Θ induce, by passage to quotients, iso-morphisms*

$$\mathcal{R}: \quad W^S \xrightarrow{\sim} H_{DR}^{n+1}(\mathbf{P} - X/T),$$
$$\Theta: \quad W^S \xrightarrow{\sim} \mathrm{Prim}^n(X/T),$$

which are compatible with the connections.

Proof. — We first show that $\mathcal{L}^S \cap \sum D_i \mathcal{L}$ lies in the kernel of $\Theta : \mathcal{L}^S \to \mathrm{Prim}^n(X/T)$. As $\mathrm{Prim}^n(X/T)$ is locally free and commutes with base change, it suffices to show that for every $s \in T$, the corresponding map $\Theta_s : \mathcal{L}^S \to \mathrm{Prim}^n(X_s)$ of $K(s)$ vector spaces annihilates $\mathcal{L}^S \cap \sum D_i \mathcal{L}$. By ([7], th. 1.17), the composition

$$\mathcal{L}^S \xrightarrow{\Theta_s} \mathrm{Prim}^n(X_s) \xrightarrow{\text{restriction}} H_{DR}^n(X_s^{\emptyset})$$

(here $X_s^{\emptyset}$ is the open subset of X_s where all coordinates X_i are invertible) annihilates $\mathcal{L}^S \cap \sum D_i \mathcal{L}$. By ([7], cor. 1.11), temporarily writing $X_s^{(1)}$ for the open subset of X_s where the first coordinate X_1 is invertible, the restriction map

$$H^n(X_s^{(1)}) \to H^n(X_n^{\emptyset})$$

is injective. And

$$H^n(X_s) = \mathrm{Prim}^n(X_s) \oplus \text{Kernel of } (H^n(X_s) \to H^n(X_s^{(1)})),$$

whence $\mathrm{Prim}^n(X_s) \to H_{DR}^n(X_s^{\emptyset})$ is injective.

To show that $\Theta : W^S \to \mathrm{Prim}^n(X/T)$ is an isomorphism, it suffices, since both W^S and $\mathrm{Prim}^n(X/T)$ are locally free, to show it over the residue field of each $s \in T$. This is so in virtue of ([7], th. 1.17). That $\mathcal{R}$ (and hence Θ) transforms σ into the Gauss-Manin connection is seen directly

$$\mathcal{R}(\sigma_D(X^w)) = \mathcal{R}(F^D X^w) = (-1)^{w_0} w_0! \frac{F^D X^w}{F^{1+w_0}} \frac{d(X_1/X_{n+2})}{X_1/X_{n+2}} \wedge \cdots \wedge \frac{d(X_{n+1}/X_{n+2})}{X_{n+1}/X_{n+2}}$$

while

$$D(\mathcal{R}(X^w)) = D\left\{ (-1)^{w_0-1} (w_0-1)! \frac{X^w}{F^{w_0}} \frac{d(X_1/X_{n+2})}{X_1/X_{n+2}} \wedge \cdots \wedge \frac{d(X_{n+1}/X_{n+2})}{X_{n+1}/X_{n+2}} \right\}$$

and, as D annihilates all X_i/X_{n+2}, $i = 1, \ldots, n+1$, this *is*

$$(-1)^{w_0} w_0! \frac{F^D X^w}{F^{w_0+1}} \frac{d(X_1/X_{n+1})}{X_1/X_{n+1}} \wedge \cdots \wedge \frac{d(X_{n+1}/X_{n+2})}{X_{n+1}/X_{n+2}}.$$

<div align="right">Q. E. D.</div>

4. The Intersection Matrix. — Any T-bilinear pairing

$$\mathrm{Prim}^n(X/T) \times \mathrm{Prim}^n(X/T) \to H^{2n}_{DR}(X/T) \simeq \mathcal{O}_T$$

may be thought of as a global section of the locally free sheaf

$$Hom_{\mathcal{O}_T}\left(\mathrm{Prim}^n(X/T) \otimes_{\mathcal{O}_T} \mathrm{Prim}^n(X/T), \mathcal{O}_T\right).$$

Because T is (reduced and) irreducible, for any non-void open subsets $U \subset V$ of T, and any locally free sheaf F on T, the restriction mapping on sections $\Gamma(V, F) \to \Gamma(U, F)$ is injective. Fixing a point $s \in T$, we have $\Gamma(V, F) \subset \varinjlim_{U \in s} \Gamma(U, F) = F_s$, the stalk at s. Since the local ring $\mathcal{O}_{T,s}$ at s is noetherian, it follows by Krull's Intersection Theorem that $\mathcal{O}_{T,s}$ is a subring of its completion $\hat{\mathcal{O}}_{T,s}$; F_s being free over $\mathcal{O}_{T,s}$, we have $F_s \subset \hat{F}_s = F_s \otimes_{\mathcal{O}_{T,s}} \hat{\mathcal{O}}_{T,s}$. Denote by $\hat{X}_s$ the fibre product $X \times_T \hat{\mathcal{O}}_{T,s}$, which may be thought of as a tubular neighborhood of the special fibre $X_s = X \times_T K(s)$.

PROPOSITION 4. — *Let $s \in T$ be a closed point which is* Q-*rational. Then the $\hat{\mathcal{O}}_{T,s}$-module* $\mathrm{Prim}^n(\hat{X}_s/\hat{\mathcal{O}}_{T,s})$ *admits a basis of horizontal (for the Gauss-Manin connection) elements.*

Proof. — The hypersurface $\hat{X}_s$ is given by an equation

$$X_1^d + \sum_w (a_w + A_w) X^w = 0,$$

where the sum is over all monomials X^w of degree d other than X_a^d, and the A_w are independent variables, with $\hat{\mathcal{O}}_{T,s} = \mathbf{Q}[[A_w]]$. The hypersurface X_s is given by $X_1^d + \sum a_w X^w = 0$.

[We have used the fact that all the monomials X_i^d necessarily occur in the equation of an X in general position.] Let $\{\eta_i\}$ be a basis of $\mathrm{Prim}^n(\hat{X}_s/\hat{\mathcal{O}}_{T,s})$. Each $\frac{\partial}{\partial A_w}$ acts; say $\frac{\partial \eta_i}{\partial A_w} = \sum_j B_{i,j}(w) \eta_j$. If it possible to find a basis of horizontal sections τ_i, which agree with the η_i over the residue field of s, they will necessarily be linear combinations of the η_i,

$$\tau_i = \sum_j u_{ij} \eta_j, \qquad u_{ij} \in \mathbf{Q}[[A_w]],$$

where the matrix $\mathrm{U} = (u_{ij})$ satisfies the equation

$$\begin{cases} \dfrac{\partial \mathrm{U}}{\partial \mathrm{A}_w} = -\,\mathrm{UB}\,(w) & \text{for each } \mathrm{A}_w; \quad \text{here} \quad \mathrm{B}\,(w) = (\mathrm{B}_{i,j}\,(w)), \\[2mm] \mathrm{U}\,(\mathrm{o}) = \mathrm{I}. \end{cases}$$

That this system has a unique solution results from the formal version of the Frobenius theorem on " complete integrability ", applicable precisely because the Gauss-Manin connection is integrable ([1], p. 3o4).

<div align="right">Q. E. D.</div>

Remark. — The inverse P of the matrix U, by means of which the η_i may be expressed in terms of the horizontal basis, is nothing other than the classical period matrix. Let us explain why this is so. Consider a proper and smooth (holomorphic) map $f : \mathrm{Y} \to \Delta$, where Δ is the unit disc $|t| < \mathrm{I}$. Replacing Δ if necessary by a smaller disc around o, we may suppose that $f : \mathrm{Y} \to \Delta$ is a differentiably trivial fibre bundle. Choose a C^∞ trivialization $\gamma : \mathrm{Y} \overset{\sim}{\to} \mathrm{Y}_0 \times \Delta$ [here $\mathrm{Y}_0 = f^{-1}(\mathrm{o})$]. Let $\{\eta_i(t)\}$ be a basis of $\mathrm{H}_{\mathrm{DR}}^{\mathrm{anal}}(\mathrm{Y}/\Delta)$. The horizontal basis $\{\tau_i(t)\}$ is obtained as follows; for each $t \in \Delta$, γ induces $\gamma_t : \mathrm{Y}_t \overset{\sim}{\to} \mathrm{Y}_0$, and we take $\tau_i(t) = \gamma_t^\star(\eta_i(\mathrm{o}))$. Let $\alpha_i(\mathrm{o})$ be the basis of $\mathrm{H}_\cdot(\mathrm{Y}_0, \mathbf{C})$ dual to the $\eta_i(\mathrm{o})$, and define a basis $\alpha_i(t)$ of $\mathrm{H}_\cdot(\mathrm{Y}_t, \mathbf{C})$ by $\alpha_i(t) = \gamma_t^{-1}(\alpha_i(\mathrm{o}))$. Clearly we have

$$\int_{\alpha_j(t)} \tau_i(t) = \int_{\alpha_j(t)} \gamma_t^\star(\eta_i(\mathrm{o})) = \int_{\gamma_t(\alpha_j(t))} \eta_i(\mathrm{o}) = \int_{\alpha_i(\mathrm{o})} \eta_i(\mathrm{o}) = \delta_{i,j}.$$

Hence $\eta_i(t) = \displaystyle\sum_j \left(\int_{\alpha_j(t)} \eta_i(t) \right) \tau_j(t)$ in $\mathrm{H}_\cdot(\mathrm{Y}_t, \mathbf{C})$, as both sides have the same integral over each cycle $\gamma_j(t)$. Thus the " variable " sections $\eta_i(t)$ are expressed in terms of the horizontal sections $\tau_j(t)$ by means of the matrix of periods $\displaystyle\int_{\alpha_j(t)} \eta_i(t)$.

PROPOSITION 5. — *Assumptions as in Proposition 4, under the cup-product pairing*

$$\mathrm{Prim}^n\big(\hat{\mathrm{X}}_s/\hat{\mathcal{O}}_{\mathrm{T},s}\big) \times \mathrm{Prim}^n\big(\hat{\mathrm{X}}_s/\hat{\mathcal{O}}_{\mathrm{T},s}\big) \to \mathrm{H}_{\mathrm{DR}}^{2n}\big(\hat{\mathrm{X}}_s/\hat{\mathcal{O}}_{\mathrm{T},s}\big) \simeq \hat{\mathcal{O}}_{\mathrm{T},s}$$

the cup-product of two horizontal classes, $\tau_i \wedge \tau_j$, is a rational number.

Proof. — Each $\dfrac{\partial}{\partial \mathrm{A}_w}$ acts, through the Gauss-Manin connection, as a *derivation* of the cohomology algebra [9], and the connection on $\mathrm{H}_{\mathrm{DR}}^{2n}\big(\hat{\mathrm{X}}_s/\hat{\mathcal{O}}_{\mathrm{T},s}\big) \simeq \hat{\mathcal{O}}_{\mathrm{T},s}$ is just ordinary differentiation. Thus $\tau_i \wedge \tau_j$ " is " an element τ_{ij} of $\mathbf{Q}[[\mathrm{A}_w]]$, and

$$\frac{\partial}{\partial \mathrm{A}_w}\,(\tau_{ij}) = \frac{\partial}{\partial \mathrm{A}_w}\,(\tau_i \wedge \tau_j) = \frac{\partial \tau_i}{\partial \mathrm{A}_w} \wedge \tau_j + \tau_i \wedge \frac{\partial \tau_j}{\partial \mathrm{A}_w} = \mathrm{o} + \mathrm{o}.$$

<div align="right">Q. E. D.</div>

COROLLARY. — *Let $\{\eta_i\}$ be a basis of $\mathrm{Prim}^n(\hat{\mathrm{X}}_s/\hat{\mathcal{O}}_{\mathrm{T},s})$, and $\{\tau_i\}$ the basis of horizontal sections such that for each i, $\tau_i\text{-}\eta_i$ induces 0 in $\mathrm{Prim}^n(\mathrm{X}_s/\mathrm{K}(s))$. Let $\mathrm{P} = (\mathrm{P}_{ij})$ be the period matrix*

$$\eta_i = \sum \mathrm{P}_{ij}\tau_j.$$

The intersection matrix $\mathrm{M} = (\mathrm{M}_{ij})$, $\mathrm{M}_{ij} = \eta_i \wedge \eta_j$, is given by

(4.1) $$\mathrm{M} = \mathrm{PJP}',$$

where J is the value of M at s, i. e. the matrix of rational numbers $\tau_{ij} = \eta_i(s) \wedge \eta_j(s)$ ($\eta_i(s) \in \mathrm{Prim}^n(\mathrm{X}_s/\mathbf{Q})$ being the class induced by η_i).

5. THE PAIRING OF DWORK. — The pairing was constructed in a bootstrap way (historically). Near the point $s \in \mathrm{T}$ which corresponding to the diagonal hypersurface of equation $\sum\limits_{i=1}^{n+2} \mathrm{X}_i^d = 0$, a basis of W^{s} is given by the monomials

$$\mathrm{X}^w \text{ satisfying} \begin{cases} dw_0 = \sum\limits_{i=1}^{n+2} w_i, & w_0 \in \mathbf{Z}, \\ 1 \leq w_i \leq d-1 & \text{for } i = 1, \ldots, n+2. \end{cases}$$

Such an exponent system $w = (w_1, \ldots, w_{n+2})$ is called admissible. For each admissible exponent system $w = (w_1, \ldots, w_{n+2})$, define the *dual* system by $\bar{w} = (d - w_1, \ldots, d - w_{n+2})$. Clearly $w \mapsto \bar{w}$ is an involution of admissible exponent systems.

The pairing ([4], p. 248-251) on the $\mathbf{Q}$-vector space $\mathrm{W}^{\mathrm{s}}(s)$ is defined by

$$(\mathrm{X}^u, \mathrm{X}^v)_s = \begin{cases} (-1)^{u_0} & \text{if } u = \bar{v}, \\ 0 & \text{if not.} \end{cases}$$

It is extended "locally" ([4], p. 266-267), i. e. to $\mathrm{W}^{\mathrm{s}} \otimes \hat{\mathcal{O}}_{\mathrm{T},s}$ by requiring $\hat{\mathcal{O}}_{\mathrm{T},s}$ bilinearity and that, for any two *horizontal* elements τ_1 and τ_2, the product (τ_1, τ_2) is the rational number defined above by $(\tau_1(s), \tau_2(s))$. This does define a pairing, since, just as in Proposition 4, $\mathrm{W}^{\mathrm{s}} \otimes_{\mathrm{T}} \hat{\mathcal{O}}_{\mathrm{T},s}$ admits an $\hat{\mathcal{O}}_{\mathrm{T},s}$ base of horizontal elements. Let τ_w be the horizontal section whose specialization at s is X^w.

By transport of structure by Θ, the expression of the elements X^w in terms of the horizontal elements τ_w is given by a matrix $\mathrm{P} = (\mathrm{P}_{w,u})$:

$$\mathrm{X}^w = \sum_u \mathrm{P}_{w,u}\tau_u \quad \text{in } \mathrm{W}^{\mathrm{s}} \otimes \hat{\mathcal{O}}_{\mathrm{T},s}.$$

P is the *period* matrix, expressing

$$\Theta\left(X^{(w)}\right) = \sum_u P_{w,u}\,\Theta\left(\tau_u\right)$$

in $\mathrm{Prim}^n\left(\hat{X}_s/\hat{\mathcal{O}}_{T,\,s}\right)$.

Thus Dwork's local intersection matrix $M' = M'_{u,\,v} = (X^u,\,X^v)$ is given by

(5.1) $$M' = PJ'P^t,$$

where

$$J'_{u,\,v} = \begin{cases} (-1)^{u_0} & \text{if } u = v, \\ 0 & \text{if not.} \end{cases}$$

Dwork then constructs an $\mathcal{O}_T$-bilinear pairing

$$W^s \times W^s \to \mathcal{O}_T$$

which extends the one described above ([4], p. 267-282, esp. th. 6.2). As explained in section 4, such an extension is necessarily unique. Since we will make use only of local properties, we will not describe the extension.

6. The Intersection Matrix of a Diagonal Hypersurface. — This section is devoted to proving

Theorem 6. — *Dwork's pairing corresponds to (a constant multiple of) the cup-product, i. e. the following diagram is commutative for some non-zero constant $c \in \mathbf{Q}$, which depends only on the dimension, n, and the degree, d,*

$$\begin{array}{ccc}
W^s \times W^s & \xrightarrow{\ c \times (\text{Dwork's pairing})\ } & \mathcal{O}_T \\
\downarrow{\scriptstyle \Theta \times \Theta} & & \downarrow{\wr} \\
\mathrm{Prim}^n(X/T) \times \mathrm{Prim}^n(X/T) & \xrightarrow{\ \text{cup-product}\ } & H^{2n}_{DR}(X/T)
\end{array}$$

Proof. — As explained in section 4, it suffices to show this is so after the base extension $\mathrm{Spec}\left(\hat{\mathcal{O}}_{T,\,s}\right) \to T$, i. e., that it is so in the formal neighborhood of the point $s \in T$ corresponding to the hypersurface $X(n,\,d)$ of equation $\sum_{i=1}^{n+2} X_i^d = 0$. Comparing (4.1) and (5.1), the pairings are given by PJP^t and $PJ'P^t$ respectively. Thus we must show that there is a constant c such that $J' = cJ$, i. e., that under the cup-product pairing

$$\mathrm{Prim}^n(X(n,\,d)) \times \mathrm{Prim}^n(X(n,\,d)) \to \mathbf{Q}$$

we have

$$\Theta(X^u) \wedge \Theta(X^v) = c(-1)^{u_0}\delta_{u,\bar{v}},$$

for each pair $u,\,v$ of admissible exponent systems.

LEMMA 7. — *Let u and v be admissible, and $u \neq \bar{v}$. Then*

$$\Theta(X^u) \wedge \Theta(X^v) = 0.$$

Proof. — Extending the ground field from $\mathbf{Q}$ to $K = \mathbf{Q}(\exp(2\pi i/d))$, we introduce the action of the group $G = (\mu_d)^{n+2}$, which acts on $\mathbf{P}^{n+1}$, $X(n, d)$, $\mathbf{P}^{n+1} - X(n, d)$, etc., via

$$g(X_1, \ldots, X_{n+2}) = (\zeta_1 X_1, \ldots, \zeta_{n+2} X_{n+2}).$$

Here $g = (\zeta_1, \ldots, \zeta_{n+2}) \in (\mu_d)^{n+2}$, and $(X_1, \ldots, X_{n+2})$ are the homogeneous coordinates.

To each admissible exponent system w we associate the character χ_w of G defined by

$$\chi_w(g) = \zeta_1^{w_1} \ldots \zeta_{n+2}^{w_{n+2}} \quad \text{for} \quad g = (\zeta_1, \ldots, \zeta_{n+2}).$$

For each admissible system w, we claim

$$g(\Theta(X^w)) = \chi_w(g) \Theta(X^w).$$

This is so because :

1° $\Theta(X^w) = \text{residue}(\mathcal{R}(X^w))$;

2° $\mathcal{R}(X^w) = (-1)^{w_0-1}(w_0 - 1)! \dfrac{X^w}{\left(\sum X_i^d\right)^{w_0}} \dfrac{d(X_1/X_{n+2})}{(X_1/X_{n+2})} \wedge \ldots \wedge \dfrac{d(X_{n+1}/X_{n+2})}{(X_{n+1}/X_{n+2})}$,

a differential which obviously transforms under G by χ_w, and 3° the residue map $H^{\cdot}(\mathbf{P}^{n+1} - X(n, d)) \otimes K \to H^{\cdot-1}(X(n, d)) \otimes K$ commutes with the action of G.

The product $\Theta(X^u) \wedge \Theta(X^v)$ is thus an element of $H_{DR}^{2n}(X(n, d))$ which transforms under G by $\chi_u \chi_v$. But G acts *trivially* on $H_{DR}^{2n}(X(n, d)) \otimes K$, and thus either

$$\Theta(X^u) \wedge \Theta(X^v) = 0, \quad \text{or} \quad \chi_u = \chi_v^{-1},$$

in which case $u = \bar{v}$.

<div align="right">Q. E. D.</div>

Thus it remains to show that for each admissible exponent system u,

$$\Theta(X^u) \wedge \Theta(X^{\bar{u}}) = c(-1)^{u_0},$$

where c is *independent* of u. We first do this in some special cases :

I. $d = 1$, any n; then $\text{Prim}^n(X(n, 1)) = 0$;

II. $d = 2$, any n; $\text{Prim}^n(X(n, 2))$ is zero if n is odd, and is one-dimensional, spanned by $\Theta\left(\prod_{i=1}^{n+2} X_i\right)$, is n if even;

III. $d = 3$, $n = 2$. The six admissible exponent systems

$$w = (w_1, w_2, w_3, w_4)$$

are

$$(\mathrm{I}, \mathrm{I}, 2, 2), \quad (2, 2, \mathrm{I}, \mathrm{I}), \quad (\mathrm{I}, 2, \mathrm{I}, 2), \quad (2, \mathrm{I}, 2, \mathrm{I}), \quad (\mathrm{I}, 2, 2, \mathrm{I}), \quad (2, \mathrm{I}, \mathrm{I}, 2);$$

with dual pairs written together.

Let the symmetric group $\mathfrak{S}_4$ act on $\mathcal{L}^{\mathrm{S}}$, $\mathbf{P}^3$, $X(2, 3)$, ... by permuting the homogeneous coordinates :

$$\sigma(X_1, X_2, X_3, X_4) = (X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}, X_{\sigma(4)}).$$

By functoriality, the mapping residue : $H^3(\mathbf{P}^3 - X(2, 3)) \to \mathrm{Prim}^2(X(2, 3))$ commutes with the action of σ, and by direct computation

$$\mathcal{R} : \quad W^{\mathrm{S}} \to H^3(\mathbf{P}^3 - X(2, 3))$$

commutes " up to the signature ", i. e.

$$\mathcal{R}(\sigma(X^w)) = \mathrm{sgn}(\sigma)\,\mathcal{R}(X^w).$$

Thus $\Theta(\sigma(X^w)) = \mathrm{sgn}(\sigma)\,\Theta(X^w)$, and, as the involution $w \to \bar{w}$ commutes with the action of $\mathfrak{S}_4$, we have

$$\Theta(\sigma(X^u)) \wedge \Theta(\sigma(X^{\bar{u}})) = \Theta(X^u) \wedge \Theta(X^{\bar{u}}).$$

As $\mathfrak{S}_4$ acts transitively on the six admissible systems for $X(2, 3)$, this completes case III.

The proof of Theorem 6 will be completed by the next proposition, in which we return to the *universal* $X \to T$.

PROPOSITION 8. — *Suppose $n \geq \mathrm{I}$, $d \geq 3$, and either $n \neq 2$ or $d \neq 3$. For each admissible exponent system w, the element $X^w \in \mathcal{L}^{\mathrm{S}}$ determines a global section, $\Theta(X^w)$, of $\mathrm{Prim}^n(X/T)$. For any two admissible systems u and v, one has*

$$\Theta(X^u) \wedge \Theta(X^{\bar{u}}) = (-\mathrm{I})^{u_0 - v_0}\Theta(X^v) \wedge \Theta(X^{\bar{v}}) \quad in \; \mathcal{O}_{\mathrm{T}}.$$

Proof. — It suffices to demonstrate the equality over the local ring of the generic point of T, i. e. to consider the hypersurface X of equation $X_1^d + \sum\limits_{w} A_w X^w$ (where the sum is over all monomials X^w of degree d other than X_1^d, and the A_w are independent variables) over the field $K = \mathbf{Q}(A_w)$.

The spectral sequence assosciated to the filtration of $\Omega^{\cdot}_{X/K}$ by the sub-complexes

$$F^i(\Omega^{\cdot}_{X/K}) = \underbrace{0 \to 0 \to \ldots \to 0}_{i \; z\acute{e}roes} \to \Omega^i_{X/K} \to \Omega^{i+1}_{X/K} \to \ldots,$$

has $E_1^{p, q} = H^q(X, \Omega^p_{X/K}) \Rightarrow H^{p+q}_{\mathrm{DR}}(X/K)$.

By Hodge theory, this spectral sequence degenerates at E_1 for *any* projective nonsingular X over a field K of characteristic zero. This means

that the assosciated graded objects $\mathrm{gr}^p \, \mathrm{H}^{p+q}_{\mathrm{DR}}(\mathrm{X}/\mathrm{K})$ are given by $\mathrm{H}^q(\mathrm{X}, \Omega^p_{\mathrm{X}/\mathrm{K}})$. In particular, because $n = \dim \mathrm{X}/\mathrm{K}$, $\mathrm{gr}^p \, \mathrm{H}^{2n}_{\mathrm{DR}}(\mathrm{X}/\mathrm{K})$ vanishes for $p \neq n$. Thus $\mathrm{F}^p \, \mathrm{H}^{2n}_{\mathrm{DR}}(\mathrm{X}/\mathrm{K}) = 0$ for $p > n$.

Because the filtration F^i of $\Omega^{\cdot}_{\mathrm{X}/\mathrm{K}}$ is " multiplicative ",

$$\mathrm{F}^i \Omega^{\cdot}_{\mathrm{X}/\mathrm{K}} \wedge \mathrm{F}^j \Omega^{\cdot}_{\mathrm{X}/\mathrm{K}} \subset \mathrm{F}^{i+j} \Omega^{\cdot}_{\mathrm{X}/\mathrm{K}},$$

it follows that

$$\mathrm{F}^i \mathrm{H}^{\cdot}_{\mathrm{DR}}(\mathrm{X}/\mathrm{K}) \wedge \mathrm{F}^j \mathrm{H}^{\cdot}_{\mathrm{DR}}(\mathrm{X}/\mathrm{K}) \subset \mathrm{F}^{i+j} \mathrm{H}^{\cdot}_{\mathrm{DR}}(\mathrm{X}/\mathrm{K}).$$

Thus if $\xi \in \mathrm{F}^i \, \mathrm{H}^n_{\mathrm{DR}}(\mathrm{X}/\mathrm{K})$, $\eta \in \mathrm{F}^j \, \mathrm{H}^n_{\mathrm{DR}}(\mathrm{X}/\mathrm{K})$ and $i + j > n$, then $\xi \wedge \eta = 0$.

Now consider any monomial $\mathrm{X}^w \in \mathcal{L}^{\mathrm{S}}$. We have $\Theta(\mathrm{X}^w) = \mathrm{residue}\,(\mathcal{R}(\mathrm{X}^w))$. The class $\mathcal{R}(\mathrm{X}^w)$ is represented by a form holomorphic on $\mathbf{P}^{n+1} - \mathrm{X}$, which has a pole of order (at most) w_0 along X. It follows ([5], § 8] that its residue lies in $\mathrm{F}^{n+1-w_0} \, \mathrm{H}^n_{\mathrm{DR}}(\mathrm{X}/\mathrm{K})$.

Thus if u and v are admissible exponent systems with $u_0 + v_0 \leq n + 1$, one has $\Theta(\mathrm{X}^u) \wedge \Theta(\mathrm{X}^v) = 0$.

Let us say that two admissible exponent systems u and v are *connected* if either $u \geq v$ (meaning $u_i \geq v_i$ for all i) or $v \geq u$.

As a first step towards Proposition 8, we prove

Lemma 9. — *When u and v are admissible exponent systems which are connected, the conclusion of Proposition 8 is valid.*

Proof. — Say $u \geq v$; since we may clearly interpolate a sequence of admissible systems

$$u = u^{(0)} \geq u^{(1)} \geq \ldots \geq u^{(r)} = v$$

with $u_0^{(l)} = 1 + u_0^{(l+1)}$, it is enough to check the case $u_0 = 1 + v_0$. Let us write $u = v + w$; then $\bar{v} = \bar{u} + w$ (because $v + \bar{v} = u + \bar{u}$).

Recall the connection σ on W^{S}, induced by $\sigma_{\mathrm{D}} = \mathrm{D} + \mathrm{F}^{\mathrm{D}}$. In particular, taking the derivation $\dfrac{\partial}{\partial \mathrm{A}_w}$, defined by $\dfrac{\partial}{\partial \mathrm{A}_w}(\mathrm{A}_y) = \delta_{w,y}$, and writing σ_w for σ_{A_w}, we have

$$\sigma_w = \frac{\partial}{\partial \mathrm{A}_w} + \mathrm{X}^w$$

whence

$$\begin{cases} \sigma_w(\mathrm{X}^v) = \mathrm{X}^u, \\ \sigma_w(\mathrm{X}^{\bar{u}}) = \mathrm{X}^{\bar{v}} \end{cases}$$

and so

$$\begin{cases} \dfrac{\partial}{\partial \mathrm{A}_w} \Theta(\mathrm{X}^v) = \Theta(\mathrm{X}^u), \\[2mm] \dfrac{\partial}{\partial \mathrm{A}_w} \Theta(\mathrm{X}^{\bar{u}}) = \Theta(\mathrm{X}^{\bar{v}}). \end{cases}$$

Because $v_0 + \bar{u}_0 = u_0 - 1 + \bar{u}_0 = n + 1$, one has

$$\Theta(X^v) \wedge \Theta(X^{\bar{u}}) = 0.$$

Differentiating with respect to A_w,

$$0 = \frac{\partial}{\partial A_w}(\Theta(X^v)) \wedge \Theta(X^{\bar{u}}) + \Theta(X^v) \wedge \frac{\partial}{\partial A_w}(\Theta(X^{\bar{u}}))$$

whence

$$0 = \Theta(X^u) \wedge \Theta(X^{\bar{u}}) + \Theta(X^v) \wedge \Theta(X^{\bar{v}}).$$

Q. E. D.

LEMMA 10. — *Let a be the least integer with $ad \geq n + 2$. Every admissible exponent system u is connected to an admissible v having $v_0 = a$.*

Proof. — It follows from the definition of admissible that $u_0 \geq a$. Consider *all* $n + 2$-tuples $t = (t_1, \ldots, t_{n+2})$ of integers satisfying

$$0 \leq t_i \leq u_i - 1 \quad \text{for} \quad i = 1, \ldots, n + 2.$$

As t varies over such systems, the function $\sum_{i=1}^{n+2} t_i$ assumes all integral values from 0 to $\sum_{i=1}^{n+2}(u_i - 1)$. Because

$$\sum_{i=1}^{n+2}(u_i - 1) = du_0 - (n + 2) \geq du_0 - ad,$$

there exists a t as above having $\sum_{i=1}^{n+2} t_i = d(u_0 - a)$. The desired v is $u - t$.

Q. E. D.

LEMMA 11. — *If $n \geq 1$ and $d \geq 3$, then either $n + 1 \geq 2a$ or $n = 2$ and $d = 3$.*

Proof. — Suppose $n + 1 < 2a$. As $n \geq 1$, we have $a \geq 2$. As $n + 2 > (a - 1)d$ by definition of a, we have $2a \geq n + 2 > (a - 1)d$, so $2a/a - 1 > d \geq 3$, or $a/a - 1 > 3/2$, possible only if $a = 2$. Then $d < 2a/a - 1$ gives $d = 3$. Finally, $2a \geq n + 2 > (a - 1)d$ gives $n = 2$.

Q. E. D.

LEMMA 12. — *Suppose $n + 1 \geq 2a$. Then any two admissible exponent systems u and v having $u_0 = v_0 = a$ are connected to a common admissible exponent system w.*

Proof. — For $1 \leq i \leq n + 2$, put $r_i = \max(u_i, v_i)$. We have $1 \leq r_i \leq d - 1$. If $\sum_{i=1}^{n+2} r_i$ is divisible by d, take $w = (r_1, \ldots, r_{n+2})$; if not, there are positive integers e and f with

$$\sum_{1}^{n+2} r_i = de - f, \quad 1 \leq f \leq d - 1.$$

We can take

$$w = (r_1 + t_1, r_2 + t_2, \ldots, r_{n+2} + t_{n+2}),$$

if we can find $(t_1, \ldots, t_{n+2})$ satisfying $0 \leq t_i \leq d - 1 - r_i$, and $\sum_{i=1}^{n+} t_i = f$.

As $(t_1, \ldots, t_{n+2})$ varies over all $n + 2$-tuples satisfying only $0 \leq t_i \leq d - 1 - r_i$, the function $\sum_{i=1}^{n+2} t_i$ assumes all integral values between 0 and

$$\sum_{i=1}^{n+2} (d - 1 - r_i) = (n + 2)(d - 1) - \sum_{i=1}^{n+2} r_i.$$

Because $u_i \geq 1$ and $v_i \geq 1$, $r_i = \max(u_i, v_i) \leq u_i + v_i - 1$.

Thus $\sum_{i=1}^{n+2} r_i \leq du_0 + dv_0 - (n + 2) = 2ad - (n + 2)$, and so

$$(n + 2)(d - 1) - \sum_{i=1}^{n+2} r_i \geq (n + 2)(d - 1) - 2ad + (n + 2)$$

$$= (n + 2)d - 2ad = (n + 1 - 2a)d + d \geq d.$$

Hence there exists $(t_1, \ldots, t_{n+2})$ with

$$0 \leq t_i \leq d - 1 - r_i \qquad \text{and} \qquad \sum_{i=1}^{n+2} t_i = f.$$

Q. E. D.

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