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ON THE PRINCIPLE SQUARE : CODING AND EXTENDING EMBEDDINGS

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We have obtained a number of relative consistency results. These are most naturally stated as results about the construction of models with certain properties. The central technique is to adjoin, generically, a structure from which we can extract a «higher-gap» morass of the appropriate sort. We then obtain various combinatorial properties, partial solutions to the transfer problems of model theory, and the existence of various Souslin trees and Kurepa families as consequences of the existence of the higher-gap morasses. Though the morasses which are shown to exist do not have all of the properties of the morasses constructed in L, there is nothing essentially new in the derivation of the various applications from the existence of the morasses. Samples of our results are :

THEOREM 1 : Let M be countable, transitive, $M \models ZF + GCH$. Then there is a generic extension $M[G] \supseteq M$ s. t. ${}^{\omega}M \cap M[G] \subseteq M$, s. t. for all ordinals α, β, γ, of M, ⓐ $(\beta = \text{card } \alpha)^M$ iff $(\beta = \text{card } \alpha)^{M[G]}$ ⓑ $(\beta = \text{cf } \alpha)^M$ iff $(\beta = \text{cf } \alpha)^{M[G]}$ ⓒ $(\gamma = \text{card } {}^{\beta}\alpha)^M$ iff $(\gamma = \text{card } {}^{\beta}\alpha)^{M[G]}$ and ⓓ $(\alpha$ is measurable$)^M$ iff $(\alpha$ is measurable$)^{M[G]}$. In $M[G]$ for all $1 < n < \omega$, there is an $\aleph_n$-Souslin-tree, the generalized $\aleph_n$-Kurepa-hypothesis $KH_{\aleph_n \aleph_n}$ holds, strong versions of the principle $\Box_{\aleph_n}$-hold (see below), all uniform ultrafilters on $\aleph_n$ are regular, and for all infinite cardinals α of $M[G]$, all $m < \omega$, $< \alpha^{(m)}, \alpha > \longrightarrow < \aleph_{n+m}, \aleph_n >$.

THEOREM 2 : Let M be countable, transitive, $M \models ZF + GCH$. Then there is an extension $M[G] \supseteq M$, generic by a class of conditions, s. t. ${}^{\omega}M \cap M[G] \subseteq M$, s. t. for all ordinals α, β, γ of M, ⓐ - ⓒ hold ; as well as all the other conclusions of THEOREM 1, except possibly ⓓ.

In fact, for all infinite cardinals α of $M[G]$, all regular infinite cardinals κ of $M[G]$, all $n < \omega$, $< \alpha^{(n)}, \alpha > \longrightarrow < \lambda^{(n)}, \lambda >$, there is an α^+-Souslin tree, the generalized α^+-Kurepa Hypothesis $KH_{\alpha^+ \alpha^+}$ holds, as well as strong versions of the principle $\Box_\lambda$, if λ is also uncountable.

We conjecture that in the model $M[G]$ of THEOREM 2, there is a complete solution to the transfer problem, that is, for all infinite cardinals,

κ, λ of $M[G]$, all $n < \omega$ $< \kappa^{(n)}, \kappa > \longrightarrow < \lambda^{(n)}, \lambda >$ (this is a complete solution, since GCH holds in M and is preserved in passing to $M[G]$).

To obtain THEOREM 1, we add a «gap ω» universal morass at ω_1 ; to obtain THEOREM 2, we add, at all λ s.t. in M, λ uncountable, λ is inaccessible or the successor of a singular cardinal, a «gap-ω» universal morass.

The proof, in fact the notion itself of higher-gap morass, is too complex to present here. Instead, we have distilled out of several of the crucial constructions, a «gap-two» version of the principle $\square$, which should give the reader some feeling for the considerations involved. We shall introduce the principle, and motivate it as a generalization from the situation in L. This will lead us naturally to discuss what we feel is the real significance of our results - by placing them in the context of a large, but somewhat vague program. We finish by indicating the forcing proof of the relative consistency of our principle.

Let $\kappa = \lambda^+$, λ a regular cardinal. To fix ideas, take $\lambda = \omega_1$, $\kappa = \omega_2$. In all of what follows, we shall abbreviate «closed unbounded» by «cub», «primitive-recursively closed» by «p.r. closed», «order-type» by «o.t.». If X is a set of ordinals, X^* is the set of limit points of $X = \{\beta : X \cap \beta$ is unbounded in $\beta\}$. A structure $\mathfrak{a} = \langle A, \in \cap {}^2 A, R_0, \ldots, R_{n-1}\rangle$ is amenable just in case for all $X \in A$, all $i < n$, $X \cap R_i \in A$.

We begin by recalling the principle $\square_\kappa$:

There is a sequence $\vec{C} = \langle C_\alpha : \alpha \in \operatorname{dom}\vec{C}\rangle$ s.t.

ⓐ $\alpha \in \operatorname{dom}\vec{C} \Rightarrow \alpha$ is a limit ordinal,

$\operatorname{dom}\vec{C} \supseteq \{\nu : \kappa < \nu < \kappa^+,\ \nu$ a limit ordinal$\}$;

i) C_α cub $\subseteq \alpha$; if $\alpha > \kappa$, then $C_\alpha \subseteq \alpha - \kappa + 1$

ii) if $\beta \in (C_\alpha)^*$ then $\beta \in \operatorname{dom}\vec{C}$ and $C_\beta = C_\alpha \cap \beta$;

(Coherence Property)

iii)$_\kappa$ $\operatorname{cf}(\alpha) < \kappa \Rightarrow$ o.t. $C_\alpha < \kappa$.

A sequence with these properties is a $\square_\kappa$-sequence.

It follows from ii) and iii)$_\kappa$ that for $\alpha \in \operatorname{dom}\vec{C}$, o.t. $(C_\alpha) \leq \kappa$.

Evidently, the only interesting part of a $\square_\kappa$-sequence, as far as the statement of the principle $\square_\kappa$ is concerned, is the co-initial segment $\langle C_\alpha : \alpha$ limit , $\kappa < \alpha < \kappa^+\rangle$, since for α limit, $\alpha \leq \kappa$ we can simply take $C_\alpha = \alpha$.

The proof of $\square_\kappa$ in L is long and complicated. However the $\square_\kappa$-sequence constructed there has many properties in addition to those required by the principle $\square_\kappa$. For example, there is a cub subset $S^o \subseteq \kappa$, and a function S with domain $S^o \cup \{\kappa\}$,

s.t. $\operatorname{dom}\vec{C} = \bigcup_{\alpha \in S^o \cup \{\kappa\}} S_\alpha$, $S_\kappa = \{\nu : \kappa < \nu < \kappa^+, \nu$ limit$\}$,

for $\alpha, \beta \in S^o$, $\alpha < \beta$, S_α is non-empty and closed, $\max S_\alpha = \mu_\alpha > \alpha$ is p.r. closed, $J_{\mu_\alpha} \models \ulcorner\alpha$ is regular, α is the largest cardinal$\urcorner$, $S_\alpha = \{\nu : \alpha < \nu \leq \mu_\alpha,\ \nu$ limit$\}$,

$\beta > \mu_\alpha$, $\vec{C}\restriction S_\alpha \subseteq J_\beta$, $\langle J_\beta, \vec{C}\restriction \bigcup_{\alpha' \in S^o \cap \alpha+1} S_{\alpha'}\rangle$ is amenable, and for

$\nu \in S_\alpha$, $C_\nu \subseteq \nu - \alpha + 1$. Further, the sequence $\vec{C}$ has a condensation-like property. Finally, in non-trivial cases, the set C_ν codes up much of the essential information about a possibly larger model $< J_\rho, A >$, $\rho \geqq \nu$, canonically associated to ν by the theory of the fine structure.

On the other hand, the principle $\Box_\kappa$ is quite easily proved to be relatively consistent via forcing. In an attempt to «improve» on the principle $\Box_\kappa$, as far as capturing the essentials of the situation in L, Jensen also formulated a principle $(H)_\kappa$, which Devlin in ASPECTS OF CONSTRUCTIBILITY, renamed $\heartsuit$ because it is at the «heart» of proving, in L, the existence of gap-one morasses at κ. We present a generalization of the principle $(H)_\kappa$ - this may be thought of as an attempt to build into our $\Box_\kappa$-sequence $\vec{C}$ the above properties of the $\Box$-sequence constructed in L - except the «coding» property.

$(H)_\kappa$: There is a sequence $\vec{C}^2 = < C^2_\nu : \nu \in \mathrm{dom}\,\vec{C} >$, a set $A \subseteq \kappa^+$ and a sequence $\mathcal{A} = < A^\alpha : \alpha \in \mathrm{dom}\,\mathcal{A} >$ s.t.

ⓐ $L^A_\kappa = H_\kappa$, $L^{A,\mathcal{A}}_{\kappa^+} = H_{\kappa^+}$;

ⓑ $\mathrm{dom}\,\vec{C}^2 = \bigcup_{\alpha \in S^0 \cup \{\kappa\}} S_\alpha$; S^0 cub $\subseteq \kappa = \mathrm{dom}\,\mathcal{A}$,

$S_\kappa = \{ \nu : \kappa \subset \nu \subset \kappa^+, \ \nu \text{ limit} \}$; for $\alpha \in S^0$, S_α is closed

$S_\alpha = \{ \nu : \alpha < \nu \leq \mu_\alpha, \ \nu \text{ limit} \}$, $\mu_\alpha = \max S_\alpha$ is p.r. closed ;

ⓒ $L^{A^\alpha}_{\mu_\alpha} \models \ulcorner\alpha$ is the largest cardinal$\urcorner$, $A^\alpha \cap \alpha = A \cap \alpha$

$< L^{A^\alpha}_{\mu_\alpha}, A^\alpha, \vec{C}^2 \upharpoonright \mu_\alpha - \alpha + 1 >$ is amenable ; for $\beta \in S^0 - \alpha + 1$,

μ_α, $\vec{C}^2 \upharpoonright \bigcup_{\alpha' \in S^0 \cap \alpha + 1} S_{\alpha'}$, $\mathcal{A} \upharpoonright \alpha + 1 \in L^A_\beta$;

ⓘ for $\nu \in \mathrm{dom}\,\vec{C}^2$, C^2_ν cub $\subseteq \nu$; if $\nu \in S_\alpha$ then $C^2_\nu \subseteq \nu - \alpha + 1$

(note by ⓑ $\{ S_\alpha : \alpha \in S^0 \cup \{ \kappa \}\}$ is a partition of $\mathrm{dom}\,\vec{C}^2$) ;

ⓘⓘ for $\nu \in S_\alpha$, if $\tau \in (C^2_\nu)^*$, then $\tau \in S_\alpha$ and $C^2_\tau = C^2_\nu \cap \tau$;

ⓘⓘⓘ$_\alpha$ for $\alpha \in S^0$, $\nu < \mu_\alpha$, if $L^{A^\alpha}_{\mu_\alpha} \models \mathrm{cf}(\nu) < \alpha$, then

o.t. $(C^2_\nu) < \alpha$; o.t. $(C^2_{\mu_\alpha}) < \alpha$;

ⓘⓘⓘ$_\kappa$ for $\nu \in S_\kappa$, $\mathrm{cf}(\nu) < \kappa \Rightarrow$ o.t. $(C^2_\nu) < \kappa$;

(iv) for p.r. closed ν, $\nu \in (C^2_\nu)^{**}$ iff ν is a limit of p.r. closed ordinals, whenever $\nu \in \operatorname{dom} \vec{C}$; further $(C^2_\nu)^* \subseteq \{\tau : \tau < \nu, \tau \text{ p.r. closed}\}$, $C^2_\nu \subseteq \operatorname{dom} \vec{C}^2$ - (Coherence Ppty for p.r. closed ordinals);

(v) for $\alpha \in S^o$, $\beta \in (S^o \cup \{\kappa\}) - \alpha + 1$, there is $\nu \in S_\beta$ and Π s.t. $\Pi \upharpoonright \alpha = id \upharpoonright \alpha$, $\Pi(\alpha) = \beta$ and

$$\Pi : \langle L^{A^\alpha}_{\mu_\alpha}, A^\alpha, \vec{C}^2 \upharpoonright \mu_\alpha - \alpha + 1, C^2_{\mu_\alpha} \rangle \xrightarrow[\Sigma_1]{} \langle L^{A^\beta}_\nu, A^\beta \cap \nu, \vec{C}^2 \upharpoonright \nu - \beta + 1, C^2_\nu \rangle$$

(where if $\beta = \kappa$, $A^\beta = A$);

(vi) for all $\beta \in S^o \cup \{\kappa\}$, all $\nu \in S_\beta$, there is $\alpha \in S^o \cap \beta$ s.t. for all β', if $\alpha < \beta' < \beta$ and $\beta' \in S^o$ then there is $\nu' \in S_{\beta'}$ and Π s.t. $\Pi \upharpoonright \beta' = id \upharpoonright \beta'$, $\Pi(\beta') = \beta$ and

$$\Pi : \langle L^{A^{\beta'}}_{\nu'}, A^{\beta'} \cap \nu', \vec{C}^2 \upharpoonright \nu' - \beta' + 1, C^2_\nu \rangle \xrightarrow[\Sigma_1]{} \langle L^A_\nu, A^\beta \cap \nu, \vec{C}^2 \upharpoonright \nu - \beta + 1, C^2_\nu \rangle .$$

Properties (v) and (vi) are our attempts to build in as much as possible of a condensation-like structure into our sequences $\vec{C}^2$, $\mathcal{A}$. The intuitive idea is that for $\alpha \in S^o$ $\langle L^{A^\alpha}_{\mu_\alpha}, A^\alpha, \vec{C}^2 \upharpoonright \mu_\alpha - \alpha + 1, C^2_{\mu_\alpha} \rangle$ is obtained as the transitive collapse of a Σ_1-elementary substructure of a model $\langle L^A_\nu, A \cap \nu, \vec{C}^2 \upharpoonright \nu - \kappa + 1, C^2\nu \rangle$ for ν p.r. closed $\nu \in S_\kappa$, and that these layers of approximations «fit together nicely».

We can now state our principle, which for lack of a better notation, we shall denote by $\Box_{\omega_1 \omega_2}$, although, according to Rebholz this notation is already in use to denote a partition principle. Recall that in all the above we imagined $\lambda = \omega_1$ and $\kappa = \omega_2$. In addition to the properties of $(H)_{\omega_2}$, $\Box_{\omega_1 \omega_2}$ asserts the existence of a $\Box_{\omega_1}$-sequence $\vec{C}^1$, a cub subset Y of S^o on which $\vec{C}^1$ «coheres nicely», that $\vec{C}^1$ «meshes nicely» with $\vec{C}^2 \upharpoonright \bigcup_{\alpha \in S^o} S_\alpha$, that $\vec{C}^1$ codes up much of the information about $\vec{C}^2$.

$\Box_{\omega_1 \omega_2}$: There are sequences $\vec{C}^1$, $\vec{C}^2$, $\mathcal{A}$, and sets A, Y s.t. $\vec{C}^2$, A, $\mathcal{A}$ satisfy the requirements of $(H)_{\omega_2}$, except that $H_{\omega_3} = L^{A,\vec{C}^1,\mathcal{A},Y}_{\omega_3}$, and s.t.

(vii) Y cub $\subseteq (S^o)^*$; for $\alpha \in Y$, $\langle L^A_\alpha, Y \cap \alpha, \mathcal{A} \upharpoonright \alpha, \vec{C}^2 \upharpoonright \alpha, \vec{C}^1 \upharpoonright \alpha \rangle$ is amenable; if $\alpha \in Y - Y^*$, and α' is the largest element of $Y \cap \alpha$, then α' is the largest element of $(S^o)^* \cap \alpha$, o.t. $(C^1_\alpha - \alpha' + 1) = \omega$ and $\alpha' \in (C^1_\alpha)^*$;

(d) $\operatorname{dom} \vec{C}^1 \supseteq \{\alpha : \omega_1 < \alpha < \omega_2,\ \alpha \text{ limit}\}$;

(viii) for $\alpha \in \operatorname{dom} \vec{C}^1$, C^1_α cub $\subseteq \alpha$; if $\alpha > \omega_1$ then $C^1_\alpha \subseteq \alpha - \omega_1 + 1$;

$\alpha \in \operatorname{dom} \vec{C}^1 \Rightarrow \alpha$ limit, $\alpha < \omega_2$;

(ix) for $\alpha \in \operatorname{dom} \vec{C}^1$, $\operatorname{cf}(\alpha) < \omega_1 \Rightarrow$ o.t.$(C_\alpha) < \omega_1$;

(x) if $\alpha \in \operatorname{dom} \vec{C}^1$, $\beta \in (C^1_\alpha)^*$, then $\beta \in \operatorname{dom} \vec{C}^1$ and $C^1_\beta = C^1_\alpha \cap \beta$;

if $\alpha \in Y$, $C^1_\alpha \subseteq S^o$, $(C^1_\alpha)^* \subseteq Y$;

$\alpha \in Y^*$ iff $\alpha \in (C^1_\alpha)^{**}$ iff $\mu_\alpha \in (C^2_{\mu_\alpha})^{**}$;

(xi) if $\alpha \in Y^*$, then o.t.$(C^2_{\mu_\alpha})$ = o.t.(C^1_α)

and if $\beta \in (C^1_\alpha)^*$, there is Π s.t. $\Pi \upharpoonright \beta = \operatorname{id} \upharpoonright \beta$, $\Pi(\beta) = \alpha$

$$\Pi : \langle L^{A^\beta}_{\mu_\beta}, A^\beta, \vec{C}^2 \upharpoonright \mu_\beta - \beta + 1, C^2_{\mu_\beta} \rangle \xrightarrow[\Sigma_0]{} \langle L^{A^\alpha}_{\mu_\alpha}, A^\alpha, \vec{C}^2 \upharpoonright \mu_\alpha - \alpha + 1, C^2_{\mu_\alpha} \rangle .$$

Properties (vii) and (x) make precise the notion that $\vec{C}^1$ coheres nicely on Y; properties (vii), and (xi) make precise the notion that $\vec{C}^1$ codes up much information about $\vec{C}^2$, $\mathcal{A}$; more precisely, for $\alpha \in Y^*$, C^1_α together with $\vec{C}^2 \upharpoonright \alpha$, $\mathcal{A} \upharpoonright \alpha$ codes up much information about the model $\langle L^{A^\alpha}_{\mu_\alpha}, A^\alpha, \vec{C}^2 \upharpoonright \mu_\alpha - \alpha + 1, C^2_{\mu_\alpha} \rangle$; we shall see this in the next LEMMA.

LEMMA : Suppose $\alpha \in Y^*$ $\mathfrak{a}_\alpha = \langle L^A_\alpha, A \cap \alpha, Y \cap \alpha, \vec{C}^2 \upharpoonright \alpha, \vec{C}^1 \upharpoonright \alpha, \mathcal{A} \upharpoonright \alpha, C^1_\alpha \rangle$;

for $\beta \in Y \cap \alpha + 1$, set $\mathcal{L}_\beta = \langle L^{A^\beta}_{\mu_\beta}, A^\beta, \vec{C}^2 \upharpoonright \mu_\beta - \beta + 1, C^2_{\mu_\beta} \rangle$. Suppose

$X \subseteq L^A_\alpha$ and $\mathfrak{a}_\alpha \mid X \prec_{\Sigma_1} \mathfrak{a}_\alpha$. Then there is $X' \subseteq L^A_{\mu_\alpha}$ s.t.

$X' \cap L^A_\alpha = X$, $\mathcal{L}_\alpha \mid X' \prec_{\Sigma_1} \mathcal{L}_\alpha$ and s.t. $X \cap \alpha$ cofinal in α

implies $X' \cap \mu_\alpha$ cofinal in μ_α.

Proof : Set $O = (C^1_\alpha)^* \cap \alpha$.

Set $\bar{O} = O \cap X$ we note that for $\beta, \beta' \in O$, $\beta < \beta'$, there is

$\pi^{(\beta,\alpha)} : \mathcal{L}_\beta \rightarrow_{\Sigma_0} \mathcal{L}_\alpha$, $\pi^{(\beta,\alpha)} \upharpoonright \beta = \mathrm{id} \upharpoonright \beta$, $\pi^{(\beta,\alpha)}(\beta) = \alpha$, and

$\pi^{(\beta,\beta')} : \mathcal{L}_\beta \rightarrow_{\Sigma_0} \mathcal{L}_{\beta'}$, $\pi^{(\beta,\beta')} \upharpoonright \beta = \mathrm{id} \upharpoonright \beta$, $\pi^{(\beta,\beta')}(\beta) = \beta'$, s.t.

$\pi^{(\beta,\alpha)} = \pi^{(\beta',\alpha)} \cdot \pi^{(\beta,\beta')}$. Further,

$\pi^{(\beta,\alpha)}$ = the direct limit of $< \pi^{(\beta,\beta')} \; \beta' \in O - \beta + 1 >$

We note also that the system

$<< \mathcal{L}_\beta, \pi^{(\beta,\beta')} > : \beta' \in O - \beta + 1, \beta \in O>$ is $\Sigma_1(\mathfrak{a}_\alpha)$; in fact there is a single $\Sigma_1(\mathfrak{a}_\alpha)$ definition for all $\alpha \in Y^*$.

Now set $X' = \bigcup_{\beta \in \bar{O}} \pi^{(\beta,\alpha)}$ " $L^{A^\beta}_{\mu_\beta} \cap X$; clearly $X' \cap L^A_\alpha = X$, since for $\beta \in O$ $\pi^{(\beta,\alpha)} \upharpoonright L^A_\beta = \mathrm{id} \upharpoonright L^A_\beta$; further $\bar{O}$ is cofinal in $X \cap \alpha$; for $\beta \in \bar{O}$

$\mathcal{L}_\beta \mid L^{A^\beta}_{\mu_\beta} \cap X \prec_{\Sigma_\omega} \mathcal{L}_\beta$; hence $\mathcal{L}_\alpha \mid X' \prec_{\Sigma_0} \mathcal{L}_\alpha$ (since each $\pi^{(\beta,\alpha)} : \mathcal{L}_\beta \rightarrow_{\Sigma_0} \mathcal{L}_\alpha$).

By the same argument we used to see $\mathcal{L}_\alpha \mid X' \prec_{\Sigma_0} \mathcal{L}_\alpha$. it is easy to see that for $x' \in X'$, there is $\xi \in X \cap \alpha$ s.t. x' is Σ_1 $(\mathcal{L}_\alpha)$-definable in parameter ξ. It will suffice to see the converse, namely that if ξ is an element of $\alpha \cap X$ and $x' \in L^{A^\alpha}_{\mu_\alpha}$ is $\Sigma_1(\mathcal{L}_\alpha)$-definable in parameter ξ, then $x' \in X'$. Since X is closed under the formation of ordered pairs, $\alpha \cap X$ is closed under Gödel's pairing function; hence $\mathcal{L}_\alpha \mid X' \prec_{\Sigma_1} \mathcal{L}_\alpha$.

So, suppose the above situation holds; let $\varphi(v_0, v_1) = \exists v_2 \; \psi(v_0, v_1, v_2)$ be s.t. in $\mathcal{L}_\alpha$ x' is the unique solution to $\varphi(v_0, \xi)$, $\xi \in \alpha \cap X$, $\psi \; \Sigma_0$.

Then, there is $\beta \in O$, $\beta > \xi$ s.t. $x' = \pi^{(\beta,\alpha)}(\bar{x}')$ for some $\bar{x}'$ in $L^{A^\beta}_{\mu_\beta}$ and

(*) in $\mathcal{L}_\beta$, $\bar{x}'$ is the unique solution to $\varphi(v_0, \xi)$.

(*) is a $\Sigma_1(\mathfrak{a}_\alpha)$ condition.

But then there is a $\beta \in \bar{O}$, $\beta > \xi$ s.t. (*) holds. But such an $\bar{x}' \in L^{A^\beta}_{\mu_\beta}$ must be an element of X ; let $x'' = \pi^{(\beta,\alpha)}(\bar{x}')$; then $x'' = x'$ and we are finished.

Since $\alpha \in Y_\mu^*$ (xi) insures that if $X \cap \alpha$ is cofinal in α then $X' \cap \mu_\alpha$ is cofinal in μ_α, since o.t. $(C^2_{\mu_\alpha}) =$ o.t. (C^1_α), o.t. $(C^1_\alpha) \leq \omega_1$, and so o.t. $(C^1_\alpha) \cap X$ is transitive ; that is o.t. $(C^2_{\mu_\alpha}) \cap X' =$ o.t. $(C^2_{\mu_\alpha})$.

QED LEMMA

Let us now discuss the analogy between the situation of the LEMMA and the situation in L. In a certain sense, between α and μ_α, for $\alpha \in S^o$, $\vec{C}^2 \upharpoonright S_\alpha$ can be thought of representing a «fine structure» ; similarly for $\vec{C}^2 \upharpoonright S_\kappa$, between ω_2 and ω_3 - this «representation» is far from exact, but we will have a better idea of the sense of this representation, once we discuss the role of C^1_α, for $\alpha \in Y$. For such α we imagine $\mathcal{L}_\alpha$ as playing the role of $< J_\rho , A >$, where $< J_\rho , A >$ is given to us, for α, by the fine structure theory. We do not necessarily have that there is a $\Sigma_1(\mathcal{L}_\alpha)$ map of a subset of ω_1 onto α, nor do we have any way of extending X' to a Σ_n elementary substructure of some possibly larger model ; whereas , in the situation in L, we do have the certitude of making such an extension. Further, in the situation in L, it is possible that $\rho = \alpha$, while here we always have $\mu_\alpha > \alpha$.

This said, the way in which C^1_α together with $\vec{C}^2 \upharpoonright \alpha$, $\mathcal{A} \upharpoonright \alpha$ allows us to code the essential information about $\mathcal{L}_\alpha$ is in direct analogy to the way in which, in L, C^1_α is the result of a series of «foldings» of Σ_o -elementary substructures of $< J_\rho , A >$ onto elements of J_α. Similarly, given a Σ_1 embedding from some $<J_{\bar{\alpha}} , \bar{\vec{C}} , \bar{C}>$ into $< J_\alpha , \vec{C}^1 \upharpoonright \alpha , C^1_\alpha >$ we can «unfold» this embedding - extend it to a Σ_1 embedding between $< J_{\bar{\rho}} , \bar{A} >$, as given by the fine structure for $\bar{\alpha}$, and $< J_\rho , A >$, in exactly the way we obtain X' from X - as the direct limit of a system of other embeddings - which we know in advance will be well-founded. As a consequence, we obtain that, in L under the above circumstances $\bar{\vec{C}} = \vec{C}^1 \upharpoonright \bar{\alpha}$, $\bar{C} = C^1_\alpha$.

This is too much to hope for, in the general setting ; all we can do is formulate a strengthening of $\square_{\omega_1 \omega_2}$ to include considerations along the lines of $(H)_{\omega_1}$. The work involved in establishing all of the above in L is central to the proof that in L there exist gap-one morasses at all regular cardinals.

In the absence of some sort of structure resembling a $\square_{\omega_3}$-sequence we do not have any precise sense in which we intend the $\square_{\omega_2}$-sequence to be «analogous to the fine structure theory» - we are guided however, by the knowledge of the intimate connection in L between the «fine structure» and the $\square$-sequences constructed there.

In fact, this last is our very motivation for formulating principles like $\square_{\omega_1 \omega_2}$, and constructing «outer models» in which morasses exist. What does it mean, in general, for a model of set theory to have a «fine-structure», or, as we have come to say, an «L-like» structure ? Can we make such a notion precise ? Aside from the Condensation Lemma, which is, essentially a consequence of the minimality of L, the main feature of L is its uniformity - as many have remarked.

Is there a useful notion of uniformity of a model of set theory, in general ? A reasonable first approximation to such a notion is that to a very strong extent, the structure of sets of relatively high rank should be tied to the structure of sets of relatively low rank. A way of making this idea precise, in turn, is to require that for a wide class of embeddings, we can extend these embeddings to embeddings between larger models - perhaps even of embeddings of «the universe» into itself. It is striking that in Jensen's recent work, the crucial lemmas are assertions of this form ; it is exactly in the proofs of these lemmas that the ideas and techniques related to the theory of the fine structure of L play a central role !

We propose a program to investigate further these ideas. As a first step, in our thesis, we were guided by the intuition that the existence of higher gap morasses is a candidate for a least a partial notion of an L-like structure of a model of set-theory. For us, the true significance of our thesis is that, to the extent that the existence of higher-gap morasses is valuable as a candidate for such a notion, we have partially suceeded in imposing, from the outside , an L-like structure on a model of set theory. The results of THEOREMS 1 and 2 indicate that we can do so without drastically altering the ground model, and that, for skeptics, the resulting model, even from a pragmatic point of view has many points in common with L.

We close by presenting the set of conditions for proving the relative consistency of $\square_{\omega_1 \omega_2}$ by forcing. We indicate the essential properties of the set of conditions. We assume $2^{\aleph_i} = \aleph_{i+1}$, $i = 1, 2, 3$, and fix $A \subseteq \omega_3$ s.t. $H_{\omega_i} = L^A_{\omega_i}$, $i = 1,2,3$.

$< \vec{c}^1, y, \vec{\mu}, \vec{a}, \vec{c}^2, u > \; \in P$ iff

$\vec{c}^1, \vec{c}^2, \vec{a} = \; < c^1_\alpha : \alpha \in \operatorname{dom} \vec{c}^1 >, < c^2_\nu : \nu \in \operatorname{dom} \vec{c}^2 > \; < a^\alpha : \alpha \in \operatorname{dom} \vec{a} >$

$\vec{\mu} = \; < \mu_\alpha : \alpha \in \operatorname{dom} \vec{a} >$, μ_α is a p.r. closed ordinal $> \alpha$;

Y is a closed subset of $(\operatorname{dom} \vec{\mu})^*$,

$\operatorname{dom} \mu = \operatorname{dom} \vec{a}$ is a closed subset of ω_2 , $\operatorname{card}(\operatorname{dom} \mu) \leq \aleph_1$;

$\operatorname{dom} \vec{c}^1 = \{\alpha : \omega_1 < \alpha \leq \delta_1 , \text{ limit}\}; \delta_1 = \max y = \max \operatorname{dom} \vec{\mu}$;

$$\operatorname{dom} \vec{c}^2 = \bigcup_{\alpha \in \operatorname{dom} \vec{\mu}} \{\nu : \alpha < \nu \leq \mu_\alpha, \nu \text{ limit}\} \cup \{\nu : \omega_2 < \nu \leq \delta_2, \nu \text{ limit}\}$$

where $\omega_2 < \delta_2 < \omega_3$, δ_2 p.r. closed.

Then, set $s^o = \operatorname{dom} \vec{\mu}$, $s_\kappa = \{\nu : \omega_2 < \nu \leq \delta_2 , \nu \text{ limit}\}$,

for $\alpha \in s^o$, $s_\alpha = \{\nu : \alpha < \nu \leq \mu_\alpha , \nu \text{ limit}\}$;

We then require that properties (c) , (i) , (ii) , (iii) , $(\text{iii})_\kappa$,

(iv) , (xii) of $\square_{\omega_1 \omega_2}$ hold, everywhere writing $\vec{c}^1$ for $\vec{C}^1$,

$\vec{c}^2$ for $\vec{C}^2$, $\vec{a}$ for $\mathfrak{A}$, c^1_α for C^1_α , c^2_α for C^2_α , a^α for A^α ,

s^o for S^o, s_κ for S_κ , s_α for S_α , and s.t. :

$\operatorname{card}(u) \leq \aleph_1$; $u \subseteq (\operatorname{dom} \vec{\mu}) \times (\delta^2 + 1 - \omega_2 + 1)$; $\operatorname{dom} u = \operatorname{dom} \vec{\mu}$ and

$\langle \alpha, \nu \rangle \in u \Rightarrow \langle L^A_\nu , \vec{c}^2 \restriction \nu - \omega_2 + 1 \rangle$ is amenable,

ν is p.r. closed, and that for all
$\alpha' \in \operatorname{dom} u - \alpha$, there is ν'
s.t. $\langle \alpha', \nu' \rangle \in u$ and
$(\alpha' = \alpha \Rightarrow \nu' = \nu)$ and there is Π s.t.
$(\Pi \restriction \alpha' = \operatorname{id} \restriction \alpha'$, $\Pi(\alpha') = \omega_2$

$$\Pi : \langle L^{a^{\alpha'}}_{\mu_{\alpha'}} , a^{\alpha'} , \vec{c}^2 \restriction \mu_{\alpha'} - \alpha' + 1, c^2_{\mu_{\alpha'}} \rangle \xrightarrow[\Sigma_1]{} \langle L^A_{\nu'} , A \restriction \nu', \vec{c}^2 \restriction \nu' - \omega_2 + 1, c^2_{\nu'} \rangle$$

and $(\nu \neq \nu' \Rightarrow \nu < \nu'$, $\nu \in \operatorname{range} \Pi))$.

u is device for «bookkeeping», designed to guarantee that we can continually make ppties (v) and (vi) true.

Then : $\langle \vec{c}^1, y, \vec{\mu}, \vec{a}, \vec{c}^2, u \rangle \leq \langle (\vec{c}^1)', y', \vec{\mu}', \vec{a}', (\vec{c}^2)', u' \rangle$ iff

$\vec{c}^1 \supseteq (\vec{c}^1)'$, $\vec{c}^2 \supseteq (\vec{c}^2)'$, $\vec{\mu} \supseteq \vec{\mu}'$, $\vec{a} \supseteq \vec{a}'$, $u \supseteq u'$ and

y an end-extension of y', dom u an end-extension of dom u'.

THEOREM : $\mathbb{P} = \langle P, \leq \rangle$ is (ω_1, ∞) distributive and has the $\aleph_3$-c.c.