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Necessary condition for measures which are (L^q, L^p) multipliers

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Abstract

Let G be a locally compact group and ρ the left Haar measure on G. Given a non-negative Radon measure μ, we establish a necessary condition on the pairs $(q,\ p)$ for which μ is a multiplier from $L^q(G,\ \rho)$ to $L^p(G,\ \rho)$. Applied to $\mathbb{R}^n$, our result is stronger than the necessary condition established by Oberlin in [14] and is closely related to a class of measures defined by Fofana in [7].

When G is the circle group, we obtain a generalization of a condition stated by Oberlin [15] and improve on it in some cases.

Résumé

Soit G un groupe localement compact et ρ la mesure de Haar à gauche sur G. Etant donné une mesure de Radon positive μ, nous établissons une condition nécessaire sur les couples $(q,\ p)$ pour lesquels μ est un multiplicateur de $L^q(G,\ \rho)$ dans $L^p(G,\ \rho)$. Appliqué à $\mathbb{R}^n$, notre résultat est plus fort que la condition nécessaire établie par Oberlin dans [14] et est très lié à une classe de mesures définie par Fofana dans [7].
Lorsque G est le tore, nous obtenons une généralisation d'une condition énoncée par Oberlin [15] et l'améliorons dans certains cas.

1. Introduction

We suppose that G is a locally compact group and ρ is the left Haar measure on G.

For $1 \leq q < \infty$, a Radon measure μ on G is said to be L^q-improving if there exists a real number $p > q$ such that

$$\mu * f \in L^p(G,\ \rho) \quad \text{and} \quad \|\mu * f\|_{L^p(G,\ \rho)} \leq c\,\|f\|_{L^q(G,\ \rho)}$$

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for all $f \in L^q(G,\ \rho)$, where c is a real number not depending on f.

Of course absolutely continuous measures with Radon-Nikodym derivatives with respect to ρ in $L^r(G,\ \rho)$ with $\frac{1}{q}+\frac{1}{r}-1>0$ are L^q-improving. But L^q-improving singular measures also exist.

Bonami [2] showed that all tame Riesz products on the Walsh group are L^q-improving, and that was extended to all compact abelian groups by Ritter [16]. Moreover it is well known that on the circle group $\mathbb{T}=\mathbb{R}/\mathbb{Z}$, the Cantor-Lebesgue measure μ_δ^2 associated with the Cantor set of constant ratio of dissection $\delta>2$ is L^q-improving for $1<q<\infty$.(See Section 4 for a precise definition of this measure.) This result was proved by Oberlin [12] for $\delta=3$. Ritter [17], Beckner, Janson and Jerison [1] proved the same for δ rational and Christ [3] for δ irrational.

In fact, Christ has extended the result to Cantor-Lebesgue measures with variable but bounded ratios $2<\delta_t\leq c$ of dissection.

In this note, we are interested in the following problem: given a non-negative Radon measure μ on G, determine the indices $1\leq q<p<\infty$ for which there exists a non-negative constant $c(\mu,\ q,\ p)$ such that

$$\|\mu * f\|_{L^p(G,\ \rho)} \leq c(\mu,\ q,\ p)\, \|f\|_{L^q(G,\ \rho)}, \qquad f\in L^q(G,\ \rho). \tag{1.1}$$

In [15] Oberlin stated the following

Proposition 1.1. *If the Cantor-Lebesgue measure μ_3^2 associated to the middle third Cantor set satisfies (1.1), then*

$$\frac{1}{q}+\left(1-\frac{\log 2}{\log 3}\right)\left(1-\frac{1}{p}\right)\leq 1. \tag{1.2}$$

Graham, Hare and Ritter obtained in [9] the following

Proposition 1.2. *Let μ be a measure on the circle group $\mathbb{T}$ and $1\leq q<2$. If there exists a non-negative constant $c(\mu,\ q)$ such that*

$$\|\mu * f\|_{L^2(\mathbb{T})} \leq c(\mu,\ q)\, \|f\|_{L^q(\mathbb{T})}, \qquad f\in L^q(\mathbb{T}),$$

then there exists a positive real number K such that for any interval I whose endpoints are x and $x+h$, we have

$$|\mu(I)| \leq K\,|h|^{\frac{1}{q}-\frac{1}{2}}. \tag{1.3}$$

Inequality (1.3) means that μ satisfies a Lipschitz condition of order $\frac{1}{q} - \frac{1}{2}$.

Replacing $\mathbb{T}$ by $\mathbb{R}^n$, Oberlin proved a similar necessary condition (see the proof of Proposition 2 in [14]).

Proposition 1.3. *If a non-negative Radon measure on $\mathbb{R}^n$ satisfies (1.1), then there exists a positive real number K such that*

$$\mu(R) \leq K |R|^{\frac{1}{q} - \frac{1}{p}} \tag{1.4}$$

for all rectangles R in $\mathbb{R}^n$.

In the present paper, we establish the following necessary condition:

Proposition 1.4. *Suppose that μ is a non-negative Radon measure on G satisfying (1.1). Then for any subsets V and $\{x_i \ / \ i \in I\}$ of G such that*
i) V is relatively compact,
ii) I is countable and $(x_i V) \cap (x_j V) = \emptyset$ for $i \neq j$,
we have

$$\rho(V)^{\frac{1}{p}} \left(\sum_{i \in I} \mu(x_i V)^p \right)^{\frac{1}{p}} \leq c(\mu, \ q, \ p) \rho\left(V^{-1} V\right)^{\frac{1}{q}}. \tag{1.5}$$

We show that all the necessary conditions stated in Proposition 1.1, Proposition 1.2 and Proposition 1.3 follow from Proposition 1.4.

Moreover any non-negative Radon measure μ on $\mathbb{T}$ or $\mathbb{R}^n$ satisfying the conclusion of Proposition 1.4 belongs to the space $M^{p,\,\alpha}$, $\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}$ (see Notation 3.4 and Section 4 for the definition of $M^{p,\,\alpha}$). In [7], Fofana used these spaces of measures and their subspaces $(L^q, \ l^p)^\alpha$ to express a necessary condition for Fourier multipliers. He also obtained a generalization of Hausdorff-Young inequality. For other results related to these spaces see [6], [8] and [11].
Inequality (1.2) means exactly that μ_3^2 belongs to $M^{p,\,\alpha}$ where $\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}$ (see the comment after the proof of Proposition 4.2).
Applied to the Cantor-Lebesgue measure associated to the Cantor set of constant ratio of dissection $\delta > 3$, Proposition 1.4 yields the following

Proposition 1.5. *Let $\delta > 3$ and $1 < q < p < \infty$. Assume that*

$$\left\| \mu_\delta^2 * f \right\|_{L^p(\mathbb{T})} \leq c\left(\mu_\delta^2, \ p, \ q\right) \|f\|_{L^q(\mathbb{T})}, \qquad f \in L^q(\mathbb{T}).$$

Then

$$p \leq \frac{\log\left(\frac{\delta}{2}\right)}{\log\left(\frac{\delta}{3}\right)} q \tag{1.6}$$

and

$$\frac{1}{q} + \left(1 - \frac{\log 2}{\log \delta}\right)\left(1 - \frac{1}{p}\right) \leq 1. \tag{1.7}$$

Notice that (1.6) is stronger than (1.7) if $q > \frac{\log 3}{\log 2}$.
The remainder of this paper is organized as follows: in Section 2 we prove Proposition 1.4 and apply it to $G = \mathbb{R}^n$ in Section 3. In Section 4 we examine the case $G = \mathbb{T}$.

2. **Proof of Proposition 1.4**

Proof. Let V be a relatively compact subset of G. Then $f = \chi_{V^{-1}V}$ belongs to $L^q(G,\ \rho)$. We have, for all $i \in I$ and all $x \in x_i V$,

$$\mu * f(x) = \int_G f\left(y^{-1}x\right) d\mu(y) \geq \int_{x_i V} f\left(y^{-1}x\right) d\mu(y),$$

$$y \in x_i V \Longrightarrow y^{-1}x \in V^{-1}V \quad \text{and} \quad f\left(y^{-1}x\right) = 1$$

and therefore $\mu * f(x) \geq \mu(x_i V)$. It follows that

$$\int_G (\mu * f(x))^p \, d\rho(x) \geq \sum_{i \in I} \int_{x_i V} (\mu * f(x))^p \, d\rho(x) \geq \sum_{i \in I} \mu(x_i V)^p \, \rho(x_i V).$$

Therefore

$$\begin{aligned} \rho(V)^{\frac{1}{p}} \left(\sum_{i \in I} \mu(x_i V)^p\right)^{\frac{1}{p}} &\leq \|\mu * f\|_{L^p(G,\ \rho)} \\ &\leq c(\mu,\ q,\ p)\, \|f\|_{L^q(G,\ \rho)} \\ &= c(\mu,\ q,\ p)\rho\left(V^{-1}V\right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof. $\square$

3. Case $G = \mathbb{R}^n$

Notation 3.1. Let R be a rectangle in $\mathbb{R}^n$ with sides $a_i v_i$, $i = 1, \dots, n$, where $(v_i)_{1\le i\le n}$ is a direct orthonormal basis in $\mathbb{R}^n$ and $a_i > 0$, $i = 1, \dots, n$.
For any $r > 0$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, set

$$R_k^r = \left\{ \sum_{i=1}^{n} (k_i r a_i + x_i)\, v_i \;/\; 0 \le x_i < r a_i,\ i = 1, \dots, n \right\}.$$

In other words, R_k^r is a rectangle which i-th edge is parallel to the vector v_i and of length $r a_i$. Notice that for $r > 0$, the family $\{R_k^r \;/\; k \in \mathbb{Z}^n\}$ is a partition of $\mathbb{R}^n$.

Proposition 3.2. *Let $1 \le q \le p < \infty$. If a non-negative Radon measure μ on $\mathbb{R}^n$ satisfies (1.1), then for all rectangles R in $\mathbb{R}^n$*

$$\sup_{r>0} \left(r^n |R|\right)^{\frac{1}{\alpha}-1} \left(\sum_{k\in\mathbb{Z}^n} \mu\left(R_k^r\right)^p \right)^{\frac{1}{p}} \le c(\mu,\ q,\ p) 2^{\frac{n}{q}}$$

where $\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}$.

Proof. Let $r > 0$. Notice that for every $k \in \mathbb{Z}^n$ we have that $R_k^r = R_0 + u_k$, where $R_0 = \left\{ \sum_{i=1}^{n} x_i v_i \;/\; 0 \le x_i < r a_i,\ i = 1, \dots, n \right\}$ and $u_k = \sum_{i=1}^{n} k_i r a_i v_i$.
It follows from Proposition 1.4 that

$$|R_0 - R_0|^{-\frac{1}{q}} |R_0|^{\frac{1}{p}} \left(\sum_{k\in\mathbb{Z}^n} \mu\left(R_k^r\right)^p \right)^{\frac{1}{p}} \le c(\mu,\ q,\ p).$$

Since $|R_0| = r^n |R|$, we have

$$2^{-\frac{n}{q}} \left(r^n |R|\right)^{-\frac{1}{q}} \left(r^n |R|\right)^{\frac{1}{p}} \left(\sum_{k\in\mathbb{Z}^n} \mu\left(R_k^r\right)^p \right)^{\frac{1}{p}} \le c(\mu,\ q,\ p).$$

Hence

$$\left(r^n |R|\right)^{\frac{1}{\alpha}-1} \left(\sum_{k\in\mathbb{Z}^n} \mu\left(R_k^r\right)^p \right)^{\frac{1}{p}} \le c(\mu,\ q,\ p) 2^{\frac{n}{q}}.$$

The assertion follows. □

Remark 3.3. Proposition 1.3 is a direct consequence of Proposition 3.2. In fact, suppose that μ satisfies (1.1) and let R be any rectangle. As $\left\{R_k^1 \ / \ k \in \mathbb{Z}^n\right\}$ is a partition of $\mathbb{R}^n$, $R \subset \underset{k\in M}{\cup} R_k^1$, where M is a subset of $\mathbb{Z}^n$ which number of elements does not exceed 2^n. So $\mu(R) \leq \sum_{k\in M} \mu(R_k^1)$ and by Hölder inequality we have

$$\begin{aligned} |R|^{\frac{1}{p}-\frac{1}{q}} \mu(R) &\leq 2^{\frac{n(p-1)}{p}} |R|^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{k\in M} \mu\left(R_k^1\right)^p\right)^{\frac{1}{p}} \\ &\leq 2^{\frac{n(p-1)}{p}} |R|^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{k\in \mathbb{Z}^n} \mu\left(R_k^1\right)^p\right)^{\frac{1}{p}} \\ &\leq 2^{\frac{n(p-1)}{p}} \sup_{r>0} \left(r^n |R|\right)^{\frac{1}{\alpha}-1} \left(\sum_{k\in \mathbb{Z}^n} \mu\left(R_k^r\right)^p\right)^{\frac{1}{p}} \end{aligned}$$

where $\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}$. Thus, by Proposition 3.2 we obtain

$$|R|^{\frac{1}{p}-\frac{1}{q}} \mu(R) \leq 2^{n\left(1-\frac{1}{p}+\frac{1}{q}\right)} c(\mu,\ q,\ p).$$

Notation 3.4. For any $k \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$ and $r > 0$, set

$$I_k^r = \prod_{i=1}^n \left[k_i r,\ (k_i+1)\, r\right) \quad \text{and} \quad J_x^r = \prod_{i=1}^n \left(x_i - \frac{r}{2},\ x_i + \frac{r}{2}\right).$$

Let M^0 denote the space of Radon measures (not necessarily non-negative) on $\mathbb{R}^n$. For $\mu \in M^0$, $|\mu|$ stands for its total variation. Let $1 \leq \alpha,\ p \leq \infty$. For $\mu \in M^0$ and $r > 0$, we set

$$_r\|\mu\|_p = \begin{cases} \left(\sum_{k\in\mathbb{Z}^n} |\mu| \left(I_k^r\right)^p\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sup_{x\in\mathbb{R}^n} |\mu| \left(J_x^r\right) & \text{if } p = \infty \end{cases}$$

and $\|\mu\|_{p,\,\alpha} = \sup_{r>0} r^{n\left(\frac{1}{\alpha}-1\right)}\ _r\|\mu\|_p$.

We define $M^{p,\,\alpha}(\mathbb{R}^n) = \left\{\mu \in M^0 \ / \ \|\mu\|_{p,\,\alpha} < \infty\right\}$.

Another consequence of Proposition 3.2 is the following

Corollary 3.5. *Assume that $1 \le q \le p < \infty$ and μ satisfies (1.1). Then μ belongs to $M^{p,\,\alpha}(\mathbb{R}^n)$ where $\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}$.*

Proof. It follows by choosing $a_i = 1$ for $i \in \{1, \ldots, n\}$ and $(v_i)_{1 \le i \le n} = (e_i)_{1 \le i \le n}$ the usual basis of $\mathbb{R}^n$ in the definition of R_k^r in Proposition 3.2. □

4. Case $G = \mathbb{T}$

In this section we suppose that $m \ge 2$ is an integer. Let us describe the construction of the Cantor set with variable ratios of dissection and its associated Cantor-Lebesgue measure. We take the interval $[0,\ 1)$ as a model for $\mathbb{T}$. Let $\delta_t > m$ for $t = 1,\ 2,\ \ldots$. Delete from $[0,\ 1)$, $(m-1)$ left closed intervals of equal length $\frac{1}{m-1}\left(1 - \frac{m}{\delta_1}\right)$ so that the m remaining left closed intervals denoted by E_l^1, $1 \le l \le m$, are equally spaced and have the same length $\frac{1}{\delta_1}$. From each interval E_l^1, $1 \le l \le m$, delete $(m-1)$ left closed intervals of equal length $\frac{1}{(m-1)\delta_1}\left(1 - \frac{m}{\delta_1\delta_2}\right)$ so that the m remaining left closed subintervals E_l^2, $1 \le l \le m^2$, are equally spaced and have the same length $\frac{1}{\delta_1\delta_2}$. At this stage, the remaining subset of $[0,\ 1)$ is $C^m_{(\delta_1,\ \delta_2)} = \bigcup_{l=1}^{m^2} E_l^2$. By iteration, we obtain a sequence of subsets $C^m_{(\delta_1,\ \delta_2,\ \ldots\ ,\ \delta_j)} = \bigcup_{l=1}^{m^j} E_l^j$, where each E_l^j is a left closed interval of length $r_j = \prod_{t=1}^{j} \delta_t^{-1}$. $C^m_{(\delta_t)} = \bigcap_{j=1}^{\infty} C^m_{(\delta_1,\ \delta_2,\ \ldots\ ,\ \delta_j)}$ is the $(m,\ (\delta_t))$-Cantor set and the δ_t's are called its ratios of dissection. Associated to $C^m_{(\delta_t)}$ in a natural way is a probability measure $\mu^m_{(\delta_t)}$ satisfying $\mu^m_{(\delta_t)}\left(E_l^j\right) = \frac{1}{m^j}$ for $j = 1,\ 2,\ \ldots$ and for $l = 1,\ 2,\ \ldots\ ,\ m^j$. This measure is the Cantor-Lebesgue measure associated to the $(m,\ (\delta_t))$-Cantor set. When $\delta_t = \delta$, $t = 1,\ 2,\ \ldots$, we write $\mu^m_{(\delta_t)} = \mu^m_\delta$. It follows that μ^2_3 is the usual Cantor-Lebesgue measure associated to the middle third Cantor set. For a detailed exposition on Cantor sets see Zygmund [19].

Notice that if μ is a non-negative Radon measure on $\mathbb{T}$, then in a natural way, we may identify μ with a non-negative Radon measure ν on $\mathbb{R}$ having support in the interval $[0,\ 1)$. In addition, we have the following result established by Ritter in [17].

Proposition 4.1. *Let $1 \leq q \leq p < \infty$, and suppose there is a constant $K > 0$ such that*

$$\|\mu * f\|_{L^p(\mathbb{T})} \leq K \|f\|_{L^q(\mathbb{T})}, \qquad f \in L^q(\mathbb{T}).$$

Then there is a constant $K_0 > 0$ such that

$$\|\nu * f\|_{L^p(\mathbb{R})} \leq K_0 \|f\|_{L^q(\mathbb{R})}, \qquad f \in L^q(\mathbb{R}).$$

Defining, for $1 \leq \alpha,\ p \leq \infty$,

$$M^{p,\,\alpha}(\mathbb{T}) = \{\mu \in M^{p,\,\alpha}(\mathbb{R}) \ / \ \mathrm{supp}(\mu) \subset [0,\ 1)\}$$

where $\mathrm{supp}(\mu)$ denotes the support of μ, it is easy to see that Corollary 3.5 holds in this setting.
The following result gives a characterization of measures $\mu^m_{(\delta_t)}$ which belong to $M^{p,\,\alpha}(\mathbb{T})$.

Proposition 4.2. *Let $\delta_t > m$, $t = 1,\ 2,\ \ldots$. Assume that $1 < \alpha \leq p < \infty$. Then $\mu^m_{(\delta_t)}$ belongs to $M^{p,\,\alpha}(\mathbb{T})$ if and only if there exists a constant $c > 0$ such that*

$$\prod_{t=1}^{j} \delta_t \leq c m^{\frac{\alpha(p-1)j}{p(\alpha-1)}}, \qquad j = 1,\ 2,\ \ldots .$$

In particular, the Cantor-Lebesgue measure μ^m_δ of constant ratio of dissection δ belongs to $M^{p,\,\alpha}(\mathbb{T})$ if and only if

$$1 - \frac{1}{\alpha} - \frac{\log m}{\log \delta}\left(1 - \frac{1}{p}\right) \leq 0. \tag{4.1}$$

Proof. a) For all $r \geq 1$

$$r^{\frac{1}{\alpha}-1}\ {}_r\left\|\mu^m_{(\delta_t)}\right\|_p = r^{\frac{1}{\alpha}-1} \leq 1.$$

b) Let j be a positive integer and $r_j = \prod_{t=1}^{j} \delta_t^{-1}$. Recall that for $l = 1,\ 2,\ \ldots,\ m^j$, $\left|E^j_l\right| = r_j$ and $\mu^m_{(\delta_t)}\left(E^j_l\right) = \frac{1}{m^j}$. For each fixed l, put $K_l = \left\{k \in \mathbb{N} \ / \ E^j_l \cap I^{r_j}_k \neq \emptyset\right\}$. Then K_l has at most 2 elements. In the same way, for each fixed k in $\mathbb{N}$ set $L_k = \left\{l \in \{1,\ 2,\ \ldots,\ m^j\} \ / \ E^j_l \cap I^{r_j}_k \neq \emptyset\right\}$.

Then the number of elements of L_k is at most 2. We have

$$\begin{aligned} m^j m^{-jp} &= \sum_{l=1}^{m^j} \mu^m_{(\delta_t)} \left(E^j_l\right)^p \\ &= \sum_{l=1}^{m^j} \left(\sum_{k\in K_l} \mu^m_{(\delta_t)} \left(E^j_l \cap I^{r_j}_k\right)\right)^p \\ &\leq 2^{p-1} \sum_{l\in L_k} \sum_{k\in K_l} \mu^m_{(\delta_t)} \left(E^j_l \cap I^{r_j}_k\right)^p \\ &= 2^{p-1} \sum_{k\in\mathbb{N}} \sum_{l\in L_k} \mu^m_{(\delta_t)} \left(E^j_l \cap I^{r_j}_k\right)^p \\ &\leq 2^p \sum_{k\in\mathbb{N}} \mu^m_{(\delta_t)} \left(I^{r_j}_k\right)^p . \end{aligned}$$

Then

$$\left(r_j^{\frac{1}{j}\left(\frac{1}{\alpha}-1\right)} m^{\frac{1}{p}-1} \right)^j = \left(\prod_{t=1}^{j} \delta_t \right)^{1-\frac{1}{\alpha}} m^{-j\left(1-\frac{1}{p}\right)} \leq 2 r_j^{\frac{1}{\alpha}-1} \ {}_{r_j}\left\| \mu^m_{(\delta_t)} \right\|_p .$$

c) Let $r \in (0,\ 1)$. There exists an integer $j \geq 1$ such that $r_j \leq r < r_{j-1}$ where $r_0 = 1$ and $r_n = \prod_{t=1}^{n} \delta_t^{-1}$ for $n \geq 1$. Furthermore, each I^r_k intersects at most m intervals E^j_l. So $\mu^m_{(\delta_t)}(I^r_k) \leq m^{-j} m$. The number of I^r_k which intersect the intervals E^j_l is at most $2m^j$. It follows that

$$\sum_{k\in\mathbb{N}} \mu^m_{(\delta_t)} (I^r_k)^p \leq 2 m^{j(1-p)} m^p .$$

Hence

$$\begin{aligned} r^{\frac{1}{\alpha}-1} \ {}_r\left\| \mu^m_{(\delta_t)} \right\|_p &\leq 2^{\frac{1}{p}} r^{\frac{1}{\alpha}-1} m^{j\left(\frac{1}{p}-1\right)} m \\ &\leq 2^{\frac{1}{p}} r_j^{\frac{1}{\alpha}-1} m^{j\left(\frac{1}{p}-1\right)} m \\ &= 2^{\frac{1}{p}} m \left(\left(\prod_{t=1}^{j} \delta_t \right)^{\frac{1}{j}\left(1-\frac{1}{\alpha}\right)} m^{\frac{1}{p}-1} \right)^j . \end{aligned}$$

Finally,

$$\mu^m_{(\delta_t)} \in M^{p,\,\alpha}(\mathbb{T}) \iff \sup_j \left(\left(\prod_{t=1}^{j} \delta_t \right)^{\frac{1}{j}\left(1-\frac{1}{\alpha}\right)} m^{\frac{1}{p}-1} \right)^j < \infty$$

$$\mu^m_{(\delta_t)} \in M^{p,\,\alpha}(\mathbb{T}) \iff \prod_{t=1}^{j} \delta_t \leq c m^{\frac{\alpha(p-1)j}{p(\alpha-1)}}, \quad j = 1,\ 2,\ ...$$

where c is a positive constant not depending on j.
d) Now, let $\delta_t = \delta$ for all $t \geq 1$. From c) we know that:

$$\mu^m_\delta \in M^{p,\,\alpha}(\mathbb{T}) \iff \delta^j \leq c m^{\frac{\alpha(p-1)j}{p(\alpha-1)}}, \qquad j = 1,\ 2, ...$$

where c is a positive constant not depending on j. That means:

$$\mu^m_\delta \in M^{p,\,\alpha}(\mathbb{T}) \iff \log \delta \leq \tfrac{\alpha(p-1)}{p(\alpha-1)} \log m$$

$$\mu^m_\delta \in M^{p,\,\alpha}(\mathbb{T}) \iff 1 - \tfrac{1}{\alpha} - \tfrac{\log m}{\log \delta}(1 - \tfrac{1}{p}) \leq 0.$$

□

Notice that for $1 - \frac{1}{\alpha} = \frac{1}{q} - \frac{1}{p}$, (4.1) reduces to (1.2) when $m = 2$ and $\delta = 3$.

Proposition 4.3. *Let $\mu^m_{(\delta_t)}$ be the Cantor-Lebesgue measure with variable ratios $\delta_t > m$ of dissection. Let $1 < q < p < \infty$. Assume that*

$$\left\| \mu^m_{(\delta_t)} * f \right\|_{L^p(\mathbb{T})} \leq c\left(\mu^m_{(\delta_t)},\ p,\ q \right) \|f\|_{L^q(\mathbb{T})}, \qquad f \in L^q(\mathbb{T}).$$

Then there exists a constant $c > 0$ such that

$$\prod_{t=1}^{j} \delta_t \leq c m^{\frac{q(p-1)j}{p-q}}, \qquad j = 1,\ 2,\ ...\ .$$

In particular, if $\delta_t = \delta$ for all $t \geq 1$, then

$$\frac{1}{q} + \left(1 - \frac{\log m}{\log \delta}\right)\left(1 - \frac{1}{p}\right) \leq 1.$$

Proof. Let $1 - \frac{1}{\alpha} = \frac{1}{q} - \frac{1}{p}$. Then the desired result follows from Corollary 3.5 and Proposition 4.2. □

Proposition 1.1 is obtained from Proposition 4.3 by taking $m = 2$ and $\delta_t = 3$ for all $t \geq 1$.

Proof of Proposition 1.5. We are in the case $m = 2$ and $\delta_t = \delta > 3$ for all $t \geq 1$. Let j be a positive integer. Observe that for any non-negative integer k, any $l \in \left\{1,\ 2,\ \dots,\ 2^{j+k}\right\}$, $E_l^{j+k} = x_l^{j+k} + \left[0,\ \delta^{-j-k}\right)$ and $\mu_\delta^2\left(E_l^{j+k}\right) = 2^{-j-k}$. Set $A_0 = [0,\ \delta^{-j})$ and $B_0 = A_0 - A_0 = (-\delta^{-j},\ \delta^{-j})$. From Proposition 1.4 we obtain

$$|B_0|^{-\frac{1}{q}} |A_0|^{\frac{1}{p}} \left(\sum_{l=1}^{2^j} \mu_\delta^2\left(E_l^j\right)^p\right)^{\frac{1}{p}} \leq c\left(\mu_\delta^2,\ p,\ q\right).$$

Observe that for fixed l in $\{1,\ 2,\ \dots,\ 2^j\}$, E_l^j contains two intervals $E_{l_1}^{j+1}$ and $E_{l_2}^{j+1}$ satisfying

$$\mu_\delta^2\left(E_l^j\right) = \mu_\delta^2\left(E_{l_1}^{j+1} \cup E_{l_2}^{j+1}\right)$$

and

$$E_{l_1}^{j+1} \cup E_{l_2}^{j+1} = x_l^j + \left(\left[0,\ \delta^{-j-1}\right) \cup \left[\delta^{-j} - \delta^{-j-1},\ \delta^{-j}\right)\right).$$

Setting $A_1 = [0,\ \delta^{-j-1}) \cup [\delta^{-j} - \delta^{-j-1},\ \delta^{-j})$ and applying Proposition 1.4 we obtain

$$|A_1 - A_1|^{-\frac{1}{q}} |A_1|^{\frac{1}{p}} \left(\sum_{l=1}^{2^j} \mu_\delta^2\left(E_l^j\right)^p\right)^{\frac{1}{p}} \leq c\left(\mu_\delta^2,\ p,\ q\right).$$

But each preceding interval $E_{l_i}^{j+1}$, $i \in \{1,\ 2\}$, contains two intervals $E_{l_{i,1}}^{j+2}$ and $E_{l_{i,2}}^{j+2}$ such that

$$\mu_\delta^2\left(E_{l_i}^{j+1}\right) = \mu_\delta^2\left(E_{l_{i,1}}^{j+2} \cup E_{l_{i,2}}^{j+2}\right) = \frac{1}{2^{j+1}}.$$

Moreover $\bigcup_{i=1}^{2}\left(E_{l_{i,1}}^{j+2} \cup E_{l_{i,2}}^{j+2}\right) = x_l^j + A_2$ where

$$\begin{aligned} A_2 \ = \ & \left[0,\ \delta^{-j-2}\right) \cup \left[\delta^{-j-1} -\ \delta^{-j-2},\ \delta^{-j-1}\right) \cup \\ & \cup \left[\delta^{-j} -\ \delta^{-j-1},\ \delta^{-j} -\ \delta^{-j-1} + \delta^{-j-2}\right) \cup \left[\delta^{-j} - \delta^{-j-2},\ \delta^{-j}\right). \end{aligned}$$

This remark enables us to apply again Proposition 1.4. Thus we obtain

$$|A_2 - A_2|^{-\frac{1}{q}} |A_2|^{\frac{1}{p}} \left(\sum_{l=1}^{2^j} \mu_\delta^2\left(E_l^j\right)^p\right)^{\frac{1}{p}} \leq c\left(\mu_\delta^2,\ p,\ q\right).$$

The iteration of the process leads us to two sequences of sets $(A_k)_{k\geq 0}$ and $\left(\widetilde{A_k}\right)_{k\geq 0}$ defined by:

$$A_{k+1} = \frac{1}{\delta}A_k \cup \left(\delta^{-j} - \frac{1}{\delta}\widetilde{A_k}\right), \quad \widetilde{A_{k+1}} = \frac{1}{\delta}\widetilde{A_k} \cup \left(\delta^{-j} - \frac{1}{\delta}A_k\right)$$

with $A_0 = [0,\ \delta^{-j})$, $\widetilde{A_0} = (0,\ \delta^{-j}]$ and satisfying

$$|B_k|^{-\frac{1}{q}} |A_k|^{\frac{1}{p}} \left(\sum_{l=1}^{2^j} \mu_\delta^2 \left(E_l^j\right)^p\right)^{\frac{1}{p}} \leq c\left(\mu_\delta^2,\ p,\ q\right), \tag{4.2}$$

where $B_k = A_k - A_k$ for all $k \geq 0$.
Notice that $A_0 - A_0 = \widetilde{A_0} - \widetilde{A_0}$ and $|A_0| = \left|\widetilde{A_0}\right|$. Furthermore, for any $k \geq 0$, clearly $A_{k+1} - A_{k+1} = \widetilde{A_{k+1}} - \widetilde{A_{k+1}}$ and since $\frac{1}{\delta}A_k \cap \left(\delta^{-j} - \frac{1}{\delta}\widetilde{A_k}\right) = \emptyset = \frac{1}{\delta}\widetilde{A_k} \cap \left(\delta^{-j} - \frac{1}{\delta}A_k\right)$ we have $|A_{k+1}| = \left|\widetilde{A_{k+1}}\right|$. Thus

$$A_k - A_k = \widetilde{A_k} - \widetilde{A_k} \quad \text{and} \quad |A_k| = \left|\widetilde{A_k}\right|, \quad k \geq 0. \tag{4.3}$$

Observe that: $|A_0| = \delta^{-j}$, $|A_1| = 2\delta^{-j-1}$ and $|A_2| = 2^2\delta^{-j-2}$. Suppose that for some integer $k \geq 0$, $|A_k| = 2^k\delta^{-j-k}$. By the preceding remarks we get $|A_{k+1}| = \frac{1}{\delta}|A_k| + \frac{1}{\delta}\left|\widetilde{A_k}\right| = \frac{2}{\delta}|A_k| = 2^{k+1}\delta^{-j-(k+1)}$. We conclude that

$$|A_k| = 2^k\delta^{-j-k}, \quad k \geq 0.$$

Notice that $A_0 + \widetilde{A_0} = (0,\ 2\delta^{-j}) = \delta^{-j} - (-\delta^{-j},\ \delta^{-j}) = \delta^{-j} - (A_0 - A_0) = \delta^{-j} - B_0$. Furthermore, for any $k \geq 0$, on the one hand

$$\begin{aligned}
B_{k+1} &= \left[\frac{1}{\delta}A_k \cup \left(\delta^{-j} - \frac{1}{\delta}\widetilde{A_k}\right)\right] - \left[\frac{1}{\delta}A_k \cup \left(\delta^{-j} - \frac{1}{\delta}\widetilde{A_k}\right)\right] \\
&= \frac{1}{\delta}(A_k - A_k) \cup \left(\frac{1}{\delta}\left(A_k + \widetilde{A_k}\right) - \delta^{-j}\right) \cup \\
&\quad \cup \left(\delta^{-j} - \frac{1}{\delta}\left(\widetilde{A_k} + A_k\right)\right) \cup \frac{1}{\delta}\left(\widetilde{A_k} - \widetilde{A_k}\right) \\
&= \frac{1}{\delta}(A_k - A_k) \cup \left(\frac{1}{\delta}\left(A_k + \widetilde{A_k}\right) - \delta^{-j}\right) \cup \left(\delta^{-j} - \frac{1}{\delta}\left(\widetilde{A_k} + A_k\right)\right) \\
&\quad \text{(because of (4.3))}
\end{aligned}$$

and on the other hand

$$\begin{aligned} A_{k+1}+\widetilde{A_{k+1}} &= \frac{1}{\delta}\left(A_k+\widetilde{A_k}\right)\cup\left(\delta^{-j}+\frac{1}{\delta}\left(A_k-A_k\right)\right)\cup \\ &\cup\left(\delta^{-j}+\frac{1}{\delta}\left(\widetilde{A_k}-\widetilde{A_k}\right)\right)\cup\left(2\delta^{-j}-\frac{1}{\delta}\left(\widetilde{A_k}+A_k\right)\right) \\ &= \frac{1}{\delta}\left(A_k+\widetilde{A_k}\right)\cup\left(\delta^{-j}+\frac{1}{\delta}\left(A_k-A_k\right)\right)\cup \\ &\cup\left(2\delta^{-j}-\frac{1}{\delta}\left(\widetilde{A_k}+A_k\right)\right) \qquad \text{(because of (4.3))} \end{aligned}$$

and so $A_{k+1}+\widetilde{A_{k+1}}=\delta^{-j}-B_{k+1}$. Thus

$$|A_k-A_k|=\left|A_k+\widetilde{A_k}\right|, \qquad k\geq 0. \tag{4.4}$$

Notice that for all $k\geq 0$, the sets $\frac{1}{\delta}\left(A_k-A_k\right)$, $\frac{1}{\delta}\left(A_k+\widetilde{A_k}\right)-\delta^{-j}$ and $\delta^{-j}-\frac{1}{\delta}\left(\widetilde{A_k}+A_k\right)$ form a partition of B_{k+1}. Thus, by (4.4) we have $|B_{k+1}|=\frac{3}{\delta}\left|A_k-A_k\right|=\frac{3}{\delta}\left|B_k\right|$, $k\geq 0$. As $|B_0|=2\delta^{-j}$, we conclude that for all $k\geq 0$, $|B_k|=\left(\frac{3}{\delta}\right)^k 2\delta^{-j}$.
Finally, using inequality (4.2) we get:

$$\left(2\left(\frac{3}{\delta}\right)^k\delta^{-j}\right)^{-\frac{1}{q}}\left(2^k\delta^{-j-k}\right)^{\frac{1}{p}}2^{j\left(\frac{1}{p}-1\right)}\leq c\left(\mu_\delta^2,\ p,\ q\right), \qquad k\geq 0,\ j\geq 1$$

$$2^{-\frac{1}{q}}\left(3^{-\frac{1}{q}}\delta^{\frac{1}{q}-\frac{1}{p}}2^{\frac{1}{p}}\right)^k\left(\delta^{\frac{1}{q}-\frac{1}{p}}2^{\frac{1}{p}-1}\right)^j\leq c\left(\mu_\delta^2,\ p,\ q\right), \qquad k\geq 0,\ j\geq 1$$

$$3^{-\frac{1}{q}}\delta^{\frac{1}{q}-\frac{1}{p}}2^{\frac{1}{p}}\leq 1 \quad \text{and} \quad \delta^{\frac{1}{q}-\frac{1}{p}}2^{\frac{1}{p}-1}\leq 1$$

$$p\leq\frac{\log\left(\frac{\delta}{2}\right)}{\log\left(\frac{\delta}{3}\right)}q \quad \text{and} \quad \frac{1}{q}+\left(1-\frac{\log 2}{\log\delta}\right)\left(1-\frac{1}{p}\right)\leq 1.$$

□

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