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Trace Theorems for Sobolev Spaces on Lipschitz Domains. Necessary Conditions

Giuseppe Geymonat

Abstract

A famous theorem of E. Gagliardo gives the characterization of traces for Sobolev spaces $W^{1,p}(\Omega)$ for $1 \leq p < \infty$ when $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain. The extension of this result to $W^{m,p}(\Omega)$ for $m \geq 2$ and $1 < p < \infty$ is now well-known when Ω is a smooth domain. The situation is more complicated for polygonal and polyhedral domains since the characterization is given only in terms of local compatibility conditions at the vertices, edges, ... Some recent papers give the characterization for general Lipschitz domains for m=2 in terms of global compatibility conditions. Here we give the necessary compatibility conditions for $m \geq 3$ and we prove how the local compatibility conditions can be derived.

1. Introduction

Let Ω be a Lipschitz bounded and connected subset of $\mathbb{R}^N$ whose bounded and orientable boundary is denoted by Γ. For $1 \leq p < \infty$ and m integer $W^{m,p}(\Omega)$ denotes the Sobolev space of functions of $L^p(\Omega)$ whose distributional derivatives up to the order m also belong to $L^p(\Omega)$. For $m \geq 1$ the restriction $\gamma_0(u) = u_{|\Gamma}$ to Γ of a function $u \in W^{m,p}(\Omega)$ is well-defined and belongs to $L^p(\Gamma)$. A famous result of E. Gagliardo [6] gives, for $m = 1$, the caracterization of the range of γ_0. More precisely, Gagliardo proves that the operator γ_0 is linear and continuous from $W^{1,p}(\Omega)$ onto $W^{1-1/p,p}(\Gamma)$ for $1 \leq p < \infty$ and has a continuous right inverse for $p > 1$.

When $u \in W^{2,p}(\Omega)$ then $\frac{\partial u}{\partial x_j} \in W^{1,p}(\Omega)$ for $j = 1, \dots, N$; therefore $\gamma_1(u) = \frac{\partial u}{\partial \mathbf{n}} = \sum_{j=1}^{N} \gamma_0\left(\frac{\partial u}{\partial x_j}\right) n_j \in L^p(\Gamma)$ since $\mathbf{n} = (n_1, \dots, n_N)$ is defined almost everywhere and belongs to $(L^\infty(\Gamma))^N$. J. Nečas [9] proves that $\gamma_0(u) \in W^{1,p}(\Gamma)$ and that the map $u \longrightarrow (\gamma_0(u), \gamma_1(u))$ is linear and continuous from $W^{2,p}(\Omega)$ into $W^{1,p}(\Gamma) \times L^p(\Gamma)$. A natural question is to characterize the range of the map (γ_0, γ_1). A first answer has been obtained for polygonal-type domains $\Omega \subset \mathbb{R}^2$ by Kondratev and Grisvard (see e.g. [8] for full references) in terms of *compatibility conditions at* *the corners* and then the results have been extended to polyhedral-type domains ($N = 3$). These characterizations have been extensively used in order to give regularity results for different types of boundary-value problems.

For general Lipschitz domains a first characterization of the range of (γ_0, γ_1) has been obtained for $N = 2$ and $p = 2$ in [7] as a byproduct of the study of the Airy function; this result has been extended to general $p > 1$ by [5]. During a visit at the Istituto di Analisi Numerica del CNR in Pavia the following equivalent statement came out after some discussions the with F. Brezzi and A. Buffa.

Theorem 1.1. *The range of* (γ_0, γ_1) *is the set of* $(f_0, f_1) \in W^{1,p}(\Gamma) \times L^p(\Gamma)$ *such that:*

$$\frac{\partial f_0}{\partial t}\boldsymbol{t} + f_1 \boldsymbol{n} \in (W^{1-1/p,p}(\Gamma))^2. \tag{1.1}$$

A first consequence of (1.1) has been a general characterization of the range of (γ_0, γ_1) for $N = 3$ (see [2]) that also works for all $N \geq 3$.

Let us remark that the compatibility conditions at a corner follow from the characterization of $W^{1-1/p,p}(\Gamma)$ and the *exchange of* $\mathbf{n}$ *and* $\mathbf{t}$ at the crossing of the corner.

The statement of the theorem 1.1 allows an easy interpretation of the necessity of the condition (1.1). Indeed, when $u \in W^{2,p}(\Omega)$ then $grad\, u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right) \in \left(W^{1,p}(\Omega)\right)^2$ and $\gamma_0(grad\, u) \in \left(W^{1-1/p,p}(\Gamma)\right)^2$. Hence the necessity of (3.1) follows from

$$\gamma_0(grad\, u) = \frac{\partial \gamma_0(u)}{\partial t}\mathbf{t} + \gamma_1(u)\mathbf{n}. \tag{1.2}$$

In this paper we give general necessary conditions for the traces of the elements of $W^{m,p}(\Omega)$ for all integer $m \geq 2$, all $p > 1$ and all $N \geq 2$. The proof of [7] can be adapted in order to prove that these conditions are also sufficient when $N = 2$ and $p = 2$.

The author is indebted to many people. At first to F. Krasucki whose questions about the mechanical meaning of Grisvard results on the traces of Airy functions were the starting point for the extension of the Gagliardo theorem to $W^{m,p}(\Omega)$ in the case m=2 and p=2 for 2-dimensional domains [7]. During a visit to the IAN of the CNR the discussions with F. Brezzi and particularly with A. Buffa allowed the extension to the case m=2 and $1 < p < \infty$ for general N-dimensional domains [2]. At last the discussions

with F. Murat during the Colloque were stimulating to obtain the actual formulation of the conditions for m=3.

2. **Preliminaries**

Let Ω be a Lipschitz bounded and connected subset of $\mathbb{R}^N$ whose bounded and orientable boundary is denoted by Γ. This means (see [9], [8] and [1]) that for every $x \in \Gamma$ there exists a neighborhood $V(x)$ of x in $\mathbb{R}^N$ and a new orthonormal coordinate system $\{y_1, ..., y_N\}$ such that $V(x) = \prod_{i=1}^N]-\alpha_i, \alpha_i[= V'(x) \times]-\alpha_N, \alpha_N[$ and there exists a Lipschitz continuous function $\varphi : V'(x) \longrightarrow]-\alpha_N, \alpha_N[$ such that $|\varphi(y')| \le \alpha_N/2$ for every $y' = (y_1, ..., y_{N-1}) \in V'(x)$ and such that $\Omega \cap V(x) = \{y = (y', y_N) \in V(x) \mid y_N > \varphi(y')\}$ and $\Gamma \cap V(x) = \{y = (y', y_N) \in V(x) \mid y_N = \varphi(y')\}$.

Since Γ is compact there exists M open and connected subsets Γ_i such that $\Gamma = \bigcup_{i=1}^M \Gamma_i$ and there exists M points $a_i \in \Gamma_i$ and M Lipschitz continuous function φ_i such that $\Gamma_i = \Gamma \cap V(a_i)$. The induced parametrization (φ_i, Γ_i) of Γ is defined for $i = 1, \dots, M$ by

$$y' = (y_1, ..., y_{N-1}) \longmapsto (y', \varphi_i(y')) .$$

This parametrization induces $N-1$ linearly independent tangent vectors defined a.e. on Γ_i:

$$\mathbf{t}_k = \left(\mathbf{e}_k^{N-1}, \, \partial\varphi_i/\partial y_k\right) \, , \; k = 1, \dots, N-1$$

where $\mathbf{e}_k^{N-1} = (\delta_{h,k})_{h=1,...,N-1}$. A corresponding set of orthonormalized vectors $\mathbf{T}^k = \left(T_h^k\right)_{h=1,...,N} \in (L^\infty(\Gamma))^N$ can defined as follows for $k = 1$:

$$\mathbf{T}^1 = \frac{-1}{\sqrt{1 + (\partial\varphi_i/\partial y_1)^2}} \mathbf{t}_1 \tag{2.1}$$

and for $k = 2, \dots, N-1$,

$$T_h^k = \frac{-\partial\varphi_i/\partial y_h}{\sqrt{1 + \sum_{l=1}^{k-1} (\partial\varphi_i/\partial y_l)^2}} \frac{\partial\varphi_i/\partial y_k}{\sqrt{1 + \sum_{l=1}^{k} (\partial\varphi_i/\partial y_l)^2}} \, , \; h = 1, \dots, k-1, \tag{2.2}$$

$$T_k^k = \frac{\sqrt{1 + \sum_{l=1}^{k-1} (\partial\varphi_i/\partial y_l)^2}}{\sqrt{1 + \sum_{l=1}^{k} (\partial\varphi_i/\partial y_l)^2}} \tag{2.3}$$

$$T_h^k = 0 \ , \ h = k, \dots, N-1 \tag{2.4}$$

$$T_N^k = \frac{1}{\sqrt{1+\sum_{l=1}^{k-1}\left(\partial\varphi_i/\partial y_l\right)^2}} \frac{\partial\varphi_i/\partial y_k}{\sqrt{1+\sum_{l=1}^{k}\left(\partial\varphi_i/\partial y_l\right)^2}}. \tag{2.5}$$

At a.e. point $x = (y', \varphi_i(y')) \in \Gamma_i$ the vectors $(\mathbf{T}^1, \dots, \mathbf{T}^{N-1})$ span the tangent space $T_x\Gamma$ who is an hyperplane of $\mathbb{R}^N$. Any other orthonormal basis of $T_x\Gamma$ is obtained applying a rotation to the previous one. The unit outward normal vector $\mathbf{n} = (n_1, n_2, \dots, n_N) \in (L^\infty(\Gamma))^N$ is defined a.e. on Γ_i by

$$y' \longmapsto \frac{(\partial\varphi_i/\partial y_1, \dots, \partial\varphi_i/\partial y_{N-1}, \ -1)}{\sqrt{1+\sum_{k=1}^{N-1}\left(\partial\varphi_i/\partial y_k\right)^2}}.$$

For a.e. $x \in \Gamma$ the vectors $(\mathbf{T}^1, \dots, \mathbf{T}^{N-1}, \mathbf{n})$ are a positively oriented basis of $T_x\Omega \approx \mathbb{R}^N$. As in the case of regular domains the definitions are intrinsic since not depending from the choice of the parametrization (φ_i, Γ_i).

Since Γ is a Lipschitz-continuous manifold without boundary the Sobolev spaces $W^{s,p}(\Gamma)$ are well defined (independently from the parametrization (φ_i, Γ_i)) for $-1 \le s \le 1$ and $1 < p < \infty$. More precisely, $\psi \in L^p(\Gamma)$ means that for $i = 1, \dots, M$ one has $\psi(y', \varphi_i(y')) \in L^p(V'(x_i))$ and $\psi \in W^{1,p}(\Gamma)$ when $\psi(y', \varphi_i(y')) \in W^{1,p}(V'(x_i))$. This means that for $k = 1, \dots, N-1$ one has $\partial(\psi(y', \varphi_i(y')))/\partial y_k \in L^p(V'(x_i))$ where

$$\frac{\partial\psi(y', \varphi_i(y'))}{\partial y_k} = \left(\frac{\partial\psi}{\partial y_k}\right)(y', \varphi_i(y')) + \left(\frac{\partial\psi}{\partial y_N}\right)(y', \varphi_i(y'))\frac{\partial\varphi_i(y')}{\partial y_k}. \tag{2.6}$$

Using the tangent fields $\mathbf{T}^k$ we define the tangential derivatives

$$\partial_{\mathbf{T}^k}\psi = \sum_{h=1}^{N}\left(\frac{\partial\psi}{\partial y_h}\right)(y', \varphi_i(y'))\, T_h^k \tag{2.7}$$

and for $\psi \in W^{1,p}(\Gamma)$ the tangential vector field $\nabla_\Gamma\psi = \sum_{k=1}^{N-1}(\partial_{\mathbf{T}_k}\psi)\mathbf{T}^k$. It follows that $\nabla_\Gamma\psi$ belongs to $(L^p(\Gamma))^N$ and its definition does not depend on the parametrization (φ_i, Γ_i).

For $u \in W^{m,p}(\Omega)$, $m \ge 1$, the restriction to Γ is defined on every Γ_i by $u_{|\Gamma_i} = u(y', \varphi_i(y'))$. This restriction, denoted $\gamma_0(u) = u_{|\Gamma}$, belongs to $L^p(\Gamma)$.

When $m = 2$ then $Du = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right) \in \left(W^{1,p}(\Omega)\right)^N$. Hence it follows from (2.6) that $\gamma_0(u) \in W^{1,p}(\Gamma)$ since for $i = 1, \dots, M$ on Γ_i one has for $k = 1, \ldots, N-1$:

$$\frac{\partial \gamma_0(u)\left(y', \varphi_i\left(y'\right)\right)}{\partial y_k} = \gamma_0\left(\frac{\partial u}{\partial y_k}\right) + \gamma_0\left(\frac{\partial u}{\partial y_N}\right)\frac{\partial \varphi_i}{\partial y_k} \in L^p\left(\Gamma_i\right). \tag{2.8}$$

The exterior normal derivative of u on Γ, denoted $\gamma_1(u)$, is defined a.e. on Γ_i by:

$$\left(\frac{\partial u}{\partial \mathbf{n}}\right)\left(y', \varphi_i\left(y'\right)\right) = \sum_{j=1}^{N}\left(\frac{\partial u}{\partial y_j}\right)\left(y', \varphi_i\left(y'\right)\right) n_j\left(y'\right). \tag{2.9}$$

The normal derivative $\gamma_1(u)$ so defined belongs to $L^p(\Gamma)$ since the unit normal vector $\mathbf{n} = (n_1, \ldots, n_N)$ belongs to $\left(L^\infty(\Gamma)\right)^N$.

In a similar way, when $m = 3$, $Du \in \left(W^{2,p}(\Omega)\right)^N$ and the hessian $Hu = \left(D^\alpha u\right)_{|\alpha|=2}$ belongs to $W^{1,p}\left(\Omega; M^N_{sym}\right)$ where M^N_{sym} denotes the vector space of symmetric $N \times N$ matrices. The second order exterior normal derivative, denoted $\gamma_2(u)$, is defined on Γ_i by:

$$\left(\frac{\partial^2 u}{\partial \mathbf{n}^2}\right)\left(y', \varphi_i\left(y'\right)\right) = \sum_{l,j=1}^{N}\left(\frac{\partial^2 u}{\partial y_l \partial y_j}\right)\left(y', \varphi_i\left(y'\right)\right) n_l\left(y'\right) n_j\left(y'\right). \tag{2.10}$$

Hence $\gamma_2(u) \in L^p(\Gamma)$.

More generally, when $u \in W^{m,p}(\Omega)$, $m \geq 4$, all $\gamma_0\left(D^\alpha u\right) \in L^p(\Gamma)$ for $|\alpha| \leq m-1$ and then the exterior normal derivatives $\gamma_k(u)$ up to the order $k = m-1$, are defined on Γ_i by:

$$\left(\frac{\partial^k u}{\partial \mathbf{n}^k}\right)\left(y', \varphi_i\left(y'\right)\right) = \sum_{|\alpha|=k}\frac{k!}{\alpha!}\left(D^\alpha u\right)\left(y', \varphi_i\left(y'\right)\right) n^\alpha\left(y'\right) \tag{2.11}$$

for all $k \geq 1$, $\gamma_k(u) \in L^p(\Gamma)$.

3. Necessary conditions

From the previous definitions it follows that $u \longmapsto \left(\gamma_0(u), \ldots, \gamma_{m-1}(u)\right)$ is a linear and continuous map from $W^{m,p}(\Omega)$ into $W^{1,p}(\Gamma) \times \left(L^p(\Gamma)\right)^{m-1}$. A natural question is the characterization of the range of such map. When the boundary Γ of Ω is more regular (for instance of class C^∞), the extension of the Gagliardo's theorem states that, for $p > 1$, $u \longmapsto \left(\gamma_0(u), \ldots, \gamma_{m-1}(u)\right)$ is a linear and continuous map from $W^{m,p}(\Omega)$ onto $\prod_{k=0}^{m-1} W^{m-k-1/p,p}(\Gamma)$ and it has a continuous right inverse.

For polygonal-type domains (when $N = 2$) and for polyhedral-type domains (when $N = 3$) the caracterization of the range of the map $(\gamma_0, \ldots, \gamma_{m-1})$ has been obtained in terms of *compatibility conditions*.

A first step toward the characterization of the range for general Lipschitz domains is the obtention of the necessary conditions.

Proposition 3.1. *Let be $(f_0, f_1) \in W^{1,p}(\Gamma) \times L^p(\Gamma)$. A necessary condition in order that $(f_0, f_1) \in range(\gamma_0, \gamma_1)$ is:*

$$\nabla_\Gamma f_0 + f_1 \boldsymbol{n} \in (W^{1-1/p,p}(\Gamma))^N. \tag{3.1}$$

Proof. When $u \in W^{2,p}(\Omega)$ then $Du = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right) \in \left(W^{1,p}(\Omega)\right)^N$. It follows from (2.8), the definition of ∇_Γ and of $\mathbf{T}^k$ that in every point $(y', \varphi_i(y')) \in \Gamma_i$ one has for $h = 1, \ldots, N-1$:

$$(\nabla_\Gamma \gamma_0(u))_h = \frac{\partial u}{\partial y_h} - \left(\sum_{l=1}^{N-1} \frac{\partial u}{\partial y_l}\frac{\partial \varphi_i}{\partial y_l} - \frac{\partial u}{\partial y_N}\right) \frac{\partial \varphi_i / \partial y_h}{1 + \sum_{k=1}^{N-1} (\partial \varphi_i / \partial y_k)^2}$$

and

$$(\nabla_\Gamma \gamma_0(u))_N = \frac{\partial u}{\partial y_N} + \left(\sum_{l=1}^{N-1} \frac{\partial u}{\partial y_l}\frac{\partial \varphi_i}{\partial y_l} - \frac{\partial u}{\partial y_N}\right) \frac{1}{1 + \sum_{k=1}^{N-1} (\partial \varphi_i / \partial y_k)^2}.$$

Hence from (2.9) and the definition of $\mathbf{n}$ it follows that

$$\nabla_\Gamma \gamma_0(u) + \gamma_1(u)\mathbf{n} = \gamma_0(Du). \tag{3.2}$$

One obtains (3.1) since $\gamma_0(Du) \in (W^{1-1/p,p}(\Gamma))^N$. $\square$

Remark 3.2. For Lipschitz domains the two terms of the sum in (3.1) belong separately only to $(L^p(\Gamma))^N$.

When $m = 3$, then $Du \in \left(W^{2,p}(\Omega)\right)^N$ and hence $\gamma_0(Du) \in (W^{1,p}(\Gamma))^N$. From the previous proposition it then follows the following result.

Lemma 3.3. *Let be $(f_0, f_1, f_2) \in W^{1,p}(\Gamma) \times L^p(\Gamma) \times L^p(\Gamma)$. A necessary condition in order that $(f_0, f_1, f_2) \in range(\gamma_0, \gamma_1, \gamma_2)$ is:*

$$\nabla_\Gamma f_0 + f_1 \boldsymbol{n} \in (W^{1,p}(\Gamma))^N. \tag{3.3}$$

In order to state the second necessary condition, let be

$$\mathbf{g}^0 = \nabla_\Gamma f_0 + f_1 \mathbf{n} = \left(g_h^0\right)_{h=1,\ldots,N} \in (W^{1,p}(\Gamma))^N$$

and let define for $k = 1, \ldots, N-1$ the vector $\partial_{\mathbf{T}^k}\mathbf{g}^0 \in (L^p(\Gamma))^N$ whose components are

$$\left(\partial_{\mathbf{T}^k}\mathbf{g}^0\right)_h = \partial_{\mathbf{T}^k} g_h^0, \quad h = 1, \ldots, N \tag{3.4}$$

where the tangential derivatives are defined in (2.7).

Theorem 3.4. *Let be $(f_0, f_1, f_2) \in W^{1,p}(\Gamma) \times L^p(\Gamma) \times L^p(\Gamma)$. The necessary conditions in order that $(f_0, f_1, f_2) \in range(\gamma_0, \gamma_1, \gamma_2)$ are (3.3) and:*

$$\begin{aligned}
&\sum_{k=1}^{N-1}\left(\partial_{\boldsymbol{T}^k}\boldsymbol{g}^0 \cdot \boldsymbol{T}^k\right)\left(\boldsymbol{T}^k \otimes \boldsymbol{T}^k\right) + \\
&+ \sum_{1 \le k < p \le N-1}\left(\partial_{\boldsymbol{T}^k}\boldsymbol{g}^0 \cdot \boldsymbol{T}^p\right)\left(\boldsymbol{T}^k \otimes \boldsymbol{T}^p + \boldsymbol{T}^p \otimes \boldsymbol{T}^k\right) + \\
&+ \sum_{k=1}^{N-1}\left(\partial_{\boldsymbol{T}^k}\boldsymbol{g}^0 \cdot \boldsymbol{n}\right)\left(\boldsymbol{T}^k \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{T}^k\right) + \\
&+ f_2 \boldsymbol{n} \otimes \boldsymbol{n} \in W^{1-1/p,p}\left(\Gamma; M_{sym}^N\right).
\end{aligned} \tag{3.5}$$

Proof. Since $\mathbf{g}^0 = \gamma_0(Du) \in (W^{1,p}(\Gamma))^N$ it follows from (3.4) and the definition of $\mathbf{T}^k$ that on $V'(x_i)$ one has for $h = 1, \ldots, N$:

$$\partial_{\mathbf{T}^k} g_h^0 = \partial_{\mathbf{T}^k}\frac{\partial u}{\partial y_h} = \sum_{s=1}^{N}\left(\frac{\partial^2 u}{\partial y_h \partial y_s}\right)\left(y', \varphi_i(y')\right) T_s^k$$

A simple computation then gives:

$$\partial_{\mathbf{T}^k}\mathbf{g}^0 \cdot \mathbf{T}^p = \frac{1}{2}\gamma_0(Hu) : \left(\mathbf{T}^k \otimes \mathbf{T}^p + \mathbf{T}^p \otimes \mathbf{T}^k\right) \tag{3.6}$$

and

$$\partial_{\mathbf{T}^k}\mathbf{g}^0 \cdot \mathbf{n} = \frac{1}{2}\gamma_0(Hu) : \left(\mathbf{T}^k \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{T}^k\right) \tag{3.7}$$

where $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^N a_{ij} b_{ij}$ denotes the scalar product of the symmetric matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$. Since the vectors $(\mathbf{T}^1, \ldots, \mathbf{T}^{N-1}, \mathbf{n})$ are an orthonormal basis of $T_x\Omega$, an orthonormal basis of the symmetrized tensor product $T_x\Omega \otimes_S T_x\Omega$ is given by:

$$\left\{\mathbf{T}^k \otimes \mathbf{T}^k\right\}_{k=1,\ldots,N-1}, \ \mathbf{n} \otimes \mathbf{n}$$

$$\left\{\tfrac{1}{\sqrt{2}}\left(\mathbf{T}^{k}\otimes\mathbf{T}^{p}+\mathbf{T}^{p}\otimes\mathbf{T}^{k}\right),\ \tfrac{1}{\sqrt{2}}\left(\mathbf{T}^{k}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{T}^{k}\right)\right\}_{1\le k<p\le N-1}$$

A development of $\gamma_0(Hu)$ with respect to this basis and the use of (3.6) and (3.7) gives immediately (3.5). $\square$

Remark 3.5. Let us once more remark that each term of the sum in (3.5) only belongs to $L^p\left(\Gamma; M^N_{sym}\right)$.

Remark 3.6. With the same procedure one can write the necessary compatibility conditions for the traces of $u \in W^{m,p}(\Omega)$, $m \geq 4$.

4. **Examples**

In order to avoid inessential technicalities, we consider for $N = 2$ the case where $\Omega = \{(x, y)\,; x > 0, y > 0\}$ is the first quadrant and hence the corner has opening $\pi/2$. On the vertical (resp. horizontal) side of the corner $\Gamma_1 = \{(0, y)\,; y \geq 0\}$, resp. $\Gamma_2 = \{(x, 0)\,; x \geq 0\}$, one has $\mathbf{T}^1 = (0, -1)$ and $\mathbf{n} = (-1, 0)$, resp.$\mathbf{T}^1 = (-1, 0)$ and $\mathbf{n} = (0, -1)$.

Example 4.1. We prove that the usual compatibility conditions in a corner (see e.g. [8]) for $u \in W^{2,p}(\Omega)$ follow from (3.1). Indeed this condition becomes:

$$\mathbf{g}^0 = (\partial_{\mathbf{T}^1} f_0)\,\mathbf{T}^1 + f_1\mathbf{n} \in (W^{1-1/p,p}(\Gamma))^2\,.$$

Since the definition of $W^{1-1/p,p}(\Gamma)$ is invariant under the Lipschitz transform $\Gamma \longrightarrow \mathbb{R}$, the previous condition means that:

$$W^{1-1/p,p}(\mathbb{R}) \ni g_1^0 = \begin{cases} -f_1 & \text{for} \quad x < 0 \\ -df_0/dx & \text{for} \quad x > 0 \end{cases}$$

and

$$W^{1-1/p,p}(\mathbb{R}) \ni g_2^0 = \begin{cases} -df_0/dx & \text{for} \quad x < 0 \\ -f_1 & \text{for} \quad x > 0\,. \end{cases}$$

Since obviously, $df_0/dx, f_1 \in W^{1-1/p,p}(\mathbb{R}^{\pm})$, one can verify these conditions with the help of the following proposition, whose proof can be found e.g. in [8].

Proposition 4.2. *Let be $H^{\pm} \in W^{1-1/p,p}(\mathbb{R}^{\pm})$; and let define:*

$$H(x) = \begin{cases} H^-(x) & \textit{for} \quad x < 0 \\ H^+(x) & \textit{for} \quad x > 0\,. \end{cases} \tag{4.1}$$

Then $H \in W^{1-1/p,p}(\mathbb{R})$

(i) *for* $1 < p < 2$ *without any other condition;*

(ii) *for* $p = 2$ *if and only if*

$$\int_0^{+\infty} \left|H^+(t) - H^-(-t)\right|^2 dt/t < +\infty;$$

(ii) *for* $p > 2$ *if and only if*

$$H^+(0) = H^-(0).$$

Example 4.3. In order to prove that the usual compatibility conditions in a corner for $u \in W^{3,p}(\Omega)$, follow from (3.3)and (3.5) one has at first to express the conditions $g_h^0 \in W^{1,p}(\mathbb{R})$ for $h = 1, 2$ with an analogous of proposition 4.2. Since the orthonormal basis of $T_x\Omega \otimes_S T_x\Omega$ is now given by:

$$\mathbf{T}^1 \otimes \mathbf{T}^1, \quad \mathbf{n} \otimes \mathbf{n}, \quad \frac{1}{\sqrt{2}}\left(\mathbf{T}^1 \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{T}^1\right),$$

it is a simple exercice to express the different terms of these symmetric matrices on Γ_1, Γ_2 and so find the compatibility conditions, always thank to the proposition 4.2.

When $N = 3$ with an analogous method one can express the compatibility conditions for a vertex or an edge of a polyhedral-type domain. Once more since the definition of $W^{1-1/p,p}(\Gamma)$ and $W^{1,p}(\Gamma)$ are invariant under a Lipschitz transform one can reduce the study of the vertex behaviour to the following problems. Let be $\mathbb{R}^2$ divided in Λ non overlapping sectors S_λ, $\lambda = 1, \ldots, \Lambda$, with vertex at the origin. Let be given $H^\lambda \in W^{1-1/p,p}(S_\lambda)$ (resp.$H^\lambda \in W^{1,p}(S_\lambda)$) for $\lambda = 1, \ldots, \Lambda$; find the conditions such that:

$$H(x,y) = \begin{cases} H^1(x,y) & \text{for} \quad (x,y) \in S_1 \\ \cdots & \\ H^\Lambda(x,y) & \text{for} \quad (x,y) \in S_\Lambda \end{cases}$$

belongs to $W^{1-1/p,p}(\mathbb{R}^2)$, resp.to $W^{1,p}(\mathbb{R}^2)$. Some partial answers can be found in [3] and in [5].

5. Concluding remarks

The examples suggest that the necessary conditions (3.3) and (3.5) are also necessary. Indeed in the case $N = 2$ and $p = 2$ it is possible to adapt the reasoning used in [7] to prove the sufficiency of these conditions and the arguments used in [5] will also give the proof of the sufficiency for all $p > 1$.

The proof of the sufficiency in the general case $N \geq 3$ seems more delicate. The extension of the previous results to fractional Sobolev Spaces $W^{s,p}(\Omega)$ seems open (for the case $p = 2$ and $\frac{1}{2} < s < \frac{3}{2}$ see however [4]).

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Trace Theorems. Necessary Conditions

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