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The Affine Frame in p-adic Analysis

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Abstract

In this paper, we will introduce the concept of affine frame in wavelet analysis to the field of p-adic number, hence provide new mathematic tools for application of p-adic analysis.

1 Introduction

The concept of affine frame was introduced to wavelet analysis at first in reference [2]. And the theory of affine frame was studied in reference [3] and [4] further. In this paper, we introduce the concept of affine frame in wavelet analysis to the field of p-adic number as the discrete formula of wavelet transform. Before now, we have introduced the wavelet transform to the field of p-adic analysis([1]). General knowledge on p-adic analysis see [6].

It is known that if $x = p^r \sum_{k=0}^{n} x_k p^{-k} \in R^+ \cup \{0\},\ x_0 \neq 0.\ 0 \leq x_k \leq p-1, k = 1, 2, \cdots$, then there is another expression for x.

$$x = p^r(\sum_{k=0}^{n-1} x_k p^{-k} + (x_n - 1)p^{-n} + (p-1)\sum_{k=n+1}^{\infty} p^{-k}) \tag{1.1}$$

We don't adopt this expression. Let M_R be the set of numbers in the form of (1.1). An mapping ρ was introduced in reference [5], which $\rho : \mathbf{Q_p} \rightarrow R^+ \cup \{0\}$, for $x = p^{-r} \sum_{x=0}^{\infty} x_k p^k,\ x_0 \neq 0,\ 0 \leq p-1,\ k = 1, 2, \cdots$

$$\rho(x) = p^{-r-l} \sum_{k=0}^{\infty} x_k p^{-k} \tag{1.2}$$

For $a_R, b_R \in R^+ \cup \{0\}$, mapping (1.2) follows that

$$b_R - a_R = \mu([a_p, b_p]),\ a_p = \rho^{-1}(a_R), b_p = \rho^{-1}(b_R)$$

and then we have the following lemma.

Lemma 1.1: Let $f(x_p) \in L^2(Q_p)$ and $f_{_R}(x_{_R}) \overset{def}{=} f(\rho(x_p)), x_p \in Q_p, , x_{_R} = \rho(x_p)$ then

$$\int_{a_p}^{b_p} f(x_p)\mathrm{d}x_p = \int_{a_R}^{b_R} f_{_R}(x_{_R})\mathrm{d}x_{_R},$$

where $a_{_R} = \rho(a_p), b_{_R} = \rho(b_p)$.

Remark: $\mu\{[a_p,\ b_p]\}$ is defined as the infimum of measure of all the disjoint discs covering $\{B_{r_i}(a_i)\}$ for interval $[a_p,\ b_p]$, and for complex-valued function f, the integral of f is defined by

$$\int_{[a_p,b_p]} f\mathrm{d}x_p \overset{def}{=} \inf_{\{B_{r_i}(a_i)\}} \sum_i f(a_i)\mu\big(B_{r_i}(a_i)\big).$$

2 The Frame in $L^2(\mathbf{Q_p})$

Let $f, h \in L^2(\mathbf{Q_p})$. The following we will discuss conditions of $\{h_{m,n}\}$ becomes to the frame of $L^2(\mathbf{Q_p})$. Here, the so-called frame defined as: $\exists A, B > 0$ such that

$$A\|f\|^2_{L^2(\mathbf{Q_p})} \le \sum_{m,n} |(f, h_{mn})|^2_{L^2(\mathbf{Q_p})} \le B\|f\|^2_{L^2(\mathbf{Q_p})}$$

where

$$\|f\|^2_{L^2(\mathbf{Q_p})} = \int_{\mathbf{Q_p}} |f|^2\mathrm{d}x_p$$

$$(f, h_{mn})_{L^2(\mathbf{Q_p})} = \int_{\mathbf{Q_p}} f\overline{h}_{mn}\mathrm{d}x_p$$

It is known that if $\{h_{mn}\}$ is the frame of $L^2(Q_p)$, then the function $f \in L^2(\mathbf{Q_p})$ can be express as

$$f(x_p) = \sum_{m,n}(f, h^*_{mn})_{L^2(\mathbf{Q_p})}h_{mn}(x_p) = \sum_{m,n}(f, h_{mn})_{L^2(\mathbf{Q_p})}h^*_{mn} \tag{2.1}$$

here $h^*_{mn} = S^{-1}h_{mn}$is the dual frame of h_{mn},and $S : L^2(\mathbf{Q_p}) \to L^2(\mathbf{Q_p})$ is the frame operator.

$$Sf = \sum_{m,n}(f, h_{mn})_{L^2(\mathbf{Q_p})}h_{mn}$$

Theorem2.1. Let $f, h \in L^2(\mathbf{Q_p})$ be complex-value function, and $h(0) = 0$, $x_{_R} = \rho(x_p)$, $x_p \in Q_p \setminus M$, here $M = \rho^{-1}(M_{_R})$, and measure of set M be equal to 0. Select $b_0 = p^{r(b_0)}$, $r(b_0) \in \mathbf{Z}$, $a_0 = p^{r(a_0)}, r(a_0) \in \mathbf{Z}$. If

(1) $\exists A, B > 0$ such that for $\omega \neq 0$

$$A \leq G(\omega) \stackrel{\text{def}}{=} \sum_{m\in\mathbf{Z}} \left|\widehat{h}_1(\frac{\omega}{a_0^m})\right|^2 \leq B,$$

(2) $\text{supp}\widehat{h}_1 \subset [-\frac{1}{2b_0}, \frac{1}{2b_0}]$

then

$$\left\{ h_{mn}(x_p) \stackrel{def}{=} a_0^{m/2} h\left(\frac{\alpha_n^{(m)}(x_p) - x_p}{a_0^{-m}}\right)\right\}_{n,m\in\mathbf{Z}}$$

is a frame of $L^2(\mathbf{Q_p})$, where

$$\alpha_n^{(m)}(x_p) = \begin{cases} \rho^{-1}(x_{_R} + a_0^{-m}nb_0) + x_p\ , & x_{_R} + a_0^{-m}nb_0 > 0 \\ x_p\ , & x_{_R} + a_0^{-m}nb_0 \leq 0 \end{cases}$$

the sign$\widehat{\ }$ denotes the Fourier transform

$$\widehat{f}(w) = \int_R f(x) e^{-2\pi i\omega x} \mathrm{d}x$$

and h_1 is defined from(2.3).

Proof. From the definition of $\alpha_n^{(m)}(x_p)$, we have

$$\begin{aligned}
(Sf, f)_{L^2(Q_p)} &= \sum_{m,n\in\mathbf{Z}} \left|(f, h_{mn})_{L^2(Q_p)}\right|^2 \\
&= \sum_{m,n\in\mathbf{Z}} a_0^m \left|\int_{Q_p\setminus M} f(x_p) h\left\{\frac{\alpha_n^{(m)}(x_p) - x_p}{a_0^{-m}}\right\} \mathrm{d}x_p\right|^2 \\
&= \sum_{m,n\in\mathbf{Z}} a_0^m \left|\int_{\mathbf{R}^+\cup\{\mathbf{0}\}} f_{_R}(x_{_R}) h_1\left\{\frac{x_{_R} + a_0^{-m}nb_0}{a_0^{-m}}\right\} \mathrm{d}x_{_R}\right|^2 \qquad (2.2)
\end{aligned}$$

where $x_{_R} = \rho(x_p)$, $x_p \in Q_p \setminus M$, $f_{_R}(x_{_R}) = f(\rho^{-1}(x_{_R}))$,

$$h_1(x_{_R}) = \begin{cases} h(\rho^{-1}x_{_R}), & x_{_R} > 0 \\ h(0), & x_{_R} \leq 0 \end{cases} \qquad (2.3)$$

Let

$$f^+_{_R}(x_{_R}) = \begin{cases} f_{_R}(x_{_R}), & x_{_R} \geq 0 \\ 0, & x_{_R} < 0 \end{cases}$$

Thus, from(2.2), the following equalities hold

$$\sum_{m,n\in Z} \left|(f, h_{mn})_{L^2(Q_p)}\right|^2$$

$$= \sum_{m,n\in Z} a_0^m \left| \int_R f^+_{_R}(x_{_R}) h_1 \Big(\frac{x_{_R} + a_0^{-m} n b_0}{a_0^{-m}} \Big) \mathrm{d}x_{_R} \right|^2$$

$$= \sum_{m,n\in Z} a_0^{-m} \left| \int_R \widehat{f^+_{_R}}(\omega) \widehat{h}_1 (\frac{\omega}{a_0^m}) \exp(2\pi i a_0^{-m} n b_0 \omega) \mathrm{d}\omega \right|^2$$

$$= \sum_{m,n\in Z} a_0^{-m} \left| \int_{-\frac{a_0^m}{2b_0}}^{\frac{a_0^m}{2b_0}} \widehat{f^+_{_R}}(\omega) \widehat{h}_1 (\frac{\omega}{a_0^m}) \exp(2\pi i a_0^{-m} n b_0 \omega) \mathrm{d}\omega \right|^2 \qquad (2.4)$$

In the last equality, we used supp$\widehat{h_1} \subset [-\frac{a_0^m}{2b_0}, \frac{a_0^m}{2b_0}]$. But

$$\frac{1}{a_0^m/b_0} \int_{-\frac{a_0^m}{2b_0}}^{\frac{a_0^m}{2b_0}} \widehat{f^+_R}(\omega) \widehat{h}_1 (\frac{\omega}{a_0^m}) \exp\big(2\pi i a_0^{-m} n b_0 \omega\big) \mathrm{d}\omega$$

is the Fourier coefficient of the function $\widehat{f^+_{_R}}(\omega)\widehat{h}_1(\frac{\omega}{a_0^m})$. Denotes this coefficient by c_{-n}. By the *Parseval* equality, we have

$$\sum_{n\in\mathbf{Z}} \left|c_n\right|^2 = \frac{1}{a_0^{2m}/b_0^2} \sum_{n\in\mathbf{Z}} \left| \int_{-\frac{a_0^m}{2b_0}}^{\frac{a_0^m}{2b_0}} \widehat{f^+_R}(\omega) \widehat{h}_1 (\frac{\omega}{a_0^m}) \exp\big(2\pi i a_0^{-m} n b_0 \omega\big) \mathrm{d}\omega \right|^2$$

$$= \frac{b_0}{a_0^m} \int_{-\frac{a_0^m}{2b_0}}^{\frac{a_0^m}{2b_0}} \left| \widehat{f^+_R}(\omega) \widehat{h}_1 (\frac{\omega}{a_0^m}) \right|^2 \mathrm{d}\omega \qquad (2.5)$$

By (2.4), we have

$$\sum_{m,n\in Z} \left| (f, h_{mn})_{_{L^2(Q_p)}} \right|^2 = \frac{1}{b_0} \sum_{m\in\mathbf{Z}} \int_{-\frac{a_0^m}{2b_0}}^{\frac{a_0^m}{2b_0}} \left| \widehat{f^+_R}(\omega) \widehat{h}_1 (\frac{\omega}{a_0^m}) \right|^2 \mathrm{d}\omega$$

$$= \frac{1}{b_0} \int_R |\widehat{f_R^+}(\omega)|^2 |\widehat{h}_1(\frac{\omega}{a_0^m})|^2 \mathrm{d}\omega \tag{2.6}$$

Here

$$G(\omega) = \sum_{m \in Z} \left| \widehat{h}_1(\frac{\omega}{a_0^m}) \right|^2 \tag{2.7}$$

Finally, by the condition of theorem $0 < A \leq |G(\omega)|^2 \leq B$ and formula (2.6), we have

$$\sum_{m,n \in Z} |(f, h_{mn})_{L^2(Q_p)}|^2 = \begin{cases} \geq \frac{1}{b_0} A \int_R |\widehat{f}_R^+(\omega)|^2 \mathrm{d}\omega \\ \leq \frac{1}{b_0} B \int_R |\widehat{f}_R^+(\omega)|^2 \mathrm{d}\omega \end{cases}$$

But

$$\int_R |\widehat{f}_R^+(\omega)|^2 \mathrm{d}\omega = \int_R |f_R^+(x_R)|^2 \mathrm{d}x_R = \int_{R^+ \cup \{0\}} |f_R(x_R)|^2 \mathrm{d}x_R = \int_{Q_p} |f(x_p)|^2 \mathrm{d}x_p$$
$$= \|f\|^2_{L^2(Q_p)}$$

Hence, the theorem follows. $\square$

3 Dual frame h^*_{mn}

It is known that if $\{h_{mn}\}_{mn}$ construct a frame of $L^2(Q_p)$, then expression (2.1) is valid. From (2.6), we have

$$\begin{aligned} (Sf, f)_{L^2(Q_p)} &= \frac{1}{b_0} \int_R |\widehat{f}_R^+(\omega)|^2 G(\omega) \mathrm{d}\omega \\ &= \frac{1}{b_0} \int_R \left(\widehat{f_R^+ G}\right)^\vee (x_R) \overline{f}_R^+(x_R) \mathrm{d}x_R \\ &= \frac{1}{b_0} \int_{R^+ \cup \{0\}} (\widehat{f}_R^+ G)^\vee (x_R) \overline{f}_R(x_R) \mathrm{d}x_R \\ &= \frac{1}{b_0} \int_{Q_p} \left(\widehat{f_R^+ G}\right)^\vee \big(\rho(x_p)\big) \overline{f}(x_p) \mathrm{d}x_p \end{aligned} \tag{3.1}$$

where $x_p = \rho^{-1}(x_R), x_R \in R^+ \cup \{0\}$, and "$\vee$" is the sign of Fourier inverse transform. Here, we use lemma A.

(3.1) can be rewritten as

$$(Sf,f)_{L^2(Q_p)} = \frac{1}{b_0}\Big((\hat{f}_R^{+}G)^{\vee}\big(\rho(x_p)\big), f(x_p)\Big)_{L^2(Q_p)} \tag{3.2}$$

Since f is an arbitrary function in Q_p,

$$\big(Sf\big)(x_p) = \frac{1}{b_0}\big(\hat{f}_R^{+}G\big)^{\vee}\big(\rho(x_p)\big) \tag{3.3}$$

or for $x \in R^{+} \cup \{0\}$, we conclude that

$$(Sf)_{R}(x_{R}) = \frac{1}{b_0}\big(\hat{f}_R^{+}G\big)^{\vee}(x_{R}),\ (Sf)_{R} = S(f \circ \rho^{-1}) \tag{3.4}$$

On the basis of formula (3.4), we can extend the domain of function $(Sf)_{R}(x_{R})$ onto R. Therefore

$$(Sf)_{R}^{\wedge}(\omega) = \frac{1}{b_0}\widehat{f_R^{+}}(\omega)G(\omega),\ \omega \in R^{+} \cup \{0\} \tag{3.5}$$

Replacing f by $S^{-1}f$ in the formula (3.5), we have

$$\widehat{f_{R}}(\omega_{R}) = \frac{1}{b_0}\widehat{(S^{-1}f)}_{R}^{+}(\omega)G(\omega)$$

Thus

$$\widehat{(S^{-1}f)}_{R}^{+}(\omega_{R}) = \frac{b_0\widehat{f_{R}}(\omega)}{G(\omega)}$$

or

$$(S^{-1}f)_{R}^{+}(x_{R}) = b_0\Big\{\frac{\widehat{f_{R}}}{G}\Big\}^{\vee}(x_{R})$$

For $x_{R} \geq 0$, we have

$$(S^{-1}f)_{R}(x_{R}) = b_0\Big\{\frac{\widehat{f_{R}}}{G}\Big\}^{\vee}(x_{R})$$

or

$$(S^{-1}f)(x_p) = b_0\Big(\frac{\widehat{f_{R}}}{G}\Big)^{\vee}\big(\rho(x_p)\big)$$

So for $f_{R}(x_{R}), x_{R} \geq 0$,$(S^{-1}f)(x_p) = b_0[f_{R} * (G^{-1})^{\vee}]$ is valid. Finally, We have

$$h^{*}_{mn}(x_p) = b_0[(h_{mn})_R * (G^{-1})^{\vee}](\rho(x_p)),$$

where the sign $*$ denotes convolution.

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