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# On the solution set of second-order delay differential inclusions in Banach spaces

A. Sghir

## Abstract

In this paper, we consider the second-order delay differential inclusion  $x''(t) \in Ax(t) + F(t, x_t)$  in a Banach space and we study some properties of its solution set. We prove a relaxation theorem which reveals the connection between the solution sets of a second-order delay differential inclusion and its convexified version, under some weak conditions.

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## 1 Introduction

Many problems in applied mathematics, such as those in control theory, lead to the study of second-order delay differential inclusions

$$x''(t) \in Ax(t) + F(t, x_t), \quad (1)$$

where  $A$  is the infinitesimal generator of a  $C_0$ -propagator of linear operators  $(C(t))_{t \in \mathbb{R}}$  on a Banach space  $(E, |\cdot|_E)$  and  $F$  is a nonlinear multimapping, satisfying assumptions to be specified in the third section.

As particular cases of relations of the form (1) we have:

i) The second-order delay differential equation

$$x''(t) = Ax(t) + f(t, x_t)$$

where  $F(t, x_t) = f(t, x_t)$ .

ii) The differential inequalities

$$|x''(t) - Ax(t) - f(t, x_t)|_E \leq g(t, x_t)$$

where  $F(t, x_t)$  is the ball of radius  $g(t, x_t)$  centered at  $Ax(t) + f(t, x_t)$ .

iii) Control problems where the control  $u(t)$  and the trajectory  $x(t)$  are related by the second-order delay differential equation

$$x''(t) = Ax(t) + f(t, x_t, u(t)), \quad u(t) \in U(t).$$

Here, the control function  $u(t)$  is a measurable function and  $F(t, x_t) = f(t, x_t, U(t))$ . This paper is concerned with the second-order delay differential inclusion (1) and its mild trajectories. We show that many results which allow us to apply differential inclusions, see for example [1, 3, 8, 10, 13] and references therein, are valid as well for (1). In our relaxation theorem, the assumption of integrable boundedness (condition  $(H_4)$ ) will be replaced by an integrability condition (condition  $(H'_3)$ ). We also give some properties of the solution set of the inclusion (1).

## 2 Preliminaries

For a real Banach space  $(E, |\cdot|_E)$  and  $J := [-r, 0]$  ( $r > 0$ ), let  $\mathcal{C} := C([-r, 0]; E)$  be the Banach space of continuous functions from  $J$  to  $E$  with the usual supremum norm  $\|\cdot\|$ . For any continuous function  $x \in C([-r, \omega]; E)$  ( $\omega > 0$ ) and any  $t \in I := [0, \omega]$  we denote by  $x_t$  the element of  $\mathcal{C}$  defined by  $x_t(\theta) = x(t + \theta)$ ,  $\theta \in J$ .

For a subset  $A \subset E$ ,  $coA$ ,  $\overline{co}A$  and  $clA$  are respectively the convex hull, the closed convex hull and the closure. We denote by  $\mathcal{F}(E)$  (resp.  $\mathcal{F}_c(E)$ ) the family of all nonempty closed (resp. closed convex) subsets of  $E$ , and by  $\delta$  the Hausdorff distance in  $\mathcal{F}(E)$ , i.e. for  $A, B \in \mathcal{F}(E)$

$$\delta(A, B) = \max\left[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right]$$

where  $d(a, B) = \inf_{b \in B} d(a, b)$ .

Next we present some basic concepts concerning multimappings.

Let  $X$  be another Banach space, for a multimapping  $G : X \rightarrow \mathcal{P}(E)$  (the family of all nonempty subsets of  $E$ ), we define its  $\limsup$  and  $\liminf$  at  $x \in X$  in the Kuratowski sense by

$$\limsup_{y \rightarrow x} G(y) = \{z \in E : \liminf_{y \rightarrow x} d(z, G(y)) = 0\}$$

and

$$\liminf_{y \rightarrow x} G(y) = \{z \in E : \lim_{y \rightarrow x} d(z, G(y)) = 0\}.$$

We say that the limit of  $G(y)$  as  $y$  tends to  $x$  exists in the Kuratowski sense if

$$\limsup_{y \rightarrow x} G(y) = \liminf_{y \rightarrow x} G(y).$$

We denote this limit by  $\lim_{y \rightarrow x} G(y) = G(x)$ . We say that  $G$  is upper (resp. lower) semicontinuous at  $x$  if

$$\limsup_{y \rightarrow x} G(y) \subseteq G(x) \quad (\text{resp. } G(x) \subseteq \liminf_{y \rightarrow x} G(y)).$$

If  $G$  is both upper and lower semicontinuous at  $x$  then we say that  $G$  is continuous at  $x$ . If  $G$  is continuous or semicontinuous for all  $x \in X$ , we say that  $G$  is continuous or semicontinuous on  $X$ .

Let  $G : I \rightarrow \mathcal{P}(E)$  be a multimapping. A function  $g : I \rightarrow E$  such that  $g(t) \in G(t)$  for every  $t \in I$  is called a selection of  $G$ .

$G$  is called measurable if, for almost all  $t \in I$

$$G(t) \subseteq cl\{g_n(t) : n \geq 1\}$$

where  $g_n$  are measurable selections of  $G$ . This definition of the mesurability is given by Zhu [13], when  $E$  is separable and  $G(t) \in \mathcal{F}(E)$  for every  $t \in I$  this definition is the same as the classic one (see for example [3]).

By the symbol of  $I_G^1$  we will denote the set of all Bochner integrable selections of the multimapping  $G$ , i.e.

$$I_G^1 = \{g \in L^1(I; E) : g(t) \in G(t) \text{ a.e.}\}.$$

If  $I_G^1 \neq \emptyset$ , then the measurable multimapping  $G$  is called integrable and

$$\int_I G(t)dt = \left\{ \int_I g(t)dt : g \in I_G^1 \right\}.$$

Clearly if  $G$  is measurable and integrably bounded, i.e. there exists  $\nu \in L_+^1(I)$  such that

$$\|G(t)\| := \sup\{|e|_E : e \in G(t)\} \leq \nu(t) \text{ a.e.}$$

then  $G$  is integrable. But the converse is not true.

We will also need the following properties (see [13]) which will be used later.

**Lemma 2.1** Let  $G : I \rightarrow \mathcal{P}(E)$  be a measurable multimapping. Then so is  $\overline{co}G$ .

**Lemma 2.2** Let  $G : I \rightarrow \mathcal{P}(E)$  be an integrable multimapping. Then  $cl \int_I G(t)dt$  is a convex set and

$$cl \int_I G(t)dt = cl \int_I coG(t)dt = cl \int_I \overline{co}G(t)dt.$$

**Remark** If  $G : I \rightarrow \mathcal{P}(E)$  is an integrable multimapping, then so is  $\overline{G}$  where  $\overline{G}(t) = clG(t)$  and

$$cl \int_I G(t)dt = cl \int_I \overline{G}(t)dt$$

(indeed  $cl \int_I G(t)dt \subset cl \int_I \overline{G}(t)dt \subset cl \int_I \overline{co}G(t)dt = cl \int_I G(t)dt$ ).

**Lemma 2.3** Let  $G : I \rightarrow \mathcal{P}(E)$  be a measurable multimapping and  $u : I \rightarrow E$  a measurable function. Then for any measurable function  $v : I \rightarrow \mathbb{R}^+$ , there exists a measurable selection  $g$  of  $G$  such that

$$|g(t) - u(t)|_E \leq d(u(t), G(t)) + v(t) \text{ a.e.}$$

At last, we give some important properties of a  $C_0$ -propagator and its infinitesimal generator (see [7]).

A strongly continuous propagator  $(C(t))_{t \in \mathbb{R}}$  of continuous operators on  $E$  is a family of continuous linear mappings  $C(t) : E \rightarrow E$ ,  $t \in \mathbb{R}$ , satisfying

- i)  $C(0) = I$ ;
- ii)  $C(t + s) + C(t - s) = 2C(t)C(s)$ ;

iii) for  $x \in E$ ,  $C(\cdot)x : \mathbb{R} \rightarrow E$  is continuous.

A strongly continuous propagator of continuous linear mappings is also called a  $C_0$ -propagator. A linear operator  $A$  is associated with a propagator, it plays the role of the infinitesimal generator for  $C_0$ -semigroups:

$$D(A) = \{x \in E : \lim_{h \searrow 0} \frac{2}{h^2} [C(h) - I]x \text{ exists}\}$$

and

$$Ax = \lim_{h \searrow 0} \frac{2}{h^2} [C(h) - I]x \text{ for } x \in D(A)$$

is the infinitesimal generator of the  $C_0$ -propagator  $(C(t))_{t \in \mathbb{R}}$ ,  $D(A)$  is the domain of  $A$ . We have:

- There exist constants  $\alpha \geq 0$  and  $\eta \geq 1$  such that

$$\|C(t)\| \leq \eta e^{\alpha|t|} \text{ for } t \in \mathbb{R}.$$

-  $D(A)$  is dense in  $E$  and  $A$  is a closed linear operator.

- For every  $x \in D(A)$  and  $t \in \mathbb{R}$ , then  $C(t)x \in D(A)$  and

$$\frac{d^2}{dt^2} C(t)x = AC(t)x = C(t)Ax.$$

- Let  $a, b \in E$  and  $f \in L^1(I; E)$ , the function  $u \in C(I; E)$  given by

$$u(t) = C(t)a + S(t)b + \int_0^t S(t-s)(f(s))ds, \quad t \in I$$

is the mild solution on  $I$  of the initial value problem

$$\begin{cases} u''(t) = Au(t) + f(t), & t \in I \\ u(0) = a, \quad u'(0) = b \end{cases}$$

where  $S(t) = \int_0^t C(s)ds$ . Moreover

$$\|u(t)\|_E \leq \eta e^{\alpha t} \|a\|_E + \eta \alpha^{-1} (e^{\alpha t} - 1) \|b\|_E + \eta \alpha^{-1} (e^{\alpha \omega} - 1) \|f\|_1, \quad t \in I$$

( $\alpha^{-1}(e^{\alpha t} - 1)$  is replaced by  $t$  when  $\alpha = 0$ ). If  $a = 0$  then  $u$  is continuously differentiable and

$$\|u'(t)\|_E \leq \eta e^{\alpha t} \|b\|_E + \eta e^{\alpha \omega} \|f\|_1, \quad t \in I.$$

### 3 The solution set of a second-order delay differential inclusion and a relaxation theorem

Consider the functional differential inclusion

$$x''(t) \in Ax(t) + F(t, x_t) \text{ a.e. in } I \tag{3.1}$$

**Definition 3.1** A function  $x \in C_\omega := C([-r, \omega]; E)$  is called a mild trajectory of (3.1), if there exist  $\varphi \in \mathcal{B} := \{\varphi \in C : \varphi'(0) \text{ exists}\}$  and a Bochner integrable function  $f \in L^1(I; E)$  such that

$$f(t) \in F(t, x_t) \text{ a.e. in } I \tag{2}$$

and

$$x(t) = \begin{cases} \varphi(t), & t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds, & t \in I \end{cases} \tag{3}$$

i.e.,  $f$  is a Bochner integrable selection of the multimapping  $t \mapsto F(t, x_t)$  and  $x$  is a mild solution of the initial value problem

$$(4) \begin{cases} x'(t) = Ax(t) + f(t), & t \in I \\ x_0 = \varphi, \varphi \in \mathcal{B}. \end{cases}$$

For  $\varphi \in \mathcal{B}$ , we define  $S_F(\varphi) = \{x \in C_\omega : x \text{ is a mild trajectory of (3.1) with } x_0 = \varphi\}$  to be the solution set of (3.1) from the point  $\varphi$ .

Let  $\psi \in \mathcal{B}$ ,  $g \in L^1(I, E)$  and  $y \in C_\omega$  be a mild solution of the problem

$$(C) \begin{cases} y'(t) = Ay(t) + g(t), & t \in I \\ y_0 = \psi. \end{cases}$$

Suppose that the multimapping  $F : I \times \mathcal{C} \rightarrow \mathcal{F}(E)$  satisfies the following conditions:

$H_1$ ) For every  $\phi \in \mathcal{C}$ , the multimapping  $F(\cdot, \phi)$  is measurable on  $I$ .

$H_2$ ) There is an integrable function  $k : I \rightarrow \mathbb{R}^+$  such that for every  $\phi, \xi \in \mathcal{C}$ ,

$$\delta(F(t, \phi), F(t, \xi)) \leq k(t)\|\phi - \xi\| \text{ a.e. in } I.$$

$H_3$ ) The function  $q : t \mapsto d(g(t), F(t, y_t))$  is integrable on  $I$ .

$H'_3$ ) For any function  $x \in C_\omega$ , the multimapping  $t \mapsto F(t, x_t)$  is integrable on  $I$ .

$H_4$ ) There is an integrable function  $\nu \in L^1_+(I)$  such that

$$\|F(t, \phi)\| := \sup\{|y|_E : y \in F(t, \phi)\} \leq \nu(t)$$

for all  $\phi \in \mathcal{C}$  and almost all  $t \in I$ .

**Remarks**

- When  $F$  satisfies  $(H_1)$  and  $(H_2)$ , then  $t \rightarrow F(t, y_t)$  and  $q$  are measurable on  $I$ .

- If  $q$  is measurable, then the condition  $(H'_3)$  gives  $(H_3)$ .

- When  $F$  satisfies  $(H_1)$  and  $(H_2)$  it satisfies  $(H'_3)$  if and only if it satisfies: there is  $z \in C_\omega$  such that the multimapping  $t \rightarrow F(t, z_t)$  is integrable (see [13]).

- When  $F$  satisfies  $(H_2)$ , then for every integrable function  $k' > k$  and  $\phi, \xi \in \mathcal{C}$ ,

$$F(t, \phi) \subset F(t, \xi) + k'(t)\|\phi - \xi\|B \text{ a.e. in } I$$

where  $B$  denotes the closed unit ball in  $E$ .

Next we present a useful result on the relationships between the trajectories of (3.1) and the solutions of problem (C).

**Theorem 3.1** Let  $\psi \in \mathcal{B}$ ,  $g \in L^1(I; E)$  and  $y \in \mathcal{C}_\omega$  be a mild solution of problem (C). Assume that  $(H_1) - (H_3)$  hold true and let  $\mu \geq 0$ . Then for all  $\varphi \in \mathcal{B}$  with  $\|\varphi - \psi\| \leq \mu$ ,  $|\varphi'(0) - \psi'(0)|_E \leq \mu$  and for all integrable function  $v : I \rightarrow \mathbb{R}^+$ , there exist  $x \in \mathcal{C}_\omega$  and  $f \in L^1(I; E)$  satisfying (2), (3) and

$$\|x - y\|_\omega \leq K(\omega)m(\omega), \quad \|f - g\|_1 \leq K(\omega)m(\omega)$$

where  $M = \eta(e^{\alpha\omega} + \frac{e^{\alpha\omega}-1}{\alpha})$ , ( $\frac{e^{\alpha\omega}-1}{\alpha}$  is replaced by  $\omega$  when  $\alpha = 0$ ),

$$K(t) = M \exp M \int_0^t 2k(s)ds, \quad m(t) = \mu + \int_0^t (q(s) + v(s))ds.$$

**Proof.** By lemma 2.3, there is a measurable selection  $f_1$  of the multimapping  $t \mapsto F(t, y_t)$  such that, for almost all  $t \in I$ ,

$$\begin{aligned} |f_1(t) - g(t)|_E &\leq d(g(t), F(t, y_t)) + v(t) \\ &\leq q(t) + v(t) \end{aligned}$$

and then  $f_1 \in L^1(I; E)$ . Set

$$x^1(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_1(s))ds & \text{if } t \in I \end{cases}$$

we have  $x^1 \in \mathcal{C}_\omega$  and for all  $t \in I$ ,

$$\begin{aligned} \|x_t^1 - y_t\| &= \sup_{\theta \in J} |x^1(t+\theta) - y(t+\theta)|_E \\ &\leq M(\mu + \int_0^t |f_1(s) - g(s)|_E ds) \\ &\leq M(\mu + \int_0^t (q(s) + v(s))ds). \end{aligned}$$

By using lemma 2.3, there is a measurable selection  $f_2$  of the multimapping  $t \mapsto F(t, x_t^1)$  such that, for almost all  $t \in I$ ,

$$\begin{aligned} |f_2(t) - f_1(t)|_E &\leq 2d(f_1(t), F(t, x_t^1)) \\ &\leq 2\delta(F(t, y_t), F(t, x_t^1)) \\ &\leq 2k(t)\|x_t^1 - y_t\| \end{aligned}$$

and then  $f_2 \in L^1(I; E)$ . Set

$$x^2(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_2(s))ds & \text{if } t \in I. \end{cases}$$

Thus, we can define by induction two sequences  $(x^n)$  and  $(f_n)$  with  $x^n \in \mathcal{C}_\omega$  and  $f_n \in L^1(I; E)$  such that:

i)  $x^0 = y$  and for all  $n \geq 1$ ,

$$x^n(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_n(s))ds & \text{if } t \in I; \end{cases}$$

ii)  $f_0 = g$  and for all  $n \geq 1$

$$f_n(t) \in F(t, x_t^{n-1}) \text{ a.e. in } I;$$

iii) for almost all  $t \in I$  and  $n \geq 1$ ,

$$|f_{n+1}(t) - f_n(t)|_E \leq 2k(t)\|x_t^n - x_t^{n-1}\|.$$

It follows then from (iii) that  
iv) for all  $t \in I$  and  $n \geq 1$ ,

$$\begin{aligned} \|x_t^{n+1} - x_t^n\| &\leq M \int_0^t |f_{n+1}(t_1) - f_n(t_1)|_E dt_1 \\ &\leq M \int_0^t 2k(t_1)\|x_{t_1}^n - x_{t_1}^{n-1}\| dt_1 \\ &\leq M \int_0^t 2k(t_1) \left[ M \int_0^{t_1} 2k(t_2)\|x_{t_2}^{n-1} - x_{t_2}^{n-2}\| dt_2 \right] dt_1 \\ &\vdots \\ &\leq M^n \int_0^t 2k(t_1) \int_0^{t_1} 2k(t_2) \cdots \int_0^{t_{n-1}} 2k(t_n)\|x_{t_n}^1 - y_{t_n}\| dt_n \cdots dt_1 \\ &\leq M \left[ \eta + \int_0^t (g(s) + v(s)) ds \right] \cdot \frac{[M \int_0^t 2k(s) ds]^n}{n!}. \end{aligned}$$

Then, for all  $n \geq 1$

$$\begin{aligned} \|x^{n+1} - x^n\|_\omega &: = \max(\|x^{n+1} - x^n\|, \sup_{t \in I} |x^{n+1}(t) - x^n(t)|_E) \\ &= \sup_{t \in I} |x^{n+1}(t) - x^n(t)|_E \\ &\leq \sup_{t \in I} \|x_t^{n+1} - x_t^n\| \\ &\leq Mm(\omega) \frac{[M \int_0^\omega 2k(t) dt]^n}{n!} \end{aligned}$$

By (iv) we obtain for all  $t \in I$  and  $n \geq 1$ ,

$$\begin{aligned} \|x_t^{n+1} - y_t\| &\leq \|x_t^1 - y_t\| + \sum_{i=1}^n \|x_t^{i+1} - x_t^i\| \\ &\leq Mm(t) \left[ 1 + \sum_{i=1}^n \frac{[M \int_0^t 2k(s) ds]^i}{i!} \right] \\ &\leq K(t)m(t). \end{aligned}$$

We deduce that  $(x^n)$  is a Cauchy sequence of a continuous functions, converging uniformly to a function  $x \in C_\omega$  and for almost all  $t \in I$ ,  $(f_n(t))$  is a Cauchy sequence in



$E$ , hence  $(f_n(\cdot))$  converges pointwise almost everywhere to a measurable function  $f(\cdot)$  in  $E$ . But for almost all  $t \in I$  and  $n \in \mathbb{N}$

$$\begin{aligned} |f_{n+1}(t) - g(t)|_E &\leq \sum_{i=1}^n |f_{i+1}(t) - f_i(t)|_E + |f_1(t) - g(t)|_E \\ &\leq 2k(t) \sum_{i=1}^n \|x_t^i - x_t^{i-1}\| + q(t) + v(t) \\ &\leq 2k(t)K(\omega)m(\omega) + q(t) + v(t) \end{aligned}$$

hence,  $|f_{n+1}(t)|_E \leq |g(t)|_E + 2k(t)K(\omega)m(\omega) + q(t) + v(t)$ , thus  $(f_n)$  converges to  $f$  in  $L^1(I; E)$  and then  $(x^n(t))$  ( $t \in [-r, \omega]$ ) converges in  $E$  to

$$\begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds & \text{if } t \in I, \end{cases}$$

we obtain

$$x(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds & \text{if } t \in I. \end{cases}$$

Furthermore, for almost all  $t \in I$

$$\begin{aligned} d(f(t), F(t, x_t)) &\leq |f(t) - f_n(t)|_E + d(f_n(t), F(t, x_t)) \\ &\leq |f(t) - f_n(t)|_E + \delta(F(t, x_t^{n-1}), F(t, x_t)) \\ &\leq |f(t) - f_n(t)|_E + k(t)\|x_t^{n-1} - x_t\|. \end{aligned}$$

The right hand side tends to zero almost everywhere on  $I$  as  $n \rightarrow +\infty$ . Thus, for almost all  $t \in I$ ,  $f(t) \in F(t, x_t)$ .

Consequently  $x \in S_F(\varphi)$ , moreover, for all  $n \in \mathbb{N}$

$$\begin{aligned} \|x^{n+1} - y\|_\omega &\leq \sup_{t \in I} \|x_t^{n+1} - y_t\| \\ &\leq K(\omega)m(\omega). \end{aligned}$$

Taking limits in the precedent inequality, we have  $\|x - y\|_\omega \leq K(\omega)m(\omega)$ .

We now show  $\|f - g\|_1 \leq K(\omega)m(\omega)$ .

For almost all  $t \in I$  and  $n \in \mathbb{N}$ , we have

$$|f_{n+1}(t) - g(t)|_E \leq q(t) + v(t) + 2k(t)Mm(\omega) \sum_{i=1}^n \frac{[M \int_0^t 2k(s)ds]^{i-1}}{(i-1)!}$$

thus,

$$\begin{aligned} \|f_{n+1} - g\|_1 &\leq m(\omega)[1 + \sum_{i=1}^n \frac{[M \int_0^\omega 2k(t)dt]^i}{i!}] \\ &\leq m(\omega)K(\omega). \end{aligned}$$

Taking the limit in the above inequality, we obtain  $\|f - g\|_1 \leq m(\omega)K(\omega)$ . ■

In the next theorem we compare trajectories of (3.1) and of the convexified (relaxed) second-order delay differential inclusion  $x''(t) \in Ax(t) + \overline{co}F(t, x_t)$  (3.2).

For  $\varphi \in \mathcal{B}$ , we put

$$S_{\overline{co}F}(\varphi) = \{x \in \mathcal{C}_\omega : x \text{ is a trajectory of (3.2) with } x_0 = \varphi\}.$$

**Theorem 3.2** Assume that  $F$  satisfies conditions  $(H_1)$ ,  $(H_2)$  and  $(H_3)$ . Then, for all  $\varphi \in \mathcal{B}$ ,

$$clS_F(\varphi) = clS_{\overline{co}F}(\varphi).$$

**Proof.** It is easy to see that  $clS_F(\varphi) \subset clS_{\overline{co}F}(\varphi)$ . Conversely, we shall show that  $S_{\overline{co}F}(\varphi) \subset clS_F(\varphi)$ . Let  $y \in S_{\overline{co}F}(\varphi)$ , then there exists  $g \in L^1(I; E)$  such that

$$y(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(g(s))ds & \text{if } t \in I \end{cases}$$

where  $g(s) \in \overline{co}F(s, y_s)$  a.e. in  $I$ .

The following result follows immediately from [3 p. 85].

**Lemma 3.1**

Let  $G : I \rightarrow P(E)$  be a measurable multimapping, then so is  $s \rightarrow S(t-s)G(s)$ . Moreover if  $f(s) \in S(t-s)G(s)$  then, there exists a measurable selection  $g(s) \in G(s)$  such that  $f(s) = S(t-s)g(s)$  a.e. in  $I$ .

By  $(H'_3)$  for all fixed  $t$  in  $I$ , the multimapping  $s \mapsto S(t-s)F(s, y_s)$  is integrable on  $I$  and by lemma 2.2 and its remark we obtain

$$s \mapsto clS(t-s)F(s, y_s) \text{ and } s \mapsto \overline{co}S(t-s)F(s, y_s)$$

are also integrable on  $I$  and

$$\begin{aligned} cl \int_I S(t-s)F(s, y_s)ds &= cl \int_I clS(t-s)F(s, y_s)ds \\ &= cl \int_I \overline{co}S(t-s)F(s, y_s)ds \end{aligned}$$

but,  $\overline{co}S(t-s)F(s, y_s) = clS(t-s)\overline{co}F(s, y_s)$ , indeed

$$S(t-s)F(s, y_s) \subset clS(t-s)\overline{co}F(s, y_s)$$

which is a closed convex set and then

$$\overline{co}S(t-s)F(s, y_s) \subset clS(t-s)\overline{co}F(s, y_s),$$

conversly, it suffice to see that

$$S(t-s)\overline{co}F(s, y_s) \subset \overline{co}S(t-s)F(s, y_s)$$

let  $f(s) \in S(t-s)\overline{co}F(s, y_s)$ , then there exists  $g(s) \in \overline{co}F(s, y_s)$  such that  $f(s) = S(t-s)g(s)$  hence, there exists a sequence  $(g_n(s))$  such that  $g_n(s) \in coF(s, y_s)$  and  $\lim_{n \rightarrow +\infty} g_n(s) = g(s)$ , we put

$$f_n(s) = S(t-s)g_n(s) \in S(t-s)coF(s, y_s) = coS(t-s)F(s, y_s)$$

and taking the limit as  $n \rightarrow +\infty$ , we obtain

$$f(s) = S(t-s)g(s) \in cl\ coS(t-s)F(s, y_s)$$

thus,

$$\begin{aligned} cl \int_I S(t-s)F(s, y_s)ds &= cl \int_I clS(t-s)\overline{co}F(s, y_s)ds \\ &= cl \int_I S(t-s)\overline{co}F(s, y_s)ds \end{aligned}$$

(see remark of lemma 2.2).

By lemma 3.1, we obtain for all  $\varepsilon > 0$  an integrable selection  $h(s) \in F(s, y_s)$  a.e. such that

$$\left| \int_I S(t-s)(g(s))ds - \int_I S(t-s)(h(s))ds \right|_E < \frac{\varepsilon}{K(\omega)(\|k\|_1 + \omega) + 1},$$

set

$$z(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(h(s))ds & \text{if } t \in I \end{cases}$$

then  $z$  is a mild solution of problem

$$\begin{cases} z'(t) = Az(t) + h(t) \\ z_0 = \varphi. \end{cases}$$

Moreover by assumption  $(H'_3)$ , the function  $t \mapsto q(t) = d(h(t), F(t, z_t))$  is integrable on  $I$ . It follows from theorem 3.1 for  $\mu = 0$  and  $v(t) = \frac{\varepsilon}{K(\omega)(\|k\|_1 + \omega) + 1}$  there exists  $x \in S_F(\varphi)$  such that

$$\begin{aligned} \|x - z\|_\omega &\leq K(\omega) \left[ \int_0^\omega q(t)dt + \int_0^\omega v(t)dt \right] \\ &\leq \frac{\varepsilon K(\omega)(\|k\|_1 + \omega)}{K(\omega)(\|k\|_1 + \omega) + 1} \end{aligned}$$

thus,

$$\begin{aligned} \|x - y\|_\omega &\leq \|x - z\|_\omega + \|z - y\|_\omega \\ &\leq \frac{\varepsilon K(\omega)(\|k\|_1 + \omega)}{K(\omega)(\|k\|_1 + \omega) + 1} + \frac{\varepsilon}{K(\omega)(\|k\|_1 + \omega) + 1} \\ &\leq \varepsilon. \blacksquare \end{aligned}$$

### 4 Some properties of the solution set

In this section, we discuss the continuous dependence of the solution set on parameters and initial value. We suppose that  $E$  is a reflexive Banach space.

**Theorem 4.1.** Let  $(\Lambda, d_\Lambda)$  be a metric space,  $F_\lambda : I \times \mathcal{C} \rightarrow \mathcal{F}_c(E)$  a family of multimappings satisfying conditions  $(H_1), (H_2)$  with the same function  $k$  and  $(H_4)$  for the same function  $\nu$ . If for any  $(t, \phi) \in I \times \mathcal{C}$ ,  $\lim_{\lambda \rightarrow \lambda_0} \delta(F_\lambda(t, \phi), F_{\lambda_0}(t, \phi)) = 0$ , then for all  $\varphi \in \mathcal{B}$ ,  $\lambda \mapsto S_{F_\lambda}(\varphi)$  is upper semicontinuous at  $\lambda_0$ .

**Proof.** Let  $x \in \limsup_{\lambda \rightarrow \lambda_0} S_{F_\lambda}(\varphi)$ , there exists a sequence  $(\lambda_n)$  such that  $\lim_{n \rightarrow +\infty} \lambda_n = \lambda_0$  and  $x^{\lambda_n} \in S_{F_{\lambda_n}}(\varphi)$  such that  $\lim_{n \rightarrow +\infty} x^{\lambda_n} = x$  in  $\mathcal{C}_\omega$ , hence

$$x^{\lambda_n}(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_{\lambda_n}(s))ds & \text{if } t \in I \end{cases}$$

where  $f_{\lambda_n}(s) \in F_{\lambda_n}(s, x_s^{\lambda_n})$  a.e. in  $I$ .

The sequence  $(f_{\lambda_n})$  is integrably bounded and  $E$  is reflexive, then by the Dunford-Pettis theorem [12], taking a subsequence and keeping the same notation, we may assume that it converges weakly in  $L^1(I; E)$  to some function  $f \in L^1(I; E)$ . For each  $t \in I$ , the mapping

$$g \in L^1(I; E) \rightarrow \int_0^t S(t-s)(g(s))ds$$

is a continuous linear operator from  $L^1(I; E)$  into  $E$ . It remains continuous if these spaces are endowed with the weak topologies [2]. Therefore for each  $t \in I$ , the sequence  $(x^{\lambda_n}(t))$  converges weakly to  $C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds$ . Since by assumption  $(x^{\lambda_n}(t))$  converges to  $x(t)$  in  $E$ , we have

$$x(t) = C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds.$$

We claim that  $f(s) \in F_{\lambda_0}(s, x_s)$  a.e. According to Mazur's theorem [6], the weak convergence implies the existence of the double sequence of nonnegative numbers  $(\alpha_{m,n})$  such that

i)  $\alpha_{m,n} = 0$  for  $n \geq n_0(m)$ ;

ii)  $\sum_{n=m}^{n_0(m)} \alpha_{m,n} = 1$  for  $m \in \mathbb{N}$ ;

iii) the sequence  $(\tilde{f}_m)$ , where  $\tilde{f}_m(t) = \sum_{n=m}^{n_0(m)} \alpha_{m,n} f_{\lambda_n}(t)$ , converges to  $f$  with respect to the norm of the space  $L^1(I, E)$ . Passing if necessary to a subsequence we can assume that  $(\tilde{f}_{m_j})$  converges to  $f$  almost everywhere on  $I$ . Moreover for almost everywhere  $s \in I$

$$\begin{aligned} d(f_{\lambda_n}(s), F_{\lambda_0}(s, x_s)) &\leq \delta(F_{\lambda_n}(s, x_s^{\lambda_n}), F_{\lambda_0}(s, x_s)) \\ &\leq \delta(F_{\lambda_n}(s, x_s^{\lambda_n}), F_{\lambda_n}(s, x_s)) + \delta(F_{\lambda_n}(s, x_s), F_{\lambda_0}(s, x_s)) \\ &\leq k(s) \|x_s^{\lambda_n} - x_s\| + \delta(F_{\lambda_n}(s, x_s), F_{\lambda_0}(s, x_s)) \end{aligned}$$

and since  $\lim_{\lambda \rightarrow \lambda_0} \delta(F_\lambda(t, \phi), F_{\lambda_0}(t, \phi)) = 0$ , then

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N, f_{\lambda_n}(s) \in F_{\lambda_0}(s, x_s) + 2\varepsilon B \text{ a.e. in } I$$

where  $B$  is the closed unit ball in  $E$ , and then, for all  $n > N$

$$\tilde{f}_{m_j}(s) \in \sum_{n=m_j}^{n_0(m_j)} \alpha_{m_j, n} (F_{\lambda_0}(s, x_s) + 2\varepsilon B) = F_{\lambda_0}(s, x_s) + 2\varepsilon B$$

taking the limit in the above formula, we deduce that for all  $\varepsilon > 0$ ,  $f(s) \in F_{\lambda_0}(s, x_s) + 2\varepsilon B$  a.e. in  $I$ , and then

$$f(s) \in F_{\lambda_0}(s, x_s) \text{ a.e. in } I.$$

**Remark** Since, in the theorem 4.1, the assumption  $E$  is reflexive is used only for deducing the sequence  $(f_{\lambda_n})$  converges weakly in  $L^1(I; E)$ , it may be replaced by the following assumption: there exists a  $k \geq 0$  such that for all bounded subset  $\Omega \subset \mathcal{C}$

$$\chi(F(t, \Omega)) \leq k\chi_0(\Omega) \text{ for all } t \in I$$

where  $\chi$  (resp.  $\chi_0$ ) is the measure of noncompactness in  $E$  (resp.  $\mathcal{C}$ ) (see for example [4, 11]). In this case, we obtain

$$\chi(\{f_{\lambda_n}(t) : n \in \mathbb{N}\}) \leq k\chi_0(\{x_t^{\lambda_n} : n \in \mathbb{N}\}) = 0$$

for almost all  $t \in I$ , i.e. the set  $\{f_{\lambda_n}(t) : n \in \mathbb{N}\}$  is relatively compact in  $E$  a.e. in  $I$  and since  $\sup_{n \in \mathbb{N}} \|f_{\lambda_n}\|_1 < +\infty$ , then from Diestel's theorem [4] it follows that the sequence  $(f_{\lambda_n})$  is relatively weak compact in the space  $L^1(I; E)$ .

**Theorem 4.2** ( $E$  is not reflexive). Let  $(\Lambda, d_\Lambda)$  be a metric space,  $F_\lambda : I \times \mathcal{C} \rightarrow \mathcal{F}(E)$  a family of multimappings satisfying the conditions  $(H_1)$ ,  $(H_2)$  with the same function  $k$ . If for any  $(t, \phi) \in I \times \mathcal{C}$  the multimapping  $\lambda \mapsto F_\lambda(t, \phi)$  is lower semicontinuous at  $\lambda_0 \in \Lambda$ , then for all  $\varphi \in \mathcal{B}$ ,  $\lambda \mapsto S_{F_\lambda}(\varphi)$  is lower semicontinuous at  $\lambda_0$ .

**Proof.** Since the case  $S_{F_{\lambda_0}}(\varphi) = \emptyset$  is trivial, we assume that  $S_{F_{\lambda_0}}(\varphi) \neq \emptyset$ . Let  $x \in S_{F_{\lambda_0}}(\varphi)$  then,

$$x(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f(s))ds & \text{if } t \in I \end{cases}$$

where  $f(s) \in F_{\lambda_0}(s, x_s) \subset \liminf_{\lambda \rightarrow \lambda_0} F_\lambda(s, x_s)$  a.e. in  $I$ , thus

$\lim_{\lambda \rightarrow \lambda_0} d(f(s), F_\lambda(s, x_s)) = 0$  a.e., and then for  $\varepsilon > 0$ , there exists  $\rho > 0$  such that  $d_\Lambda(\lambda, \lambda_0) < \rho$  implies  $d(f(s), F_\lambda(s, x_s)) < \frac{\varepsilon}{2\omega K(\omega)}$ . Thus for  $\lambda \in \Lambda$  such that

$$d_\Lambda(\lambda, \lambda_0) < \rho, t \mapsto d(f(t), F_\lambda(t, x_t)) = q(t)$$

is integrable and  $x$  is a mild solution of

$$\begin{cases} x''(t) = Ax(t) + f(t) \\ x_0 = \varphi \end{cases}$$

and by theorem 3.1 with  $\mu = 0$  and  $v(t) = \frac{\varepsilon}{2\omega K(\omega)}$  there exists a function  $x^\lambda \in S_{F_\lambda}(\varphi)$  (for  $d_\Lambda(\lambda, \lambda_0) < \rho$ ) such that

$$\|x^\lambda - x\|_\omega \leq K(\omega)m(\omega) = K(\omega)\left[\int_0^\omega (q(t) + v(t))dt\right] = \varepsilon,$$

hence  $x \in \liminf_{\lambda \rightarrow \lambda_0} S_{F_\lambda}(\varphi)$ . ■

Combining theorems 4.1 and 4.2, we obtain.

**Corollary** Let  $(\Lambda, d_\Lambda)$  be a metric space,  $F_\lambda : I \times \mathcal{C} \rightarrow \mathcal{F}_c(E)$  a family of multimappings satisfying the conditions  $(H_1)$ ,  $(H_2)$  with the same function  $k$  and  $(H_4)$  with the same function  $\nu$ . If for any  $(t, \phi) \in I \times \mathcal{C}$ ,  $\lim_{\lambda \rightarrow \lambda_0} \delta(F_\lambda(t, \phi), F_{\lambda_0}(t, \phi)) = 0$ , then for all  $\varphi \in \mathcal{B}$ ,  $\lambda \mapsto S_{F_\lambda}(\varphi)$  is continuous at  $\lambda_0$ .

**Theorem 4.3** Assume that  $F : I \times \mathcal{C} \rightarrow \mathcal{F}_c(E)$  satisfying the conditions  $(H_1)$ ,  $(H_2)$  and  $(H_4)$ . Then  $S_F : \mathcal{C}^1 \rightarrow \mathcal{P}(\mathcal{C}_\omega)$  is continuous on  $\mathcal{C}^1$ , where  $\mathcal{C}^1 := C^1(J; E)$  denote the Banach space of continuously differentiable  $E$ -valued functions on  $J$  with the norm  $\|\varphi\|_{\mathcal{C}^1} = \|\varphi\| + \|\varphi'\|$ .

**Proof.** For any  $\varphi_1, \varphi_2 \in \mathcal{C}^1$ , let  $F_{\varphi_2}(t, \phi) = F(t, \phi + (\tilde{\varphi}_2)_t - (\tilde{\varphi}_1)_t)$  for all  $(t, \phi) \in I \times \mathcal{C}$  then  $S_F(\varphi_2) = S_{F_{\varphi_2}}(\varphi_1) + \tilde{\varphi}_2 - \tilde{\varphi}_1$  where

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) & \text{if } t \in I \end{cases}$$

indeed,

$$x \in S_{F_{\varphi_2}}(\varphi_1) \Leftrightarrow x(t) = \begin{cases} \varphi_1(t) & \text{if } t \in J \\ C(t)\varphi_1(0) + S(t)\varphi_1'(0) + \int_0^t S(t-s)(f(s))ds & \text{if } t \in I \end{cases}$$

where  $f(s) \in F_{\varphi_2}(s, x_s)$  a.e.

$$\Leftrightarrow x(t) + \tilde{\varphi}_2(t) - \tilde{\varphi}_1(t) = \begin{cases} \varphi_2(t) \\ C(t)\varphi_2(0) + S(t)\varphi_2'(0) + \int_0^t S(t-s)(f(s))ds \end{cases}$$

where  $f(s) \in F(s, x_s + (\tilde{\varphi}_2)_s - (\tilde{\varphi}_1)_s) = F(s, (x + \tilde{\varphi}_2 - \tilde{\varphi}_1)_s)$  a.e.

$$\Leftrightarrow x + \tilde{\varphi}_2 - \tilde{\varphi}_1 \in S_F(\varphi_2).$$

Furthermore, it is clear that  $\varphi_2 \mapsto F_{\varphi_2}(t, \phi)$  (for all  $(t, \phi) \in I \times \mathcal{C}$ ) is continuous at  $\varphi_1$  and the family  $(F_{\varphi_2})_{\varphi_2 \in \mathcal{C}^1}$  satisfy the assumptions of precedent corollary, therefore for all  $\varphi \in \mathcal{C}^1$ ,  $\varphi_2 \mapsto S_{F_{\varphi_2}}(\varphi)$  is continuous at  $\varphi_1$  and then

$$\begin{aligned} \lim_{\varphi_2 \rightarrow \varphi_1} S_F(\varphi_2) &= \lim_{\varphi_2 \rightarrow \varphi_1} (S_{F_{\varphi_2}}(\varphi_1) + \tilde{\varphi}_2 - \tilde{\varphi}_1) \\ &= S_{F_{\varphi_1}}(\varphi_1) \\ &= S_F(\varphi_1). \quad \blacksquare \end{aligned}$$

**Theorem 4.4** ( $E$  is not reflexive) Assume that  $F : I \times \mathcal{C} \rightarrow \mathcal{F}_c(E)$  satisfying the conditions  $(H_1)$ ,  $(H_2)$  and  $(H'_4)$  i.e. there exists a compact  $K \subset E$  such that for every  $(t, \phi) \in I \times \mathcal{C}$ ,  $F(t, \phi) \subset K$ . Then for all  $\varphi \in \mathcal{B}$ ,  $S_F(\varphi)$  is compact.

**Proof.** We prove first that  $S_F(\varphi)$  is relatively compact. Let  $(x^n)$  be a sequence of  $S_F(\varphi)$ , then for all  $n \in \mathbb{N}$

$$x^n(t) = \begin{cases} \varphi(t) & \text{if } t \in J \\ C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)(f_n(s))ds & \text{if } t \in I \end{cases}$$

where  $f_n(s) \in F(s, x_s^n)$  a.e. in  $I$ .

We shall show that  $\mathcal{A} := \{x^n : n \in \mathbb{N}\}$  is equicontinuous. For each  $0 \leq t_0 < t \leq \omega$  and  $n \in \mathbb{N}$

$$\begin{aligned} \|x^n(t) - x^n(t_0)\|_E &\leq \|C(t)\varphi(0) - C(t_0)\varphi(0)\|_E + \|S(t)\varphi'(0) - S(t_0)\varphi'(0)\|_E + \\ &\quad \int_0^{t_0} \|S(t-s) - S(t_0-s)\| \|f_n(s)\|_E ds + \int_{t_0}^t \|S(t-s)\| \|f_n(s)\|_E ds \end{aligned}$$

but,

$$\begin{aligned} \|S(t-s) - S(t_0-s)\| &= \left\| \int_0^{t-s} C(\tau) d\tau - \int_0^{t_0-s} C(\tau) d\tau \right\| \\ &\leq \int_{t_0-s}^{t-s} \|C(\tau)\| d\tau \\ &\leq \int_{t_0-s}^{t-s} \eta e^{\alpha\tau} d\tau \\ &\leq \eta \alpha^{-1} [e^{\alpha(t-s)} - e^{\alpha(t_0-s)}] \\ &\leq \eta(t-t_0)e^{\alpha\omega} \end{aligned}$$

$(\alpha^{-1}[e^{\alpha(t-s)} - e^{\alpha(t_0-s)}])$  is replaced by  $t-t_0$  when  $\alpha = 0$ ), then

$$\int_0^{t_0} \|S(t-s) - S(t_0-s)\| \|f_n(s)\|_E ds \leq \eta(t-t_0)e^{\alpha\omega} \int_0^{t_0} \|f_n(s)\|_E ds.$$

Also,

$$\int_{t_0}^t \|S(t-s)\| \|f_n(s)\|_E ds \leq \eta(t-t_0)e^{\alpha\omega} \int_{t_0}^t \|f_n(s)\|_E ds.$$

Since  $f_n$  are integrably bounded and the maps  $t \rightarrow C(t)\varphi(0)$ ,  $t \rightarrow S(t)\varphi'(0)$  are uniformly continuous on  $I$ , we obtain that  $\mathcal{A}$  is equicontinuous, clearly it is also bounded. Now, we prove that  $\mathcal{A}(t) = \{x^n(t) : n \in \mathbb{N}\}$  is relatively compact. For all  $s \in I$ ,  $S(t-s) : E \rightarrow E$  is continuous, then by assumption  $(H'_4)$  we have that

$K_1 = \{S(t-s)f_n(s) : s \in [0, t] \text{ and } n \in \mathbb{N}\}$  is relatively compact, thus  $K_2 = \overline{\text{co}}K_1$  is compact and  $K_3 = \{tx : (t, x) \in I \times K_2\}$  is compact. Consequently

$\mathcal{A}(t) \subset C(t)\varphi(0) + S(t)\varphi'(0) + K_3$  is relatively compact. From the Ascoli theorem [4, 11]

we may assume that the sequence  $(x^n)$  converges to some  $x \in \mathcal{C}_\omega$ . We prove next that  $x \in S_F(\varphi)$ . By condition  $(H'_4)$ , the set  $\{f_n(t) : n \in \mathbb{N}\}$  is relatively compact in  $E$  and since  $\sup_{n \in \mathbb{N}} \|f_n\|_1 < +\infty$ , then from Diestel's theorem [4] it follows that the sequence

$(f_n)$  is relatively weak compact in the space  $L^1(I; E)$  and by using exactly the same method as in the proof of theorem 4.1 we obtain  $x \in S_F(\varphi)$ . ■

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