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ORTHONORMAL BASES FOR P-ADIC CONTINUOUS AND CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper we adapt the well-known Mahler and van der Put base of the Banach space of continuous functions to the case of the n-times continuously differentiable functions in one and several variables.

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1. Introduction

Let K be an algebraic extension of $\mathbb{Q}_p$, the field of p-adic numbers. As usual, we write $\mathbb{Z}_p$ for the ring of p-adic integers and $C(\mathbb{Z}_p \to K)$ for the Banach space of continuous functions from $\mathbb{Z}_p$ to K. We have the following well-known bases for $C(\mathbb{Z}_p \to K)$: on one hand, we have the Mahler base $\binom{x}{n}$ $(n \in \mathbb{N})$, consisting of polynomials of degree n and on the other hand we have the van der Put base $\{ e_n \mid n \in \mathbb{N} \}$ consisting of locally constant functions e_n defined as follows : $e_0(x) = 1$ and for $n > 0$, e_n is the characteristic function of the ball $\{ \alpha \in \mathbb{Z}_p \mid |\alpha - n| < 1/n \}$. For every f $\in C(\mathbb{Z}_p \to K)$ we have the following uniformly convergent series

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \text{ where } a_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(j)$$

$$f(x) = \sum_{n=0}^{\infty} b_n e_n(x) \text{ where } b_0 = f(0) \text{ and } b_n = f(n) - f(n_-).$$

Here n_- is defined as follows. For every $n \in \mathbb{N}_0$, we have a Hensel expansion $n = n_0 + n_1 p + ... + n_s p^s$ with $n_s \neq 0$. Then $n_- = n_0 + n_1 p + ... + n_{s-1} p^{s-1}$. We further put

$\gamma_0 = 1$, $\gamma_n = n - n_- = n_s p^s$, $\delta_0 = 1$, $\delta_n = p^s$ and $n_\sim = n - \delta_n$. Remark that $|\delta_n| = |\gamma_n|$.

In the sequel, we will also use the following notation, for $m, x \in \mathbb{Q}_p$, $x = \sum_{j=-\infty}^{\infty} a_j p^j : m \lhd x$ if $m = \sum_{j=-\infty}^{i} a_j p^j$ for some $i \in \mathbb{Z}$. We sometimes refer to the relation $\lhd$ between m and x as "m is an initial part of x" or "x starts with m".

Let $f : \mathbb{Z}_p \to K$. The (first) difference quotient $\phi_1 f : \nabla^2 \mathbb{Z}_p \to K$ is defined by $\phi_1 f(x,y) = \dfrac{f(y) - f(x)}{y - x}$, where $\nabla^2 \mathbb{Z}_p = \mathbb{Z}_p \times \mathbb{Z}_p \setminus \{(x,x) \mid x \in \mathbb{Z}_p\}$. f is called continuously differentiable (or strictly differentiable, or uniformly differentiable) at $a \in \mathbb{Z}_p$ if $\lim_{(x,y)\to(a,a)} \phi_1 f(x,y)$ exists. We will also say that f is C^1 at a. In a similar way, we may define C^n-functions as follows : for $n \in \mathbb{N}$, we define $\nabla^{n+1}\mathbb{Z}_p = \{(x_1, ..., x_{n+1}) \in \mathbb{Z}_p^{n+1} \mid x_i \neq x_j$ if $i \neq j\}$ and the n-th difference quotient $\phi_n f : \nabla^{n+1}\mathbb{Z}_p \to K$ by $\phi_0 f = f$ and

$$\phi_n f(x_1, x_2, ..., x_{n+1}) = \frac{\phi_{n-1} f(x_2, x_3, ..., x_{n+1}) - \phi_{n-1} f(x_1, x_3, ..., x_{n+1})}{x_2 - x_1}$$

. A function f is called a C^n-function if $\phi_n f$ can be extended to a continuous function $\overline{\phi_n f}$ on $\mathbb{Z}_p^{n+1}$. Recall from [4],[5] that $\overline{\phi_n f}(x, x, ..., x) = \dfrac{f^{(n)}(x)}{n!}$, for all $x \in \mathbb{Z}_p$. The set of all C^n-functions from $\mathbb{Z}_p$ to K will be denoted by $C^n(\mathbb{Z}_p \to K)$. For any C^n-function f, we define $||f||_n = max\{\, ||\phi_j f||_s \mid 0 \leq j \leq n\,\}$where $||\cdot||_s$ is the sup norm. (For $f : X \to K, ||f||_s = max_{x \in X}|f(x)|$) $||\cdot||_n$ is a norm on C^n, making C^n into a Banach space.

2. Generalization of the Mahler base for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$

One can construct other orthonormal bases of $C(\mathbb{Z}_p \to K)$ by generalizing the procedure used to define the Mahler base as did Y. Amice. In general, we have the following characterization of the polynomial sequences $e_n \in K[x], n \geq 0$ such that $deg(e_n) = n$ and which are orthonormal bases of the space $C(B \to K)$, where $B = \{\, x \in K \mid |x| \leq 1\,\}$.

Theorem : Let $(e_n)_{n \geq 0}$ be a sequence of polynomials in $K[x]$ of degree n. They form an orthonormal base of $C(B \to K)$ if and only if $||e_n||_s = 1$ and $||e_n||_G = |\text{coeff } x^n| = |\pi|^{-(n-s(n))/(q-1)}$ where π is a uniformizing parameter of K, q the cardinality of the residue class field of K and $s(n)$ the sum of the digits of n in base q. By the way, for a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$, $||f||_G = max_{i \leq n}|a_i|$.

Given an orthonormal base, we can construct other orthonormal bases by taking a certain linear combination of the given base as will be stated in the following theorem.

Theorem : Let $e_n (n \in \mathbb{N})$ be an orthonormal base of $C(\mathbb{Z}_p \to K)$ and put $p_n = \sum_{j=0}^{n} a_{n,j} e_j$ where $a_{n,j} \in K$ and $a_{n,n} \neq 0$. The $p_n (n \in \mathbb{N})$ form an orthonormal base for

$C(\mathbb{Z}_p \to K)$ if and only if $|a_{n,j}| \leq 1$ for all $j \leq n$ and $|a_{n,n}| = 1$.
We can generalize the Mahler base also by changing the degree of the polynomials as follows.

Theorem : The polynomials $q_n(x) = \binom{px}{pn}$ $(n \in \mathbb{N})$ form an orthonormal base for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$ and every continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ can be written as a uniformly convergent series $f(x) = \sum_{n=0}^{\infty} a_{pn} \binom{px}{pn}$

with $\quad a_{pn} = \sum_{k=0}^{n} (-1)^{n-k} \binom{pn}{pk} \alpha_{n-k}^{(p)} f(k)$

and $\quad \alpha_0^{(p)} = 1,\ \alpha_m^{(p)} = \sum_{\substack{l_1 \ldots l_r \\ 1 \leq l_i \leq m \\ \Sigma l_i = m}} (-1)^{r+m} \binom{pm}{pl_1 \ldots pl_r}$

If we mix the Mahler and van der Put base together, we obtain a new orthonormal base.

Theorem : The sequence $q_n(x) = \binom{x}{n} .e_n(x) (n \in \mathbb{N})$ forms an orthonormal base for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$. Moreover, every continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ can be written as a uniformly convergent series $f(x) = \sum_{i=0}^{\infty} a_i \binom{x}{i} e_i(x)$

with $\quad a_i = \sum_{j \triangleleft i} \alpha_{i,j} f(j)$

and $\quad \alpha_{i,i} = 1,\ \alpha_{i,j} = \sum_{j=k_0 \triangleleft k_1 \triangleleft \ldots \triangleleft k_n = i} (-1)^n \binom{i}{k_{n-1}} \binom{k_{n-1}}{k_{n-2}} \cdots \binom{k_1}{j}$

3. Differentiable functions

For C^n-functions the polynomials $\binom{x}{i}$ $(i \in \mathbb{N})$ still remain a base, we only have to add the factor $\gamma_i \gamma_{[i/2]} \ldots \gamma_{[i/n]}$ where $\gamma_i = i - i_-$ and $[\alpha]$ denotes the integer part of α, to obtain the orthonormal base $\gamma_i \gamma_{[i/2]} \ldots \gamma_{[i/n]} \binom{x}{i}$. The proof is based on the following lemma in case n = 2.

Lemma Let f be a continuous function with interpolation coefficients a_n. Then f is a C^2-function if and only if $\left| \frac{a_{i+j+k+2}}{(k+1)(j+k+2)} \right| \to 0$ as $i + j + k$ approach infinity.

Corollary If f is a C^2-function, then $||\phi_2 f||_s = sup_n \left| \frac{a_n}{\gamma_n \gamma_{[n/2]}} \right|$

A similar property does not hold for the van der Put base.
In case n = 1, we know that $\{ \gamma_i e_i(x) \mid i \in \mathbb{N} \} \cup \{ (x - i).e_i(x) \mid i \in \mathbb{N} \}$ is an orthonormal base for $C^1(\mathbb{Z}_p \to K)$. Therefore every continuously differentiable function f can be written

under the form $f(x) = \sum a_n e_n(x) + \sum b_n(x-n)e_n(x)$ where $a_0 = f(0)$, $a_n = f(n) - f(n_-) - (n-n_-).f'(n_-)$, $b_0 = f'(0)$ and $b_n = f'(n) - f'(n_-)$. For details we refer to [6].
The case n = 2, can be treated as follows.

Theorem : Let $f(x) = \sum_{n=0}^{\infty} a_n e_n(x) + \sum_{n=0}^{\infty} b_n(x-n)e_n(x) \in C^1(\mathbb{Z}_p \to K)$.

$f \in C^2(\mathbb{Z}_p \to K)$ if and only if $\lim_{n\to a} \frac{a_n}{\gamma_n^2}$ and $\lim_{n\to a} \frac{b_n}{\gamma_n}$ exist for all $a \in \mathbb{Z}_p$, and $\lim_{n\to a} \frac{b_n}{\gamma_n} = 2 \lim_{n\to a} \frac{a_n}{\gamma_n^2}$

Theorem : $\{\gamma_n^2 e_n(x), \gamma_n(x-n)e_n(x), (x-n)^2 e_n(x) \mid n \in \mathbb{N}\}$ is an orthonormal base for $C^2(\mathbb{Z}_p \to K)$ and for every $f \in C^2(\mathbb{Z}_p \to K)$ we have

$$f(x) = \sum_{n=0}^{\infty} a_n e_n(x) + \sum_{n=0}^{\infty} b_n(x-n)e_n(x) + \sum_{n=0}^{\infty} c_n \frac{(x-n)^2}{2} e_n(x) \text{ with}$$

$$a_0 = f(0)$$
$$a_n = f(n) - f(n_-) - (n-n_-).f'(n_-) - \frac{(n-n_-)^2}{2} f''(n_-) \qquad \text{for } n \neq 0$$
$$b_0 = f'(0)$$
$$b_n = f'(n) - f'(n_-) - (n-n_-).f''(n_-) \qquad \text{for } n \neq 0$$
$$c_0 = f''(0)$$
$$c_n = f''(n) - f''(n_-) \qquad \text{for } n \neq 0$$

The construction of this orthonormal base, which is very technical, is based on the use of an antiderivation map $P_n : C^{n-1}(\mathbb{Z}_p \to K) \to C^n(\mathbb{Z}_p \to K)$ defined by $P_n f(x) = \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{f^{(j)}(x_m)}{(j+1)!}(x_{m+1} - x_m)^{j+1}$ with $x_m = \sum_{j=-\infty}^{m} a_j p^j$ if $x = \sum_{j=-\infty}^{+\infty} a_j p^j$ and on the two following lemmas.

Lemma : For $(t_1, ..., t_k) \in \nabla^k X = \{(x_1, x_2, ..., x_k) \mid x_i \neq x_j \text{ if } i \neq j\}$ with $t_1 = x, t_i = y$ and $t_k = z$, we have

$$\phi_2 f(x,y,z) = \sum_{j=2}^{k-1} \mu_j \phi_2 f(t_{j-1} t_j, t_{j+1}) \text{ with } \mu_j = \begin{cases} \frac{(t_{j+1}-t_{j-1})(t_j-t_k)}{(z-x)(y-z)} & \text{for } j \geq i \\ \frac{(t_{j+1}-t_{j-1})(t_j-t_1)}{(z-x)(y-x)} & \text{for } j \leq i \end{cases}$$

Moreover, $\sum_{j=2}^{k-1} \mu_j = 1$

Lemma : Let S be a ball in K and $f \in C(\mathbb{Z}_p \to K)$.
Suppose that $\phi_2 f(n, n-\delta_n, n+p^k\delta_n) \in S$ for all $n \in \mathbb{N}_0, k \in \mathbb{N}$, then $\phi_2 f(x,y,z) \in S$ for all $x, y, z \in \mathbb{Z}_p, x \neq y, x \neq z, y \neq z$

4. Several variables

We can also construct the Mahler and van der Put base for functions of several variables. This brings us to the following results.

Theorem : The family $max\{\gamma_n, \gamma_m\}\cdot \binom{x}{n} \cdot \binom{y}{m}$ $(n, m \in \mathbb{N})$ forms an orthonormal base for $C^1(\mathbb{Z}_p \times \mathbb{Z}_p \to K)$.

The proof is based on

Theorem : $f(x,y) = \sum_{n,m} a_{n,m} \binom{x}{n}\binom{y}{m}$ is a C^1-function if and only if $\left|\frac{a_{i+j+1,k}}{j+1}\right| \to 0$ and $\left|\frac{a_{i,j+k+1}}{k+1}\right| \to 0$ as $i+j+k$ approach infinity or equivalently $\left|\frac{a_{n,m}}{\gamma_n}\right| \to 0$ and $\left|\frac{a_{n,m}}{\gamma_m}\right| \to 0$ as $n+m$ approach infinity.

Starting with the van der Put base $e_n (n \in \mathbb{N})$ of $C(\mathbb{Z}_p \to K)$, we get

Theorem : The family $e_n(x)e_m(y), (x-n)e_n(x)e_m(y), (y-m)e_n(x)e_m(y)$ $(n, m \in \mathbb{N})$ forms an orthogonal base for $C^1(\mathbb{Z}_p \times \mathbb{Z}_p \to K)$ and every C^1-function f can be written as $f(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j}e_i(x)e_j(y) + b_{i,j}(x-i)e_i(x)e_j(y) + c_{i,j}(y-j)e_i(x)e_j(y)$

with

$$a_{0,0} = f(0,0)$$

$$a_{n,0} = f(n,0) - f(n_-,0) - \gamma_n \frac{\partial f}{\partial x}(n_-,0) \qquad \text{for } n \neq 0$$

$$a_{0,m} = f(0,m) - f(0,m_-) - \gamma_m \frac{\partial f}{\partial y}(0,m_-) \qquad \text{for } m \neq 0$$

$$a_{n,m} = f(n,m) - f(n_-,m) - f(n,m_-) + f(n_-,m_-) - \gamma_n \left(\frac{\partial f}{\partial x}(n_-,m) - \frac{\partial f}{\partial x}(n_-,m_-)\right)$$
$$-\gamma_m \left(\frac{\partial f}{\partial y}(n,m_-) - \frac{\partial f}{\partial y}(n_-,m_-)\right) \qquad \text{for } n \neq 0 \text{ and } m \neq 0$$

$$b_{0,0} = \frac{\partial f}{\partial x}(0,0)$$

$$b_{n,0} = \frac{\partial f}{\partial x}(n,0) - \frac{\partial f}{\partial x}(n_-,0) \qquad \text{for } n \neq 0$$

$$b_{0,m} = \frac{\partial f}{\partial x}(0,m) - \frac{\partial f}{\partial x}(0,m_-) \qquad \text{for } m \neq 0$$

$$b_{n,m} = \frac{\partial f}{\partial x}(n,m) - \frac{\partial f}{\partial x}(n_-,m) - \frac{\partial f}{\partial x}(n,m_-) + \frac{\partial f}{\partial x}(n_-,m_-) \qquad \text{for } n \neq 0 \text{ and } m \neq 0$$

$$c_{0,0} = \frac{\partial f}{\partial y}(0,0)$$

$$c_{n,0} = \frac{\partial f}{\partial y}(n,0) - \frac{\partial f}{\partial y}(n_-,0) \qquad \text{for } n \neq 0$$

$$c_{0,m} = \frac{\partial f}{\partial y}(0,m) - \frac{\partial f}{\partial y}(0,m_-) \qquad \text{for } m \neq 0$$

$$c_{n,m} = \frac{\partial f}{\partial y}(n,m) - \frac{\partial f}{\partial y}(n_-,m) - \frac{\partial f}{\partial y}(n,m_-) + \frac{\partial f}{\partial y}(n_-,m_-) \qquad \text{for } n \neq 0 \text{ and } m \neq 0$$

Remark : To obtain an orthonormal base, the $e_i(x)e_j(y)$ should be multiplied by

$max\{\gamma_i, \gamma_j\}$; the $(x-i)e_i(x)e_j(y)$ by $max\left\{\frac{1}{p\gamma_i}, 1, \frac{\gamma_j}{p\gamma_i}\right\}$ in case $i \neq 0$ and by γ_j in case $i = 0$ and analogous for $(y-j)e_i(x)e_j(y)$.

Generalization : The sequence $(x-i)^k(y-j)^l e_i(x)e_j(y)$ with $0 \le k+l \le n, i \in \mathbf{N}$ and $j \in \mathbf{N}$ forms an orthogonal base for $C^n(\mathbf{Z}_p \times \mathbf{Z}_p \to K)$ whereby every C^n-function f can be written as $f(x,y) = \sum_{i,j=0}^{\infty} \sum_{k+l=0}^{n} a_{i,j}^{k,l} \frac{(x-i)^k}{k!} \frac{(y-j)^l}{l!} e_i(x)e_j(y)$ with

$$a_{i,j}^{k,l} = \frac{\partial^{k+l} f}{\partial x^k \partial y^l}(i,j) - \sum_{\alpha=0}^{n-k-l} \frac{\partial^{k+l+\alpha} f}{\partial x^{k+\alpha} \partial y^l}(i_-, j) \frac{\gamma_i^\alpha}{\alpha!} - \sum_{\beta=0}^{n-k-l} \frac{\partial^{k+l+\beta} f}{\partial x^k \partial y^{l+\beta}}(i, j_-) \frac{\gamma_j^\beta}{\beta!} + \sum_{\alpha+\beta=0}^{n-k-l} \frac{\partial^{k+l+\alpha+\beta} f}{\partial x^{k+\alpha} \partial y^{l+\beta}}(i_-, j_-) \frac{\gamma_i^\alpha \gamma_j^\beta}{\alpha! \beta!} \quad \text{for } i \neq 0 \text{ and } j \neq 0$$

$$a_{i,0}^{k,l} = \frac{\partial^{k+l} f}{\partial x^k \partial y^l}(i,0) - \sum_{\alpha=0}^{n-k-l} \frac{\partial^{k+l+\alpha} f}{\partial x^{k+\alpha} \partial y^l}(i_-, 0) \frac{\gamma_i^\alpha}{\alpha!} \quad \text{for } i \neq 0$$

$$a_{0,j}^{k,l} = \frac{\partial^{k+l} f}{\partial x^k \partial y^l}(0,j) - \sum_{\beta=0}^{n-k-l} \frac{\partial^{k+l+\beta} f}{\partial x^k \partial y^{l+\beta}}(0, j_-) \frac{\gamma_j^\beta}{\beta!} \quad \text{for } j \neq 0$$

and $a_{0,0}^{k,l} = \frac{\partial^{k+l} f}{\partial x^k \partial y^l}(0,0)$

The previous theorems show that $C^n(\mathbf{Z}_p \times \mathbf{Z}_p \to K)$ is not the complete tensor product of $C^n(\mathbf{Z}_p \to K)$ with $C^n(\mathbf{Z}_p \to K)$ as one may expect, considering the case $C(\mathbf{Z}_p \times \mathbf{Z}_p \to K)$. Therefore we define a finer structure for functions of two variables.

Definition :

$$\phi_{0,0} f(x_0, y_0) = f(x_0, y_0)$$

$$\phi_{1,0} f(x_0, x_1, y_0) = \frac{f(x_0,y_0) - f(x_1,y_0)}{x_0 - x_1} \quad \text{for } x_0 \neq x_1$$

$$\phi_{0,1} f(x_0, y_0, y_1) = \frac{f(x_0,y_0) - f(x_0,y_1)}{y_0 - y_1} \quad \text{for } y_0 \neq y_1$$

$$\vdots$$

$$\phi_{i,j} f(x_0, x_1, ..., x_i, y_0, y_1, ..., y_j)$$

$$= \frac{\phi_{i-1,j} f(x_0, ..., x_{i-2}, x_{i-1}, y_0, ..., y_j) - \phi_{i-1,j} f(x_0, ..., x_{i-2}, x_i, y_0, ..., y_j)}{x_{i-1} - x_i}$$

$$= \frac{\phi_{i,j-1} f(x_0, ..., x_i, y_0, ..., y_{j-2}, y_{j-1}) - \phi_{i,j-1} f(x_0, ..., x_i, y_0, ..., y_{j-2}, y_j)}{y_{j-1} - y_j}$$

for $(x_0, x_1, ..., x_i, y_0, y_1, ..., y_j) \in \nabla^{i+1}\mathbf{Z}_p \times \nabla^{j+1}\mathbf{Z}_p$ is the differencequotient of order i in the first variable and order j in the second variable of the function f from $\mathbf{Z}_p \times \mathbf{Z}_p$ to K.

Definition : $f : \mathbf{Z}_p \times \mathbf{Z}_p \to K$ is m times strictly differentiable in his first variable and n times strictly differentiable in his second variable (for short : a $C^{m,n}$-function) if and only if $\phi_{m,n}f$ can be extended to a continuous function $\overline{\phi_{m,n}f}$ on $\mathbb{Z}_p^{m+n+2}$. The set of all $C^{m,n}$-functions $f : \mathbb{Z}_p \times \mathbb{Z}_p \to K$ is denoted $C^{m,n}(\mathbb{Z}_p \times \mathbb{Z}_p \to K)$. For $f : \mathbb{Z}_p \times \mathbb{Z}_p \to K$, set $||f||_{m,n} = max_{\substack{0\le i\le m\\ 0\le j\le n}} ||\phi_{i,j}f||_s$.

For these functions, we get the following equivalent of the Mahler base.

Theorem : The family $\gamma_i\gamma_{[i/2]}\cdots\gamma_{[i/m]}\gamma_j\gamma_{[j/2]}\cdots\gamma_{[j/n]}\binom{x}{i}\binom{y}{j}$ $(i,j \in \mathbb{N})$ forms an orthonormal base for $C^{m,n}(\mathbb{Z}_p \times \mathbb{Z}_p \to K)$

Since it can be easily seen that there is an isometry between the complete tensor product $C^m(\mathbb{Z}_p \to K)\hat{\otimes}C^n(\mathbb{Z}_p \to K)$ and $C^{m,n}(\mathbb{Z}_p \times \mathbb{Z}_p \to K)$, the van der Put base for $C^{m,n}$-functions is given as follows.

Theorem : The family $\gamma_i^{m-k}(x-i)^k\gamma_j^{n-l}(y-j)^l e_i(x)e_j(y)$ with $0 \le k \le m$, $0 \le l \le n$, $i \in \mathbb{N}$ and $j \in \mathbb{N}$ forms an orthonormal base for $C^{m,n}(\mathbb{Z}_p \times \mathbb{Z}_p \to K)$ whereby every $C^{m,n}$-function f can be written as $f(x,y) = \sum_{i,j=0}^{\infty}\sum_{k=0}^{m}\sum_{l=0}^{n} a_{i,j}^{k,l}\frac{(x-i)^k}{k!}\frac{(y-j)^l}{l!}e_i(x)e_j(y)$ with

$$a_{i,j}^{k,l} = \frac{\partial^{k+l}f}{\partial x^k\partial y^l}(i,j) - \sum_{\alpha=0}^{m-k}\frac{\partial^{k+l+\alpha}f}{\partial x^{k+\alpha}\partial y^l}(i_-,j)\frac{\gamma_i^\alpha}{\alpha!} - \sum_{\beta=0}^{n-l}\frac{\partial^{k+l+\beta}f}{\partial x^k\partial y^{l+\beta}}(i,j_-)\frac{\gamma_j^\beta}{\beta!}$$
$$+\sum_{\alpha=0}^{m-k}\sum_{\beta=0}^{n-l}\frac{\partial^{k+l+\alpha+\beta}f}{\partial x^{k+\alpha}\partial y^{l+\beta}}(i_-,j_-)\frac{\gamma_i^\alpha\gamma_j^\beta}{\alpha!\beta!} \qquad \text{for } i \neq 0 \text{ and } j \neq 0$$

$$a_{i,0}^{k,l} = \frac{\partial^{k+l}f}{\partial x^k\partial y^l}(i,0) - \sum_{\alpha=0}^{m-k}\frac{\partial^{k+l+\alpha}f}{\partial x^{k+\alpha}\partial y^l}(i_-,0)\frac{\gamma_i^\alpha}{\alpha!} \qquad \text{for } i \neq 0$$

$$a_{0,j}^{k,l} = \frac{\partial^{k+l}f}{\partial x^k\partial y^l}(0,j) - \sum_{\beta=0}^{n-l}\frac{\partial^{k+l+\beta}f}{\partial x^k\partial y^{l+\beta}}(0,j_-)\frac{\gamma_j^\beta}{\beta!} \qquad \text{for } j \neq 0$$

and $a_{0,0}^{k,l} = \frac{\partial^{k+l}f}{\partial x^k\partial y^l}(0,0)$

REFERENCES

[1] Y. AMICE, Les nombres p-adiques, Presses Universitaires de France, 1975.

[2] S. DE SMEDT, Some new bases for p-adic continuous functions, Indagationes Mathematicae N.S. 4(1), 1993, p.91-98.

[3] S. DE SMEDT, The van der Put base for C^n-functions, Bulletin of the Belgian Mathematical Society 1(1), 1994, p.85-98.

[4] S. DE SMEDT, p-adic continuously differentiable functions of several variables, Collectanea Mathematica, To appear.

[5] W.H. SCHIKHOF, Non-Archimedean calculus (Lecture notes), Report 7812, Katholieke Universiteit Nijmegen, 1978.

[6] W.H. SCHIKHOF, Ultrametric Calculus : an introduction to p-adic analysis, Cambridge University Press, 1984.

[7] M. VAN DER PUT, Algèbres de functions continues p-adiques, Thèse Université d'Utrecht, 1967.

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